MAPPINGS OF SOME HYPERSPACES

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Abstract

For a given continuum we study mappings between some of its hyperspaces. In particular we consider retractions and selections. Several related questions are asked.

1. Introduction

Let X be a metric continuum, and let 2^X (respectively, C(X)) be the hyperspace of all its nonempty closed subsets (respectively, subcontinua) of X, equipped with the Hausdorff metric. Nadler, Jr. asks (see [49, 3.1, p. 193] and [51, 4.4 and 4.5, p. 243]) the following question.

Problem 1.1 (Nadler). When is X a continuous image of 2^X or of C(X)?

In the same paper [49] he gives some necessary and some sufficient

Received November 27, 2003

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 $^{2000\} Mathematics\ Subject\ Classification:\ 54B20,\ 54C05,\ 54C15,\ 54F15,\ 54F50.$

Key words and phrases: compactification, continuum, hyperspace, mapping, retraction, selection.

conditions for the existence of a mapping from 2^X or C(X) onto X. For example, the existence of such a mapping implies that X is weakly chainable (in the sense of [40] or — equivalently — X is a continuous image of the pseudo-arc; compare [18]) and that X is the union of two proper subcontinua each of which is arcwise connected [49, Theorem 3.2, p. 193]. In the case when the continuum X is arc-like (circle-like) such a mapping exists if and only if X is an arc (a simple closed curve, respectively) [49, 3.3, p. 193]. These results are extended below, see Theorems 3.6 and 3.7.

A space is said to be *g*-contractible [1, p. 15] provided that there exists a surjective mapping from the space onto itself which is homotopic to a constant mapping (if the space is contractible, then the surjection can be chosen to be the identity mapping). If a continuum X is *g*-contractible, then there is a mapping from C(X) onto X (see [1, Lemma II, p. 15]) and from 2^X onto X [49, Theorem 3.4, p. 193].

Since the hyperspace $F_1(X)$ of singletons of X is a subspace of 2^X and is homeomorphic to X, the continuum X can be considered as naturally embedded in C(X). Thus, identifying X with $F_1(X)$, we have $X \subset C(X) \subset 2^X$. So, the following is a particular case of Problem 1.1 (see [47, p. 413] and also (51, (6.2), p. 270]).

Problem 1.2 (Nadler). What are necessary and/or sufficient conditions in order that a continuum X be a retract of 2^X or C(X)?

Some partial answers to this question, especially in its part concerning 2^X , in particular for curves (i.e., one-dimensional continua) are presented in [10] and in [11, Chapters 5 and 6].

Let *n* be a positive integer, *X* be a continuum, and $C_n(X) \subset 2^X$ denote the hyperspace of all members of 2^X that have no more than *n* components (again equipped with the inherited topology induced by the Hausdorff metric). Identifying C(X) with $C_1(X)$, we have

$$X \subset C_1(X) \subset \dots \subset C_n(X) \subset C_{n+1}(X) \subset \dots \subset 2^X.$$

$$(1.3)$$

In the light of the above inclusions Nadler's Problems 1.1 and 1.2 can be extended as follows.

Problems 1.4. What are necessary and/or sufficient conditions in order that a continuum X be a (continuous) image (a retract, in particular) of $C_n(X)$ for some positive integer n (or for all n)?

Conditions mentioned in Problems 1.2 and 1.4 are known in a very particular case when X is locally connected. Namely the following result is known (see [47, p. 413], and [51, Theorem 6.4, p. 270]).

Theorem 1.5 (Nadler). A locally connected continuum X is a retract of 2^X if and only if X is an absolute retract.

As a consequence we get a corollary.

Corollary 1.6 (Nadler). A locally connected curve X is a retract of 2^X if and only if X is a dendrite.

The above quoted results were recently enriched by the second named author's studies in Chapter 4 of [43] as follows (see [43, Theorems 4.13-4.15, p. 269]).

Theorem 1.7. For any locally connected continuum X and for any positive integer n the following conditions are equivalent:

(1.7.1) X is an absolute retract;

(1.7.2) X is a retract of $C_n(X)$;

(1.7.3) X is a deformation retract of $C_n(X)$;

(1.7.4) X is a strong deformation retract of $C_n(X)$.

Note that the above theorem provides partial solutions to Problem 1.4.

Theorem 1.8. For any locally connected continuum X and for any positive integer n the following conditions are satisfied:

(1.8.1) $C_n(X)$ is a retract of 2^X ;

(1.8.2) $C_n(X)$ is a deformation retract of 2^X ;

(1.8.3) $C_n(X)$ is a strong deformation retract of 2^X .

Theorem 1.9. For any locally connected continuum X and for any integer n > 1 the following conditions are satisfied:

(1.9.1) C(X) is a retract of $C_n(X)$;

(1.9.2) C(X) is a deformation retract of $C_n(X)$;

(1.9.3) C(X) is a strong deformation retract of $C_n(X)$.

For all continua, not necessarily locally connected ones, the situation is much more complicated and it does not seem likely to be clarified soon. However, there are many partial results, examples in particular, which describe various situations. Many interrelations between several conditions concerning mappings between hyperspaces, in particular hyperspace retractions, as well as suitable examples, are presented in the two books on hyperspaces, [51] and [32].

It is known that if a one-dimensional continuum X is a retract of either 2^X or C(X), then it is a dendroid, i.e., it is arcwise connected and hereditarily unicoherent [26, p. 122]. Goodykoontz, Jr. shows in [25] (in [24]) an example of a nonlocally connected continuum X which is a smooth dendroid such that C(X) is (is not, respectively) a retract of 2^X . Illanes in [30] constructs an example of a continuum X which is a retract of C(X) but not of 2^X , and which does not admit any mean. In [31] two examples of dendroids are constructed concerning the existence of a selection and of a retraction on C(X).

