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A NOTE ON THE RELATIONSHIPS BETWEEN SPEARMAN'S ρ AND KENDALL'S τ UNDER BIVARIATE HOMOGENEOUS SHOCK MODEL

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Abstract

This note studies Spearman's ρ and Kendall's τ under the so-called bivariate homogeneous shock (BHS) model. We derive simple expressions of ρ and τ, and based on them, we reveal some relationships between ρ and τ under BHS model.

1. Introduction

Spearman's ρ and Kendall's τ are two most commonly used nonparametric association measures. The relationship between ρ and τ has received considerable attention recently. For example, Hutchinson and Lai [1] conjectured that $-1 + \sqrt{1 + 3\tau} \leq \rho \leq \min\{3\tau/2, 2\tau - \tau^2\}$ for stochastically increasing random variables. Hürlimann [2] has shown that the entire Hutchinson and Lai conjecture holds for bivariate extreme value distributions. Munroe et al. [3] have shown that part of the Hutchinson

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and Lai's conjecture, $1 + 3\tau \leq (1 + \rho)^2$, does not hold. Denote $X_{(1)} = \min\{X_1, ..., X_n\}$ and $X_{(n)} = \max\{X_1, ..., X_n\}$ of an i.i.d. sample $X_1, ..., X_n$ and let $\rho_n = \rho(X_{(1)}, X_{(n)})$ and $\tau_n = \tau(X_{(1)}, X_{(n)})$. Schmitz [4] conjectured that $\lim_{n\to\infty} \rho_n / \tau_n = 3/2$, and this conjecture was proved by Li and Li [5]. Chen [6] established some inequality relationships between ρ_n and τ_n. Fredricks and Nelsen [7] showed that, under mild regularity conditions, the limit of the ratio ρ/τ is $3/2$ as the joint distribution of the random variables approaches to independence. Capéraà and Genest [8] have shown that $\rho \geq \tau \geq 0$ when one variable is simultaneously left-tail decreasing and the other right-tail increasing. Fredricks and Nelson [7] provided an elegant analytical proof for the Capéraà-Genest inequality. Genest and Nešlehová [9] gave a short analytical proof for classical Daniels' inequality $|3\tau - 2\rho| \leq 1$, found by Daniels [10].

Marshall and Olkin [11] introduced a bivariate exponential (BVE) model, which is one of the few known bivariate models derived from realistic events. Wang and Li [12] generalized the BVE model to the so-called bivariate homogeneous shock (BHS) model.

In this note, we study the relationships between ρ and τ under BHS model. A prominent feature of the BHS model is its singularity. Simple expressions of these measures have been derived, and with these elegant expressions, we easily show that most established relationships under usual continuous model, such as, the Capéraà-Genest inequality, the Daniels' inequality, will still hold under BHS model.

2. Main Results

For any pair of random variables (X, Y), let $H(x, y)$ be its bivariate cumulative distribution function (c.d.f.) and $\overline{H}(x, y) = 1 - H_1(x) - H_2(y) + H(x, y)$ be its joint survival function, where H_1 and H_2 are the marginal c.d.f'.s. The classical Spearman's ρ of (X, Y) is defined as

$$\rho = -3 + 12\iint H_1(x)H_2(y)dH(x,\ y),$$

and the Kendall's τ is defined as

$$\tau = 4\iint H(x,\ y)dH(x,\ y) - 1.$$

Consider a two-component series system subjected to some fatal shocks. Assume there are three kinds of fatal shocks. Shock A governed by random variable U destroys component 1, shock B governed by random variable V destroys component 2, and shock C governed by random variable W destroys both components simultaneously. As in Wang and Li [12], we refer to such a system as bivariate homogeneous shock (BHS) model. Clearly, under this model the lifelength of component 1 is $X = \min(U, W)$ and that of component 2 is $Y = \min(V, W)$. Especially, if the random variables U, V and W are all exponential, the BHS model is just the BVE model proposed by Marshall and Olkin [11].

A prominent feature of BHS model is its singularity. More specifically, even though U, V and W are all continuous random variables, the joint distribution of X and Y is usually discontinuous.

Denote the survival functions of U, V and W as $u(x) = \mathrm{pr}(U > x)$, $v(x) = \mathrm{pr}(V > x)$ and $w(x) = \mathrm{pr}(W > x)$, respectively. Under BHS model, we find that ρ and τ have simple closed expressions, as stated in the following theorem.

Theorem 1. *Let* ρ *and* τ *be Spearman's* ρ *and Kendall's* τ, *respectively. Under BHS model, we have*

$$\rho = -\int_0^\infty u^2 v^2 dw^3 \tag{1}$$

and

$$\tau = -\int_0^\infty u^2 v^2 dw^2. \tag{2}$$

Based on these simple closed expressions of ρ and τ, we then can easily prove the results listed in the following theorem.

Theorem 2. *Under BHS model, we have*

(i) $\rho \geq \tau$.

