JP Jour. Geometry & Topology **4**(1) (2004), 35-52

PERIODIC LINKS AND MANIFOLDS

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Abstract

We consider orientable closed connected 3-manifolds obtained by Dehn surgery with rational coefficients along the components of certain periodic links. These manifolds were introduced in [Osaka J. Math. 39 (2002), 705-721] as natural generalizations of Takahashi manifolds [Tsukuba J. Math. 13 (1989), 175-189]. In this note, we re-obtain the results of [Osaka J. Math. 39 (2002), 705-721] by a different approach based on a group-theoretic argument from [Tsukuba J. Math. 22 (1998), 723-739; Corrigendum, Tsukuba J. Math 24 (2000), 433-434]. This permits to simplify some proofs of [Osaka J. Math. 39 (2002), 705-721], and to obtain some new related results.

2000 Mathematics Subject Classification: 57M12, 57M50, 57R65, 20F05.

Key words and phrases: 3-manifolds, cyclic branched coverings, periodic links and manifolds, Dehn surgery.

Work performed under the auspices of the G.N.S.A.G.A. of the C.N.R. (National Research Council) of Italy, partially supported by the MURST (Ministero per la Ricerca scientifica e Tecnologica) of Italy within the project "Proprietà Geometriche delle Varietà Reali e Complesse", and by a Research grant of the University of Modena and Reggio Emilia.

Received October 18, 2003

1. Surgery Manifolds

Generalized Takahashi manifolds are closed connected orientable 3-manifolds introduced in [15] as generalizations of certain manifolds first defined by Takahashi in [19], and successively studied by several authors (see [8], [14], and [17]). Takahashi manifolds are very interesting since they include well-known families of manifolds (many of them having hyperbolic structures), treated in the literature, as for example *Fibonacci manifolds* [4], [6], [7], *generalised Fibonacci manifolds* [10], [11], *fractional Fibonacci manifolds* [9], and *Sieradski manifolds* [2], [3]. Furthermore, many Takahashi manifolds are also examples of *maximally symmetric* 3-*manifolds* in the sense of [20]. Generalized Takahashi manifolds are represented in [15] by Dehn surgery with rational coefficients along an n-periodic $2nm$-component oriented link in the oriented 3-sphere. For any pair of positive integers n and m, let us consider the n-periodic oriented link $L_{n,m} \subset \mathbb{S}^3$ of $2nm$-components depicted in Figure 1 (case $n = m = 3$). All the components $c_{i,j}$ of $L_{n,m}$, $1 \le i \le 2n, 1 \le j \le m,$ are unknotted circles, and they form $2n$ subfamilies $c_i = \{c_{i,j}\}_{j=1,\dots,m}$ of m unlinked circles $c_{i,j}$ with common center.

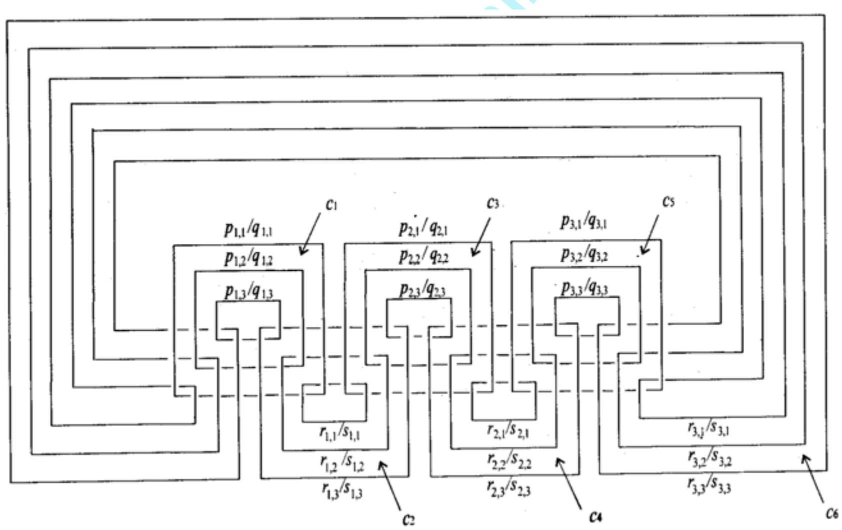


Figure 1. Dehn surgery description of the generalized Takahashi manifold $M_{n,m}(p_{i,j}/q_{i,j};\, r_{i,j}/s_{i,j})$ (case $n = m = 3$)