In [26] a complete discussion is given about all the implications concerning the existence of retractions, deformation retractions and strong deformation retractions between $F_1(X)$, C(X) and 2^X ; conclusions are collected in Table I of [26, p. 130]. Results obtained later are presented in a table in [32, p. 372].

Hyperspace retractions for a class of half-line compactifications are studied and many very interesting results are obtained by Curtis in [14]. He studies a special type (called "regular") of half-line compactifications with a locally connected remainder and shows that if X is such, then a retraction from 2^X onto C(X) exists if and only if X is homeomorphic to one of continua X_m for a nonnegative integer m as defined (and pictured) in [14, p. 31]. It is tempting to know whether or not this result can be extended so that 2^X is replaced by $C_n(X)$. Thus we have the following question.

Question 1.10. What results of [14] can be extended from 2^X to $C_n(X)$ for n > 1?

In the present paper we give some partial solutions to this question. The paper consists of five chapters. After an introduction, definitions and basic properties of the considered concepts are collected in the second chapter. We hope that general results which are gathered at the end of this chapter are of some independent value and can be useful in further studies in the area. Next, mappings between hyperspaces are studied in Chapter 3. The results contained there generalize ones known for mappings from or onto the hyperspace C(X) of subcontinua of a given continuum X. Hyperspace retractions are studied in Chapter 4. We start with showing that if a curve X admits such a retraction from $C_n(X)$ onto X, then X is a uniformly arcwise connected dendroid (Theorem 4.2). This is an extension of [10, Theorem 3.1, p. 9]. Other results of this chapter give some conditions for the existence of a retraction from $C_n(X)$ onto C(X) for some special continua X. The results are related to the ones presented by Curtis in [14]. The fifth chapter is devoted to selections for the hyperspace $C_n(X)$. It is shown that if n > 1, then a selection for $C_n(X)$ does exist if and only if X is an arc.

2. Preliminaries and General Results

All considered spaces are assumed to be metric. We denote by \mathbb{N} the set of all positive integers, and by \mathbb{R} the space of reals. A *continuum* means a compact connected space, and a *mapping* means a continuous function.

A *curve* means a one-dimensional continuum. A continuum is said to be unicoherent provided that the intersection of any two of its subcontinua whose union is the whole continuum is connected. A property of a continuum X is said to be *hereditary* provided that the whole space X has the property, as well as every subcontinuum of X also has this property. Thus, in particular, a continuum is said to be hereditarily unicoherent provided that each of its subcontinua is unicoherent. If S is an arbitrary set in a continuum X, we denote by I(S)a continuum in X containing S such that none of its proper subcontinua contains S, i.e., I(S) means a continuum in X which is *irreducible about* the set $S \subset X$. It is known (see [6, Theorem T1, p. 187]) that in hereditarily unicoherent continua X the continuum I(S) is unique (equal to the intersection of all subcontinua containing S); moreover, the above uniqueness characterizes hereditarily unicoherent continua (see [46, Theorem 1.1, p. 179). Therefore, for hereditarily unicoherent continua, the assignment I described above can be considered as a function $I: 2^X \to C(X)$. For its application to characterizations of some curves see [23, Theorems 1 and 8, p. 3 and 7].

If any two points of a space can be joined by an arc lying in the space, then the space is said to be *arcwise connected*. A space is said to be *uniquely arcwise connected* provided that for every two its points there is exactly one arc joining these points. A *dendrite* means a locally connected continuum that contains no simple closed curve. A *dendroid* means an arcwise connected and hereditarily unicoherent continuum. It follows that the concept of a dendrite coincides with that of a locally connected dendroid. An *end point* of a dendroid X is defined as a point p of X which is an end point of each arc containing p. By a *ramification point* of a dendroid X we understand a point which is the center of a simple triod contained in X. A dendroid having at most one ramification point v is called a *fan*, and v is called its *top*. The cone over the closure of the harmonic sequence of points is called the *harmonic fan*. The cone over the Cantor middle-thirds set is called the *Cantor fan*. A dendrite is said to be *finite* provided that the set of all its end points is finite.

A continuum X is said to be uniformly pathwise connected provided

that it is a continuous image of the Cantor fan. The original definition of this concept, given in [37, p. 316], is more complicated, but it agrees with the above one by Theorem 3.5 of [37, p. 322]. A space X is said to be *uniformly arcwise connected* provided that it is arcwise connected and that for each $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that every arc in X contains k points that cut it into subarcs of diameters less than ε . By Theorem 3.5 in [37, p. 322], each uniformly arcwise connected continuum is uniformly pathwise connected (but not conversely) and it is easy to see that for uniquely arcwise connected continua these two notions coincide (compare [37, p. 316]). In particular the coincidence holds for dendroids.

A dendroid X is said to be *smooth at a point* $v \in X$ provided that for each sequence of points $a_n \in X$ which converges to a point $a \in X$ the sequence of arcs $va_n \subset X$ converges to the arc va. A dendroid X is said to be *smooth* provided it is smooth at some point $v \in X$. Then the point vis called an *initial point* of X. It is known that every smooth dendroid is uniformly arcwise connected (see [12, Corollary 16, p. 318]).

A continuum X is said to be *arcwise decomposable* provided that there exist arcwise connected proper subcontinua A and B of X such that $X = A \cup B$ (see [51, 0.30, p. 16]).

Given a continuum X with a metric d, we let 2^X denote the hyperspace of all nonempty closed subsets of X equipped with the Hausdorff metric H defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \, \sup\{d(b, A) : b \in B\}\}$$
(2.1)

(see, e.g., [51, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by C(X) the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X .

Given a continuum X, a hyperspace of X means any subspace $\mathcal{H}(X)$ of

 2^X equipped with the inherited topology (thus induced by the Hausdorff metric *H* defined by (2.1)). For the reader convenience we recall definitions of the most important ones, which have appeared in the literature.