(ii) $0 \leq 3\tau - 2\rho \leq 1$.

(iii) *Hutchinson and Lai's conjecture will not hold. Especially, the inequality* $(1 + 3\tau) \leq (1 + \rho)^2$ *will not hold.*

(iv) *The limit of the ratio* ρ/τ *is* $3/2$ *as the joint distribution of the random variables approaches to independence.*

3. Proofs

Proof of Theorem 1. Let $A = \{(x,\ y) : x > y\}$, $B = \{(x,\ y) : x < y\}$ and $C = \{(x,\ y) : x = y\}$. We have

$$\rho = -3 + 12\iint H_1(x)H_2(y)dH(x,\ y)$$

$$= -3 + 12\iint H_1(x)H_2(y)d\overline{H}(x,\ y)$$

$$= -3 + 12\left[\iint_A + \iint_B + \iint_C H_1(x)H_2(y)d\overline{H}(x,\ y)\right]$$

$$= -3 + 12[I + II + III].$$

It turns out

$$I = \iint_A H_1(x)H_2(y)d\overline{H}(x,\ y)$$

$$= \iint_A \{1 - u(x)w(x)\}\{1 - v(y)w(y)\}d\{u(x)w(x)\}dv(y)$$

$$= \int_0^\infty \{1 - v(y)w(y)\}dv(y)\int_y^\infty \{1 - u(x)w(x)\}d\{u(x)w(x)\}$$

$$= -\frac{1}{2}\int_0^\infty \{1 - v(y)w(y)\}u(y)w(y)\{2 - u(y)w(y)\}dv(y)$$

$$= -\frac{1}{2}\int_0^\infty (1 - vw)uw(2 - uw)dv.$$

Symmetrically, we have

$$II = -\frac{1}{2}\int_0^\infty (1 - uw)vw(2 - vw)du.$$

On the line C, we have $d\overline{H}(x, y) = -u(x)v(x)dw(x)$. Hence

$$III = \iint_C H_1(x)H_2(y)d\overline{H}(x, y)$$

$$= -\int_0^\infty \{1 - v(x)w(x)\}\{1 - u(x)w(x)\}u(x)v(x)dw(x)$$

$$= -\int_0^\infty (1 - vw)(1 - uw)uvdw.$$

Therefore,

$$I + II + III$$

$$= -\frac{1}{2}\int_0^\infty (1 - vw)uw(2 - uw)dv - \frac{1}{2}\int_0^\infty (1 - uw)vw(2 - vw)du$$

$$- \int_0^\infty (1 - vw)(1 - uw)uvdw$$

$$= -\frac{1}{2}\int_0^\infty [(1 - vw)(2 - uw)uwdv + (1 - uw)(2 - vw)vwdu$$

$$+ 2(1 - vw)(1 - uw)uvdw]$$

$$= -\frac{1}{2}\int_0^\infty \left\{2d(uvw) - d[w^2(u^2v + uv^2)] + \frac{1}{2}d(u^2v^2w^3) + \frac{1}{6}u^2v^2dw^3\right\}$$

$$= -\frac{1}{2}\left\{-2 + 2 - \frac{1}{2} + \frac{1}{6}\int_0^{\infty} u^2v^2dw^3\right\}$$

$$= \frac{1}{4} - \frac{1}{12}\int_0^{\infty} u^2v^2dw^3.$$

Thus, we get

$$\rho = -\int_0^{\infty} u^2v^2dw^3.$$

The expression of τ has been given in Wang and Li [12]. Theorem 1 is thus proved.

Proof for Theorem 2. (i) We have

$$\rho - \tau = \int_0^{\infty} u^2v^2(2w - 3w^2)dw.$$

Since u, v and w all are monotonic decreasing functions, we can regard u, v are functions of w. Assume $u^2v^2 = f(w)$, then the function $f(t)$ is increasing satisfying $f(0) = 0$ and $f(1) = 1$. Therefore,

$$\begin{aligned}
\rho - \tau &= \int_0^{\infty} u^2v^2(2w - 3w^2)dw(x) \\
&= -\int_0^{1} f(t)(2t - 3t^2)dt \\
&= -\int_0^{\frac{2}{3}} f(t)(2t - 3t^2)dt - \int_{\frac{2}{3}}^{1} f(t)(2t - 3t^2)dt \\
&= -f(\xi_1)\int_0^{\frac{2}{3}}(2t - 3t^2)dt - f(\xi_2)\int_{\frac{2}{3}}^{1}(2t - 3t^2)dt \\
&= \frac{4}{27}[f(\xi_2) - f(\xi_1)] \geq 0.
\end{aligned}$$

(ii) As in (i), assume $u^2v^2 = f(w)$, then

$$3\rho - 2\tau = -\int_0^\infty u^2v^2(6w - 6w^2)\,dw(x)$$

$$= 6\int_0^1 f(t)t(1-t)\,dt$$

$$\leq 6\int_0^1 t(1-t)\,dt = 1.$$

Clearly, $3\tau - 2\rho \geq 0$. Thus, the inequality $0 \leq 3\rho - 2\tau \leq 1$ holds under BHS model.