The link $L_{n,m}$ is n-periodic since it has the cyclic symmetry of order n which sends c_{2i-1} and c_{2i} to c_{2i+1} and c_{2i+2}, respectively, for $i = 1, \ldots, n$, where the indices are taken mod $2n$. A *generalized Takahashi manifold* $M_{n,m}(p_{i,j}/q_{i,j}; r_{i,j}/s_{i,j})$ is the closed connected orientable 3-manifold obtained by Dehn surgeries along the components of the link $L_{n,m}$ so that the surgery coefficients $p_{i,j}/q_{i,j}$ and $r_{i,j}/s_{i,j}$ correspond to the components $c_{2i-1,j}$ and $c_{2i,j}$, respectively, where $1 \leq i \leq n$, $1 \leq j \leq m$, $\gcd(p_{i,j}, q_{i,j}) = 1$ and $\gcd(r_{i,j}, s_{i,j}) = 1$ (see Figure 1). For $m = 1$, we get the *Takahashi manifolds* introduced in [19], and intensively studied in a series of papers (see above). A generalized Takahashi manifold is called *periodic* if the surgery coefficients are n-periodic, that is, $p_{i,j} = p_j$, $q_{i,j} = q_j$, $r_{i,j} = r_j$ and $s_{i,j} = s_j$, for $1 \leq i \leq n$ and $1 \leq j \leq m$. In this case, we shall briefly write $M_{n,m}(p_j/q_j; r_j/s_j)$. The topological structure of (periodic) generalized Takahashi manifolds as 2-fold branched coverings of $\mathbb{S}^3$ was given in [15]. To state that result, let us denote by $K_{n,m}(p_{i,j}/q_{i,j}; r_{i,j}/s_{i,j})$ the closure of the rational braid on $2m+1$ strings with rational tangles $p_{i,j}/q_{i,j}$ and $r_{i,j}/s_{i,j}$ (see Figure 2 for the case $n = m = 3$). This link is n-periodic when $p_{i,j} = p_j$, $q_{i,j} = q_j$, $r_{i,j} = r_j$ and $s_{i,j} = s_j$ as above, and we write simply $K_{n,m}(p_j/q_j; r_j/s_j)$.