For each $n \in \mathbb{N}$, let

$$F_n(X) = \{A \in 2^X : \text{card } A \le n\}$$

denote the *n*-fold symmetric product of X. Thus 1-fold symmetric product of X is the hyperspace of singletons of X. The concept of the symmetric product has been introduced by Borsuk and Ulam in [4]. See [51, (0.48), p. 23] and [32, p. 6 and 7] for more information. Further, define the hyperspace $F_{\infty}(X)$ of finite subsets of X by

$$F_{\infty}(X) = \{ A \in 2^X : A \text{ is finite} \},\$$

or, equivalently, by $F_{\infty}(X) = \bigcup \{F_n(X) : n \in \mathbb{N}\}\$ (see [32, Definition 1.8, p. 7]). Note that for a continuum X all hyperspaces $F_n(X)$ are continua (see [45, 2.4.2, p. 156, and Theorem 4.10, p. 165]; compare [4, p. 877]) and $F_{\infty}(X)$ is a dense (and connected) subset of 2^X (see [45, 2.4.1, p. 156]), but neither it is locally compact at any of its points nor is a G_{δ} -subset of 2^X (see [20, Teorema 2.11, p. 26 and Proposición 2.2, p. 28]).

Similarly, for each $n \in \mathbb{N}$, define

 $C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},\$

 $C_{\infty}(X) = \{A \in 2^X : A \text{ has finitely many components}\},\$

whence it follows that

$$C_{\infty}(X) = \bigcup \{ C_n(X) : n \in \mathbb{N} \}.$$
(2.2)

The reader is referred to [42] and [43] for basic information about these hyperspaces. In particular, for each $n \in \mathbb{N}$ the hyperspaces $C_n(X)$ are (arcwise connected) continua (see [42, Theorem 3.1]), while $C_{\infty}(X)$ is neither locally compact nor a G_{δ} -subset of 2^X (see [42, Theorems 8.3 and 8.1]).

Consider the following hyperspaces $\mathcal{H}(X)$ of a continuum X, where $n \in \mathbb{N}$.

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$$\mathcal{H}(X) \in \{2^X, F_n(X), F_{\infty}(X), C_n(X), C_{\infty}(X)\}.$$
(2.3)

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Obvious inclusions for these hyperspaces are:

(2.3.1)
$$F_n(X) \subset F_{n+1}(X) \subset F_{\infty}(X);$$

(2.3.2) $C_n(X) \subset C_{n+1}(X) \subset C_{\infty}(X);$
(2.3.3) $F_n(X) \subset C_n(X);$
(2.3.4) $F_{\infty}(X) \subset C_{\infty}(X) \subset 2^X.$

In particular, $F_1(X)$ is the hyperspace of singletons, i.e.,

$$F_1(X) = \{ A \in 2^X : A \text{ is a singleton} \}.$$

Note that $F_1(X)$ is homeomorphic to X. We neglect the homeomorphism between X and $F_1(X)$, and we consider X as a subspace of 2^X under its natural embedding. Similarly, we think about C(X) as about a subspace of 2^X .

The following is obvious.

Proposition 2.4. Let K be a nonempty closed subset of a continuum X, and let $n \in \mathbb{N}$. Then

(2.4.1)
$$2^{K} \subset 2^{X}$$
;
(2.4.2) $F_{n}(K) \subset F_{n}(X)$;
(2.4.3) $F_{\infty}(K) \subset F_{\infty}(X)$;
(2.4.4) $C_{n}(K) \subset C_{n}(X)$;
(2.4.5) $C_{\infty}(K) \subset C_{\infty}(X)$.

Let a continuum X be given. By an order arc in 2^X we mean an arc Φ in 2^X such that if $A, B \in \Phi$, then either $A \subset B$ or $B \subset A$. The following fact is known (see [51, Theorem 1.8, p. 59]).

Proposition 2.5. Let X be a continuum and A, $B \in 2^X$ with $A \neq B$.

Then there exists an order arc in 2^X from A to B if and only if $A \subset B$ and each component of B intersects A.

Proposition 2.6. Let X be a continuum and, for $n \in \mathbb{N}$, let $\mathcal{H}(X)$ denote one of the hyperspaces 2^X , $C_n(X)$ or $C_{\infty}(X)$. If an order arc Φ in 2^X begins with $A \in \mathcal{H}(X)$, then $\Phi \subset \mathcal{H}(X)$.

Proof. For $\mathcal{H}(X) = 2^X$ this is Proposition 2.5. If $\mathcal{H}(X) = C_n(X)$ the result is shown in [13, Proposition 3, p. 784].

Let $\mathcal{H}(X) = C_{\infty}(X)$ and $A \in \mathcal{H}(X)$. Then by (2.2) we have $A \in C_n(X)$ for some $n \in \mathbb{N}$. Hence $\Phi \subset C_n(X)$ by the result for $C_n(X)$, and so $\Phi \subset C_{\infty}(X)$ again by (2.2).

Borsuk and Ulam have already observed in [4, 2, (a), p. 877] that if a space is arcwise connected, then $F_n(X)$ is arcwise connected for each $n \in \mathbb{N}$. On the other hand, Curtis and Nhu have shown in [15, Lemma 2.2, p. 252] that if X is a continuum and \mathcal{M} is a compact and locally connected subset of $F_{\infty}(X)$, then $\bigcup \mathcal{M}$ is a compact and locally connected subset of X. As a consequence of these two results the next proposition follows.

Proposition 2.7. Let X be a continuum and let $n \in \mathbb{N}$. Then $F_n(X)$ is arcwise connected if and only if X is arcwise connected.

Let two points p and q of a space X and a number $k \in \mathbb{N}$ be given. A finite collection of sets $\{L_i \mid i \in \{1, ..., k\}\}$ in a space X is called a *chain* from p to q provided that $p \in L_1$, $q \in L_k$ and $L_i \cap L_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A chain is called a *continuum chain* if each of its elements L_i is a continuum. A continuum chain is an ε -continuum chain if each of its elements has diameter less than ε . A continuum X is said to be *continuum chainable* provided that for each pair of points $p, q \in X$ and for each $\varepsilon > 0$ there exists in X an ε -continuum chain from p to q. The reader is referred e.g. to [28], [29] or [53] for more information about this concept.