(iii) As we easily can see, the inequality $(1 + 3\tau) \leq (1 + \rho)^2$ is equivalent to $3\tau - 2\rho \leq \rho^2$. As in (ii), it becomes

$$6\int_0^1 f(t)t(1-t)\,dt \leq 9\left(\int_0^1 (f(t)t^2)\,dt\right)^2. \tag{3}$$

Take $f(t) = 1 - (1-t)^\alpha$, then the value of the left side of (3) is $1 - \frac{6}{(\alpha+3)(\alpha+2)}$. While that on the right side is $\left(1 - \frac{6}{(\alpha+3)(\alpha+2)(\alpha+1)}\right)^2$. Take $\alpha = 0.1$, we find the inequality will not hold. Thus, we disprove the Hutchinson and Lai's conjecture, $1 + 3\tau \leq (1+\rho)^2$, under BHS model.

(iv) Replace w by w^α, then $\alpha \to 0$ implies the variables (X, Y) approach to independence. As we can easily see,

$$\lim_{\alpha\to 0}\frac{\rho}{\tau} = \lim_{\alpha\to 0}\frac{\int u^2v^2\,dw^{3\alpha}}{\int u^2v^2\,dw^{2\alpha}} = \lim_{\alpha\to 0}\frac{\int u^2v^2\,3\alpha w^{3\alpha-1}dw}{\int u^2v^2\,2\alpha w^{2\alpha-1}dw} = \frac{3}{2}.$$

So the ratio ρ/τ approaches to $3/2$ when the model approaches to independence.

4. Concluding Comments

The BHS model is derived from real events, it has clear physical basis. Despite its singularity, we find the dependency measures such as ρ and τ are of simple forms. Yanagimoto and Okamoto [13] introduced a measure of association, denoted as η, the so-called grade correlation coefficient. From Wang and Li [12], we find that under BHS model, $\eta = \rho.$ As we can see, under BHS model, a plain yet reasonable dependency measure could be in the form of $-\int_0^\infty u^l v^m dw^n$ for some positive constants *l*, *m* and *n*, as long as it has sound statistical interpretation. The measures τ and ρ happened to be the simplest two of such dependency measures. Due to the simplicity of the forms of these measures, the relationship between them can be easily established. For instance, we can easily confirm that, the ratio ρ/τ will lie in the interval $[1, 3/2],$ with the two ending points corresponding to the two extreme cases of dependency of two random variables.

References

[1] T. P. Hutchinson and C. D. Lai, Continuous Multivariate Distributions, Emphasizing Applications, Rumsby Publishing Co., Adelaide, 1990.

[2] W. Hürlimann, Hutchinson-Lai's conjecture for bivariate extreme value copulas, Statist. Probab. Lett. 61 (2003), 191-198.

[3] P. Munroe, T. Ransford and C. Genest, Un contre-exemple à une conjecture de Hutchinson et Lai, C. R. Math. Acad. Sci. Paris 348 (2010), 305-310.

[4] V. Schmitz, Revealing the dependence structure between $X_{(1)}$ and $X_{(n)}$, J. Statist. Plann. Inference 123 (2004), 41-47.

[5] X. Li and Z. Li, Proof of a conjecture on Spearman's ρ and Kendall's τ for sample minimum and maximum, J. Statist. Plann. Inference 137 (2007), 359-361.

[6] Y. P. Chen, A note on the relationship between Spearman's ρ and Kendall's τ for extreme order statistics, J. Statist. Plann. Inference 137 (2007), 2165-2171.

[7] G. Fredricks and R. B. Nelsen, On the relationship between Spearman's rho and Kendall's tau for pairs of continuous random variables, J. Statist. Plann. Inference 137 (2007), 2143-2150.

[8] P. Capéraà and C. Genest, Spearman's ρ is larger than Kendall's τ for positively dependent random variables, J. Nonparametr. Statist. 2 (1993), 183-194.

[9] C. Genest and J. Nešlehová, Analytical proofs of classical inequalities between Spearman's ρ and Kendall's τ, J. Statist. Plann. Inference 139 (2009), 3795-3798.

[10] H. E. Daniels, Rank correlation and population models, J. Roy. Statist. Soc. Ser. B 12 (1950), 171-181.

[11] A. W. Marshall and I. Olkin, A multivariate exponential distribution, J. Amer. Statist. Assoc. 62 (1967), 30-44.

[12] J. Wang and Y. Li, Dependency measures under bivariate homogeneous shock models, Statistics 39 (2005), 73-80.

[13] T. Yanagimoto and M. Okamoto, Partial orderings of permutations and monotonicity of a rank correlation statistic, Ann. Inst. Statist. Math. 21 (1969), 489-506.