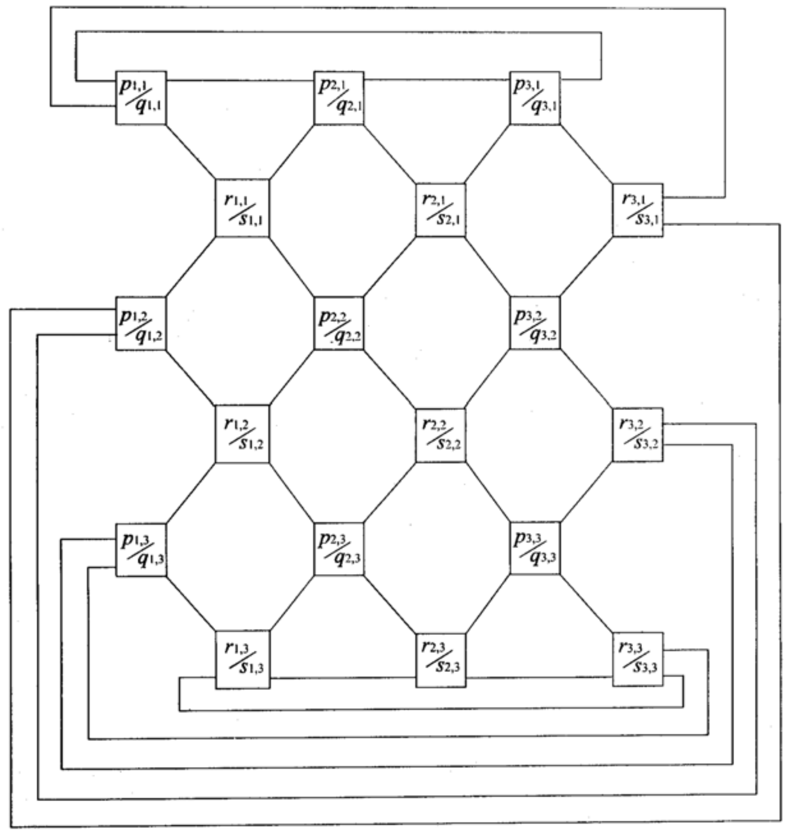


Figure 2. The link $K_{n,m}(p_{i,j}/q_{i,j};\ r_{i,j}/s_{i,j})$ which is the closure of a rational braid on $2m+1$ strings with rational tangles $p_{i,j}/q_{i,j}$ and $r_{i,j}/s_{i,j}$ (case $n = m = 3$)

The following result, proved in [15], is a generalization of theorems from [8] and [17] (case $m = 1$).

Theorem 1.1. *With the above notations, the generalized Takahashi manifold* $M_{n,m}(p_{i,j}/q_{i,j}; r_{i,j}/s_{i,j})$ *is the* 2*-fold cyclic covering of the* 3*-sphere branched along the link* $K_{n,m}(p_{i,j}/q_{i,j}; r_{i,j}/s_{i,j})$.

Furthermore, the periodic generalized Takahashi manifolds were also described in [15] as the n-fold cyclic branched coverings of the connected sum of lens spaces (including possible summands homeomorphic to either $\mathbb{S}^3$ or $\mathbb{S}^1 \times \mathbb{S}^2$). Finally, it was shown in [15] that the family of periodic generalized Takahashi manifolds contains (for particular values of the surgery coefficients) all cyclic branched coverings of 2-bridge knots (see [12] for a nice description of such manifolds as side pairings of oppositely oriented faces on the boundary of certain triangulated 3-balls). The proof follows from a long sequence of surgery moves on the components of links with rational coefficients, and using the Kirby-Rolfsen calculus. In this short note, we re-obtain all the results of [15] by a different approach which is based on the group-theoretic techniques developed in our previous paper [17]. This permits to simplify some proofs given in [15], and to obtain some new related results. We also point out that recent results about surgeries on periodic links and homology of periodic 3-manifolds can be found in a nice paper of Przytycki and Sokolov [16].

Now we prove Theorem 1.1 by using the Montesinos algorithm from [13]. To do this, let $M'_{n,m} = M'_{n,m}(p_{i,j}/q_{i,j}; r_{i,j}/s_{i,j})$ denote the 2-fold covering of $\mathbb{S}^3$ branched over the link $K_{n,m} = K_{n,m}(p_{i,j}/q_{i,j}; r_{i,j}/s_{i,j})$. We consider the neighborhoods $B_{i,j}$, $1 \le i \le 2n$, $1 \le j \le m$, which contain cross-points of the diagram of $K_{n,m}$, pictured in Figure 3 (case $n = m = 3$).