We will show that an analog of Proposition 2.7 is true if arcwise

connectedness is replaced by continuum chainability. Recall that, given a compactum X, the union map for 2^{2^X} is the function $u: 2^{2^X} \to 2^X$ defined by $u(\mathcal{A}) = \bigcup \mathcal{A}$ for each $\mathcal{A} \in 2^{2^X}$ (see [32, Exercise 11.5, p. 91]).

Proposition 2.8. Let X be a continuum, $u: 2^{2^X} \to 2^X$ be the union map and let $n \in \mathbb{N}$. Then

(2.8.1) $u(C(F_n(X))) = C_n(X);$

(2.8.2) for each $\mathcal{A} \in C(F_n(X))$ each member of \mathcal{A} intersects every component of $u(\mathcal{A})$.

Proof. One inclusion in (2.8.1) follows from [42, Lemma 7.2, p. 250]. To show the opposite inclusion, take $A \in C_n(X)$ and let $A_1, ..., A_k$ be components of A for some natural $k \leq n$. Consider the family

$$\mathcal{E}_k = \{ \{x_1, ..., x_k\} : x_i \in A_i \text{ for } i \in \{1, ..., k\} \}.$$

Then $A = A_1 \cup \cdots \cup A_k = u(\mathcal{E}_k)$.

Let $q: X^k = X \times \cdots \times X \to F_k(X)$ be the natural quotient mapping defined by $q((x_1, ..., x_k)) = \{x_1, ..., x_k\}$. Thus $\mathcal{E}_k = q(A_1 \times \cdots \times A_k)$ is a continuum (since $A_1 \times \cdots \times A_k$ is a continuum), so $\mathcal{E}_k \in C(F_n(X))$, whence $A = u(\mathcal{E}_k) \in u(C(F_n(X)))$. So (2.8.1) is shown.

To verify (2.8.2) let $\mathcal{A} \in C(F_n(X))$ and let $S \in \mathcal{A}$. In order to show that S intersects every component of $u(\mathcal{A})$ take an open and closed subset V of $u(\mathcal{A})$ such that $S \subset V$. Then the sets

$$\{H \in \mathcal{A} : H \subset V\}$$
 and $\{H \in \mathcal{A} : H \cap (u(\mathcal{A}) \setminus V) \neq \emptyset\}$

are open, disjoint, and their union is \mathcal{A} . Since \mathcal{A} is connected and S is in the former set, the latter one is empty. Therefore S must intersect each component of $u(\mathcal{A})$, as needed. The proof is complete.

Theorem 2.9. Let X be a continuum and let $n \in \mathbb{N}$. Then $F_n(X)$ is continuum chainable if and only if X is continuum chainable.

Proof. Since continuum chainability of compact spaces is obviously an invariant under continuous mappings, we conclude (similarly to an argument used in [4, 2, (a), p. 877]) that the continuum chainability of Ximplies the one of the hyperspace $F_n(X)$.

Let $F_n(X)$ be continuum chainable. We can assume that n > 1, and let $\varepsilon > 0$ be given. We claim the following assertion.

(2.9.1) If $\mathcal{A} \subset F_n(X)$ is a continuum with diam $(\mathcal{A}) < \varepsilon$, and K is a component of $u(\mathcal{A})$, then diam $(K) < 2n\varepsilon$.

Indeed, let $S = \{x_1, ..., x_n\} \in \mathcal{A}$ be fixed. Let $B(x, \varepsilon)$ stand for the open ball of center x and radius ε . Then

$$K \subset u(\mathcal{A}) \subset B(x_1, \varepsilon) \cup B(x_n, \varepsilon)$$

by (2.8.2) of Proposition 2.8. Since K is connected, there are balls $B(x_{k_1}, \varepsilon), ..., B(x_{k_m}, \varepsilon)$ for some integer $m \le n$ which cover K and such that $B(x_{k_j}, \varepsilon) \cap B(x_{k_{j+1}}, \varepsilon) \ne \emptyset$ for $j \in \{1, ..., m-1\}$. Thus (2.9.1) is shown.

To prove that X is continuum chainable let us consider two points $x, y \in X$ and $\varepsilon > 0$. Let $\{A_1, ..., A_k\}$ be a chain of subcontinua of $F_n(X)$ having diameters smaller than $\frac{1}{2n}\varepsilon$ and joining $\{x\}$ and $\{y\}$. The set $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$ is a subcontinuum of $F_n(X)$. Since $\{x\}, \{y\} \in \mathcal{A},$ the set $u(\mathcal{A})$ is connected according to (2.8.1) of Proposition 2.8. Observe that the continuum $u(\mathcal{A})$ is the union of the finite family of all components of sets $u(\mathcal{A}_j)$ for $j \in \{1, ..., k\}$, so this family is a closed cover of the connected set $u(\mathcal{A})$. Therefore there is a chain of these components joining x and y. By (2.9.1) it is an ε -chain. The proof is complete.

If $Y \subset X$, then Y is a *retract* of X means that there exists a mapping $r: X \to Y$ (called a *retraction*) such that the restriction r|Y is the identity on Y. Hence, according to the above mentioned identification of $F_1(X)$ and X, we shall consider a retraction $r: 2^X \to X$ rather than a

retraction $r: 2^X \to F_1(X)$, although the former notation is perhaps less formal. The reader is referred to [2] for needed information on retracts and related concepts.

Let a mapping $f: X \to Y$ be given. For a fixed hyperspace $\mathcal{H}(X) \subset 2^X$ define the \mathcal{H} -induced mapping $\mathcal{H}(f): \mathcal{H}(X) \to \mathcal{H}(Y)$ (where $\mathcal{H}(Y) \subset 2^Y$ is the corresponding hyperspace of Y) by

 $\mathcal{H}(f)(A) = f(A)$ for each $A \in \mathcal{H}(X)$

(if $\mathcal{H}(X) = 2^X$, then $\mathcal{H}(f)$ is usually denoted by 2^f). Since $\mathcal{H}(X) \subset 2^X$ simply by the definition, and since $\mathcal{H}(f) = 2^f | \mathcal{H}(X)$, the continuity of 2^f (see [32, Lemma 13.3, p. 106] and compare [45, 5.10.1 of Theorem 5.10, p. 170]) implies the continuity of each $\mathcal{H}(f)$. Similarly, if f is a homeomorphism, then $\mathcal{H}(f)$ is a homeomorphism, too. The reader is referred to the authors' paper [13] and to [9] for various results on induced mappings between hyperspaces.

Easy proofs of the next two propositions are left to the reader.

Proposition 2.10. For any continua X and Y, for every mapping $f : X \to Y$ and for each $n \in \mathbb{N}$ we have:

(2.10.1) $2^{f} | C_{n}(X) = C_{n}(f);$ (2.10.2) $2^{f} | C_{\infty}(X) = C_{\infty}(f);$ (2.10.3) $2^{f} | F_{n}(X) = F_{n}(f);$ (2.10.4) $2^{f} | F_{\infty}(X) = F_{\infty}(f).$

Proposition 2.11. Let $n \in \mathbb{N}$ and let $\mathcal{H}(X)$ denote one of the hyperspaces listed in (2.3). If K is a nonempty subcontinuum of X, then for each mapping $f : X \to Y$ we have

$$\mathcal{H}(f \mid K) = \mathcal{H}(f) \mid \mathcal{H}(K).$$

3. Mappings between Hyperspaces

There is a number of results that are related to mappings from, onto or between some hyperspaces (which are subspaces of the hyperspace 2^X), in particular to hyperspace retractions. For example, C(X) always is a continuous image of 2^X [49, Theorem 3.6, p. 194] (not necessarily being its retract, see [24]). This result has been generalized by the second named author (see [43, Theorem 4.1, p. 264]) as follows.

Theorem 3.1. For each continuum X and for each $n \in \mathbb{N}$ there exists a mapping of the hyperspace 2^X onto $C_n(X)$.

Recall that Kelley has shown (see [33, Theorem 2.7, p. 25]; compare [51, Theorem 1.33, p. 81]) that C(X) is a continuous image of the Cantor fan. Moreover, the same methods can be applied to extend the result to $C_n(X)$ for any $n \in \mathbb{N}$ (see [33, Remark, p. 29]). Thus Theorem 3.1 of [42] can be formulated in a stronger way, as follows.

Theorem 3.2. For each continuum X and for each $n \in \mathbb{N}$ the hyperspace $C_n(X)$ is a continuous image of the Cantor fan, and hence it is uniformly arcwise connected.

Since the hyperspace 2^X is weakly chainable for each continuum X (see [51, (2), p. 245]) and since weak chainability of continua is obviously preserved under mappings, Theorem 3.1 implies the following extension of (2) in [51, p. 245].

Corollary 3.3. For each continuum X and for each $n \in \mathbb{N}$ the hyperspace $C_n(X)$ is weakly chainable.

The next theorem corresponds to Theorem 4.7 of [51, p. 244] (compare also [49, Theorem 3.2, p. 193]).

Theorem 3.4. If X and Y are continua such that there is a mapping from $C_n(X)$ onto Y, where $n \in \mathbb{N}$, then Y is weakly chainable and arcwise decomposable.

Proof. Let a continuum Y be a continuous image of $C_n(X)$ for a

continuum *X* and for some $n \in \mathbb{N}$. Since weak chainability is a mapping invariant, the first part of the conclusion follows from Corollary 3.3.

Since Y is an image of $C_n(X)$, it follows from Theorem 3.2 that there is a mapping f from the Cantor fan F_C onto Y. Applying the Kuratowski-Zorn lemma (see [16, p. 8] for example) we infer that there exists a subcontinuum K of F_C which is irreducible with respect to the property of being a subcontinuum of F_C which is mapped onto Y under f. If the vertex v of F_C is a cut point of K, then K is the union of two arcwise connected proper subcontinua. If v is not a cut point of K, then K is an arc. In either case, there exist arcwise connected proper subcontinua K_1 and K_2 of K such that $K = K_1 \cup K_2$. Consequently, $f(K_1)$ and $f(K_2)$ are arcwise connected, proper (by the minimality property of K) subcontinua of Y whose union is Y. Thus Y is arcwise decomposable. The proof is complete.

Remark 3.5. The idea of the above proof comes from Bellamy, see [1, Example II, p. 17]. Compare also [51, the proof of Theorem 4.7, p. 244-245].

The next two results show exactly which X are continuous images of $C_n(X)$ for arc-like and circle-like continua. They extend similar Nadler's results for the hyperspaces 2^X and C(X) (see [49, Corollary 3.3, p. 193] and [51, Theorems 4.8 and 4.9, p. 246]).

Theorem 3.6. Let $n \in \mathbb{N}$. An arc-like continuum X is a continuous image of $C_n(X)$ if and only if X is an arc.

Proof. This is a consequence of arcwise connectedness of $C_n(X)$, see Theorem 3.2, and the fact that an arc is the only arcwise connected arc-like continuum.

Theorem 3.7. Let $n \in \mathbb{N}$. A circle-like continuum X is a continuous image of $C_n(X)$ if and only if X is a simple closed curve.

Proof. It easily follows from [48, Theorem 6, p. 230] that a simple closed curve is the only circle-like arcwise decomposable continuum.

Concerning mappings in the opposite direction (i.e., from $C_n(X)$ onto 2^X) we have the following.