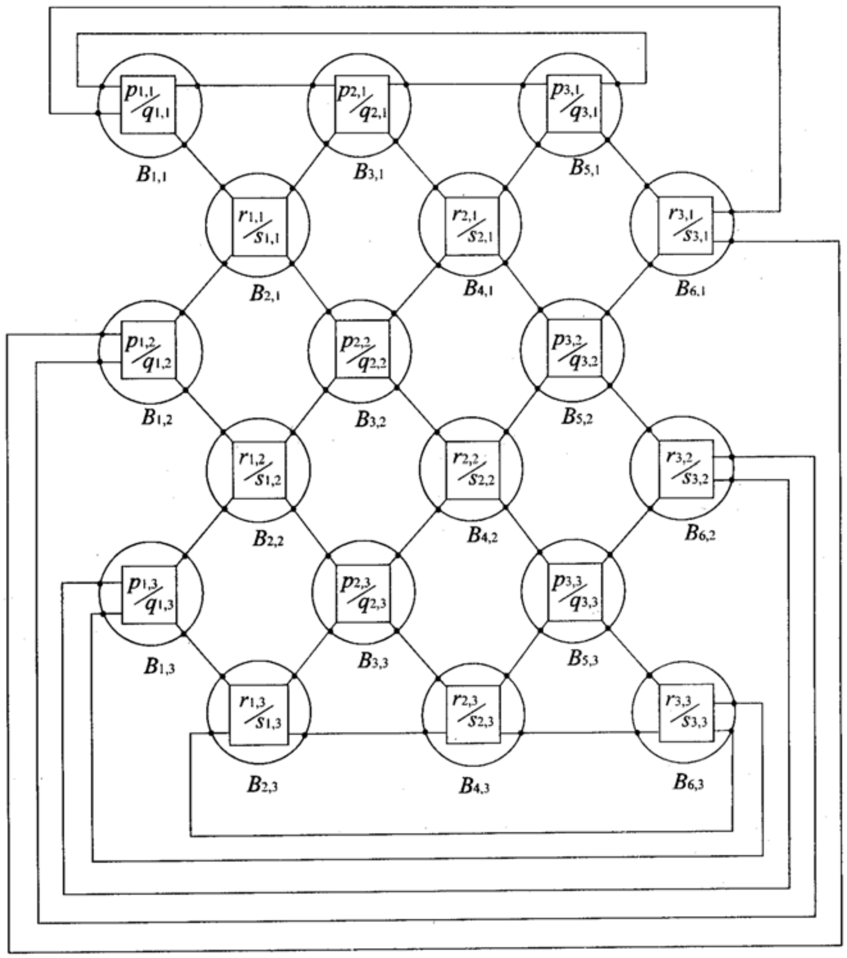


Figure 3. The balls $B_{i,j}$ containing cross-points of the diagram of $K_{n,m}(p_{i,j}/q_{i,j};\, r_{i,j}/s_{i,j})$ (case $n = m = 3$)

Each $B_{i,j}$ is a 3-ball such that $\partial B_{i,j} \cap K_{n,m}$ consists of four points which are pairwise connected by two arcs formed by $B_{i,j} \cap K_{n,m}$. Therefore $B_{i,j}$ can be considered as a Conway sphere and more exactly $B_{i,j}$ is the $p_{i,j}/q_{i,j}$-tangle if i is odd and $B_{i,j}$ is the $r_{i,j}/s_{i,j}$-tangle if i is even. Let $B'_{i,j}$ be the trivial tangles such that $\partial B'_{i,j} = \partial B_{i,j}$ and the four points on $\partial B'_{i,j}$ are pairwise connected by two arcs $a_{i,j}$ and $b_{i,j}$ inside $B'_{i,j}$. Then the set

$$(K_{n,m} \cap Ext(\cup_{i,j} B_{i,j})) \cup (\cup_{i,j} (a_{i,j} \cup b_{i,j}))$$