Theorem 3.8. For any continuum X containing an open set with uncountably many components, and for any $n \in \mathbb{N}$ there exists a mapping of $C_n(X)$ onto 2^X .

Proof. Since X contains an open set with uncountably many components, $C_n(X)$ also contains an open set with uncountably many components, see [43, Lemma 3.2, p. 259]. This suffices to the existence of a mapping from $C_n(X)$ onto 2^X (see [49, Theorem 3.5, p. 194]; compare [51, Theorem 4.2, p. 242]).

A similar result concerning the hyperspace C(X) is already known (see [43, Theorem 4.2, p. 265]).

Theorem 3.9. For any continuum X containing an open set with uncountably many components, and for any $n \in \mathbb{N}$ there exists a mapping of $C_n(X)$ onto C(X).

Since each indecomposable continuum contains an open set with uncountably many components (see [51, (*) on p. 244]), Theorems 3.8 and 3.9 imply a corollary.

Corollary 3.10. For any indecomposable continuum X and for any $n \in \mathbb{N}$ there exist mappings of $C_n(X)$ onto 2^X and onto C(X).

In connection with Theorems 3.6 and 3.7 recall that properties of hyperspaces of arc-like and circle-like continua are studied by Krasinkiewicz in [35]. Among other results he presents in [35, Theorem 4.1, p. 159] a new proof of a result of Segal [55] that for each arc-like continuum X the hyperspace C(X) has the fixed point property, and proves in [35, Theorem 4.2, p. 160] a similar result for circle-like continua. These results were independently obtained by Rogers, Jr. (see [54]). So, it is natural to ask if other hyperspaces of these continua have the fixed point property.

Question 3.11. Let a continuum X be (a) arc-like, or (b) circle-like,

and let $n \in \mathbb{N}$. Does any of the hyperspaces listed in (2.3) have the fixed point property? If not, determine sufficient and/or necessary conditions under which the property holds.

Some partial results related to this question, namely concerning the fixed point property of hyperspaces $C_n(X)$ when X is the inverse limit of an inverse sequence of locally connected continua with some special bonding mappings are obtained in [13, Theorems 32 and 33, and Corollaries 34 and 35, p. 797-798].

Quite a lot of the results about retractions from a hyperspace onto the continuum concern local connectedness of the continuum X or of a hyperspace at some of its points (see [19], [21], [22], [27], [50] for example). As early as in the end of the thirties Wojdyslawski proved that locally connected continua have contractible hyperspaces [57] and that C(X) is an absolute retract if and only if X is locally connected [58] (compare also [33, Theorem 4.4, p. 28]).

A characterization of dendrites in terms of continuity of the function $I: 2^X \to C(X)$ that assigns to a closed subset A of a hereditarily unicoherent continuum X the continuum I(A) irreducible with respect to containing A is given as Theorem 1 of [23, p. 3]. For characterizations of smooth dendroids in terms of continuity of some functions related to hyperspaces see [23, Theorem 8, p. 7]; compare also [41, Theorem 1, p. 112].

4. Hyperspace Retractions

It is observed in [26, p. 122], that if a one-dimensional continuum X is a retract of C(X), then it is a dendroid. As the only argument is used Theorem (6.9) of [51, p. 272], which leads to Vietoris homology theory or to other homology theories (see [51, Theorems (1.172)-(1.180), p. 176-179]). Another (and more complete) argument leading to an even stronger result is presented in [10, Theorem 3.1, p. 9]. We extend these results from C(X) to $C_n(X)$ for an arbitrary $n \in \mathbb{N}$. To do this we recall an auxiliary result. We say that a continuum X has *trivial shape* (in the sense of Borsuk, see [3]) provided that it is the intersection of a decreasing sequence of compact absolute retracts (equivalently, X has trivial shape if every mapping from X to any absolute neighborhood retract is homotopic to a constant mapping). Krasinkiewicz has shown (see [34, 1.9, p. 707], compare [51, 1.182, p. 180]) that C(X) has trivial shape. This result has been extended in [42, Corollary 4.6] as follows.

Proposition 4.1. For each $n \in \mathbb{N}$ and for each continuum X the hyperspace $C_n(X)$ has trivial shape.

Theorem 4.2. Let X be a curve, and let $n \in \mathbb{N}$. If

(4.2.1) there exists a retraction from $C_n(X)$ onto X,

then X is a uniformly arcwise connected dendroid.

Proof. By Proposition 4.1 the hyperspace $C_n(X)$ has trivial shape. Trivial shape is preserved under retraction (simply by definition), and therefore (4.2.1) implies that X has trivial shape. Since every curve having trivial shape is tree-like [36, Theorem 2.1 (B), p. 237], it follows that X is tree-like. It is known that every tree-like continuum is hereditarily unicoherent (see [5, Theorem 1, p. 74] and [38, Section 57, II, Theorem 2, p. 437] and note that tree-likeness is a hereditary property; compare [8, (2.1) and (4.3), p. 144 and 147]), so we infer that X is hereditarily unicoherent. Further, for each continuum X the hyperspace $C_n(X)$ is uniformly arcwise connected according to Theorem 3.2. Therefore X, being a continuous image of $C_n(X)$, is arcwise connected, so it is a dendroid. Moreover, it is uniformly arcwise connected since uniform arcwise connectedness is a continuous invariant (see [6, T19, p. 194]). Thus the proof is complete.

Question 4.3. Let $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$. Consider the following two conditions which a continuum *X* may satisfy.

(4.3.1) There exists a retraction $r_1 : C_{n_1}(X) \to X$.

(4.3.2) There exists a retraction $r_2 : C_{n_2}(X) \to X$.

Since $C_{n_1}(X) \subset C_{n_2}(X)$ according to (2.3.2), then letting $r_1 = r_2 | C_{n_1}(X)$ we see that (4.3.2) implies (4.3.1). Under what conditions concerning X the opposite implication holds?

Curtis introduced in [14, p. 30] the following concept. Let $X = [0, \infty) \cup K$ denote an arbitrary compactification of the half-line $[0, \infty)$ with a nondegenerate locally connected continuum K as the remainder. Then there always exists a retraction of X onto K. The continuum X is called a *regular* compactification provided that there exists a retraction $r: X \to K$ such that for some homeomorphism $\varphi : [0, \infty) \to [0, \infty)$ the composition mapping $r \circ \varphi : [0, \infty) \to K$ is a *periodic* surjection, i.e., there exists p > 0 such that $r(\varphi(t)) = r(\varphi(t + p))$ for all $t \in [0, \infty)$.

For each nonnegative integer m let X_m be a compactification of the half-line such that:

- X_0 is the sin (1/x)-curve;

- X_1 is the unit circle *S* with a spiral;

- if $m \ge 2$, then X_m is the regular compactification of the half-line obtained by alternately "wrapping" and "unwrapping" subintervals of $[0, \infty)$ about S, with each subinterval covering S exactly m/2 times.

See [14, p. 30] for a formal definition of X_m 's. Then Theorem 3.3 of [14, p. 40] says the following.

Theorem 4.4 (Curtis). For a regular half-line compactification X there exists a retraction $r: 2^X \to C(X)$ if and only if X is homeomorphic to some X_m for an integer $m \ge 0$.

The next theorem states that a similar result is true for $C_n(X)$ in place of 2^X , where n > 1.

Theorem 4.5. Let an integer n > 1 be given. For a regular half-line compactification X there exists a retraction $r_n : C_n(X) \to C(X)$ if and only if X is homeomorphic to some X_m for an integer $m \ge 0$.

Proof. If the continuum X is homeomorphic to some X_m for an integer $m \ge 0$, then by Theorem 4.4 there exists a retraction $r: 2^X \to C(X)$. Hence, it is enough to define $r_n = r | C_n(X) : C_n(X) \to C(X)$.

Assume now that there exists a retraction $r_n : C_n(X) \to C(X)$. Observe that the argument in the proof of Theorem 3.3 of [14, p. 40] only uses the fact that there exists a retraction of $C_2(X)$ onto C(X). Since the restriction $r_n | C_2(X) : C_2(X) \to C(X)$ is a retraction, the result follows from [14, Theorem 3.3, p. 40].

The following question is related to Theorem 4.5.

Question 4.6. Let an integer n > 1 be given, and let X denote a regular half-line compactification. Consider the following two conditions:

(4.6.1) X is homeomorphic to some X_m for an integer $m \ge 0$,

(4.6.2) there is a retraction $r_n: 2^X \to C_n(X)$.

Are these two conditions equivalent? If not, which implication is true?

The next notion is a an extension of a one defined in [14, p. 43]. Let X be a continuum and $\mathcal{H}(X) \subset 2^X$ be a hyperspace of X containing C(X). A retraction $r : \mathcal{H}(X) \to C(X)$ is said to be *conservative* provided that $A \cap r(A) \neq \emptyset$ for each $A \in \mathcal{H}(X)$.

The following result has been shown in [14, Theorem 4.1, p. 43].

Theorem 4.7 (Curtis). Let X be a regular half-line compactification for which there exists a conservative retraction $r: 2^X \to C(X)$. Then X is homeomorphic either to the $\sin(1/x)$ -curve X_0 or to the unit circle with a spiral X_1 .

Since the argument given in the proof of the above result again uses only the fact that there exists a retraction from $C_2(X)$ onto C(X), and since the restriction of this retraction to $C_n(X)$ for any integer n > 1 is obviously a retraction, the next theorem follows. **Theorem 4.8.** Let an integer n > 1 be given, and let X be a regular half-line compactification for which there exists a conservative retraction $r_n : C_n(X) \to C(X)$. Then X is homeomorphic either to the $\sin(1/x)$ -curve X_0 or to the unit circle with a spiral X_1 .

Question 4.9. Let an integer n > 1 be given, and let X denote a regular half-line compactification. Assuming that there is a conservative retraction $r_n : 2^X \to C_n(X)$, must then X be homeomorphic either to the $\sin(1/x)$ -curve X_0 or to the unit circle with a spiral X_1 ?

5. Selections

Let a continuum X and a hyperspace $\mathcal{H}(X) \subset 2^X$ be given. By a *continuous selection* (shortly a *selection*) on $\mathcal{H}(X)$ we mean a mapping $\sigma : \mathcal{H}(X) \to X$ such that $\sigma(A) \in A$ for each $A \in \mathcal{H}(X)$. Since $F_1(X)$ and X are homeomorphic, if $F_1(X) \subset \mathcal{H}(X)$, a selection on $\mathcal{H}(X)$ may be viewed as a special kind of a retraction from $\mathcal{H}(X)$ onto X.

Note the following simple observation.

Observation 5.1. Let a continuum X and two hyperspaces $\mathcal{G}(X) \subset \mathcal{H}(X) \subset 2^X$ be given. If $\sigma : \mathcal{H}(X) \to X$ is a selection, then the restriction $\sigma | \mathcal{G}(X) : \mathcal{G}(X) \to X$ also is a selection. Thus the existence of a selection is a hereditary property with respect to any (not necessarily closed) subspace of a given hyperspace.

Studying the problem of the existence of a selection on $\mathcal{H}(X)$ one can distinguish two cases. If, for a continuum X, the hyperspace $\mathcal{H}(X)$ contains the hyperspace $F_2(X)$ of singletons and doublets of X, then (thanks to several results known from the literature) the situation is completely clear: if there is a selection on $\mathcal{H}(X)$, then X is an arc and only an arc (see below, Theorem 5.2). If $\mathcal{H}(X)$ does not contain $F_2(X)$ (for example if $\mathcal{H}(X) = \mathcal{C}(X)$), then no similar result is known, and only some partial answers are known. Of course the two cases do not cover all the possibilities for $\mathcal{H}(X)$. Now we will discuss the mentioned cases separately.

Answering a question of Michael (see [45, p. 154], and compare also [17]) Kuratowski, Nadler and Young have proved in [39] that for every continuum X a selection on the hyperspace 2^X exists if and only if X is an arc. The key argument in showing this equivalence is Theorem 2 of [39, p. 5] which states that if M is a locally compact separable metric space for which there exists a selection on $F_2(M)$, then M is a subset of the real line. This result can be extended as follows.

Theorem 5.2. Let a continuum X and a hyperspace $\mathcal{H}(X)$ be given such that

$$F_2(X) \subset \mathcal{H}(X) \subset 2^X. \tag{(*)}$$

Then the following statements are equivalent:

- (5.2.1) there is a selection on $F_2(X)$;
- (5.2.2) there is a selection on $\mathcal{H}(X)$;
- (5.2.3) there is a selection on 2^X ;
- (5.2.4) X is an arc.

Proof. Assume (5.2.4), and consider a linear order on X. Then a mapping $\sigma: 2^X \to X$ defined by $\sigma(A) = \min A$ is a selection. Thus (5.2.4) implies (5.2.3).

The implications

$$(5.2.3) \Rightarrow (5.2.2) \Rightarrow (5.2.1)$$

are consequences of the inclusions (*) and of Observation 5.1. Finally the implication $(5.2.1) \Rightarrow (5.2.4)$ is just [39, Theorem 1, p. 5]. Thus the proof is complete.

Corollary 5.3. Let a continuum X and $n \in \mathbb{N}$ be given. Then the following statements are equivalent:

(5.3.1) there is a selection on $F_n(X)$ for some n > 1;

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(5.3.2) there is a selection on $F_n(X)$ for each n > 1;

(5.3.3) there is a selection on $F_{\infty}(X)$;

(5.3.4) there is a selection on $C_n(X)$ for some n > 1;

(5.3.5) there is a selection on $C_n(X)$ for each n > 1;

(5.3.6) there is a selection on $C_{\infty}(X)$;

(5.3.7) X is an arc.

Proof. Indeed, the corollary is a consequence of Theorem 5.2 and of inclusions (2.3.1)-(2.3.4).

Remark 5.4. Recall that the equivalence of (5.2.1) and (5.2.3) holds for a much more general setting, namely if *X* is a Hausdorff space having open components, see [45, 7.6, p. 177] and compare [51, Theorem 5.2 (2), p. 254].

If one seeks a continuous selection on the hyperspace C(X) of all subcontinua of X, then such a simple characterization of continua which admit a selection (they are said to be *selectible*) is not known and seems to be a rather hard problem. A very important approach to solve it was made by Nadler and Ward, who have proved that each selectible continuum is a dendroid (see [52, Lemma 3, p. 370]), and that a locally connected continuum is selectible if and only if it is a dendrite [52, Corollary, p. 371]. Since a selection on C(X) is a retraction, the above mentioned result of Nadler and Ward can be sharpened by Theorem 4.2 to the statement that if a continuum is selectible, then it is a uniformly arcwise connected dendroid [7, Proposition 2, p. 110].

A further progress was made by Ward in [56], where a concept of a rigid selection is introduced. A selection $\sigma : \mathcal{H}(X) \to X$ is said to be *rigid* provided that

if $A, B \in \mathcal{H}(X)$ and $\sigma(B) \in A \subset B$, then $\sigma(A) = \sigma(B)$.

Ward has proved the following theorem ([56, Theorem 2, p. 1043]).

Theorem 5.5 (Ward). A continuum X admits a rigid selection on C(X) if and only if X is a smooth dendroid.

Let a continuum X be hereditarily unicoherent. A retraction $r: 2^X \to X$ is said to be *internal* provided that

$$r(A) \in I(A)$$
 for each $A \in 2^X$. (5.6)

It is known that the property of having an internal retraction is hereditary (see [10, Proposition 3.9, p. 11]). The following result is shown in [10, Corollary 4.3, p. 15].

Corollary 5.7. Let X be a curve. If there exists an internal retraction $r: 2^X \to X$, then r | C(X) is a selection on C(X), and X is a selectible dendroid.

Moreover, if the retraction r satisfies the implication

(5.7.1) if
$$A, B \in 2^X$$
 and $r(B) \in A \subset B$, then $r(A) = r(B)$,

then the selection r | C(X) is rigid, and the dendroid X is smooth.

If a retraction $r: 2^X \to X$ is internal, then it need not satisfy (5.7.1), see [10, Example 4.4, p. 16].

Recall that the existence of a retraction from C(X) onto X does not suffice for the existence of a selection from C(X) onto X. Namely Illanes has constructed in [31, Section 4, p. 70] an example of a dendroid X_2 which is a retract of $C(X_2)$, while $C(X_2)$ does not admit any selection. It is shown in [10, Theorem 5.58, p. 30] that there is a retraction from 2^{X_2} onto X_2 . The dendroid X_2 coincides with the dendroid D described by Mackowiak in [44, Example, p. 321]. Its various properties are collected in [10, Theorem 5.78, p. 31].

In the same paper [31] Illanes constructs a selectible dendroid X_1 which is a retract, but not a deformation retract, of its hyperspace $C(X_1)$.

Problem 5.8. Assume that, for a continuum X, a hyperspace $\mathcal{H}(X)$ neither contains $F_2(X)$ nor is contained in C(X). Under what conditions about the continuum X and about the hyperspace $\mathcal{H}(X)$ there exists a selection on $\mathcal{H}(X)$?

Acknowledgement

The authors thank Dr. Krzysztof Omiljanowski for proving the results concerning continuum chainability. Proposition 2.8 and Theorem 2.9 are due to him. They also thank Professor Pawel Krupski for a fruitful discussion on the topic of this paper.

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