is a closed unknotted curve $\mathcal{C}$ in $\mathbb{S}^3$ (see Figure 4). We can redraw $\mathcal{C}$ as a horizontal line (see Figure 5). Let us consider the 2-fold covering of $\mathbb{S}^3$ branched over the curve $\mathcal{C}$. The 2-fold coverings of $B_{i,j}$ branched over $B_{i,j} \cap K_{n,m}$ and the 2-fold coverings of $B'_{i,j}$ branched over $B'_{i,j} \cap \mathcal{C}$ $= a_{i,j} \cup b_{i,j}$ are solid tori. Let us denote by $T_{i,j}$ a torus corresponding to $B'_{i,j}$. Then the tori $T_{i,j}$ are toroidal neighborhoods of the components of the link $L_{n,m}$ pictured in Figure 1. So the manifold $M'_{n,m}$ can be obtained by surgeries with parameters $p_{i,j}/q_{i,j}$ and $r_{i,j}/s_{i,j}$ on the components of the link $L_{n,m}$, i.e., $M'_{n,m}$ is homeomorphic to the generalized Takahashi manifold $M_{n,m}$.

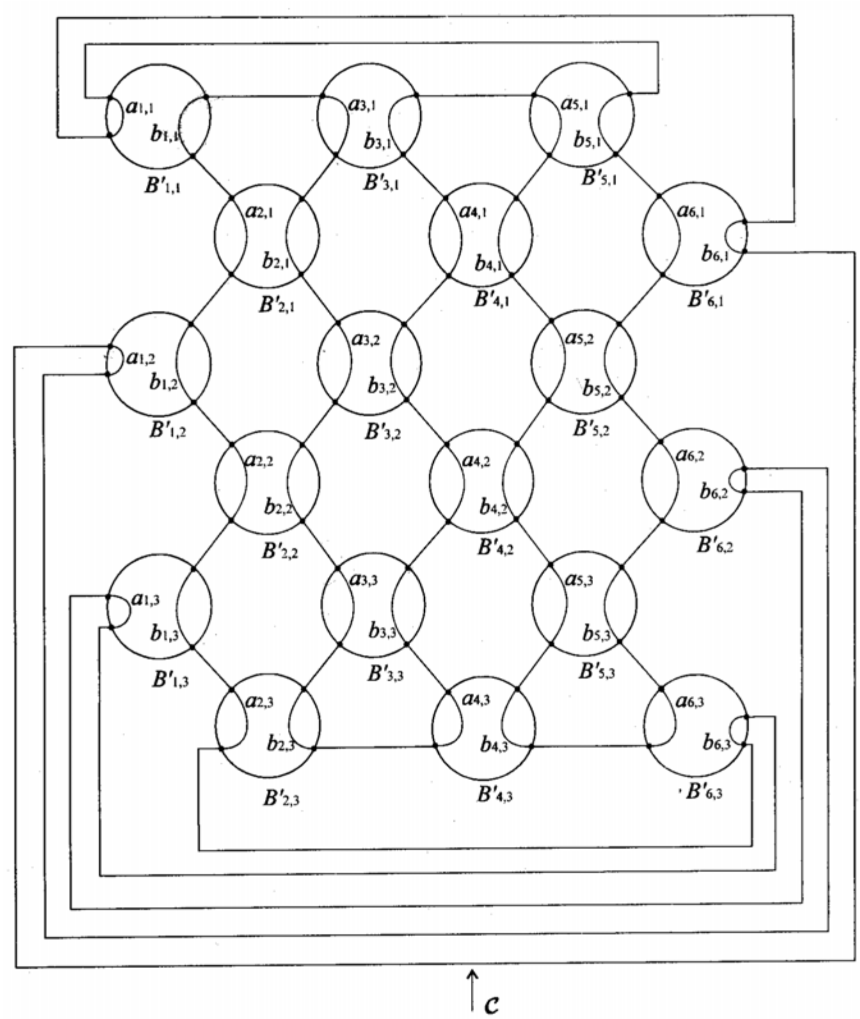


Figure 4. The trivial tangles $B'_{i,j}$ such that $\partial B'_{i,j} = \partial B_{i,j}$ and the closed unknotted curve $\mathcal{C}$

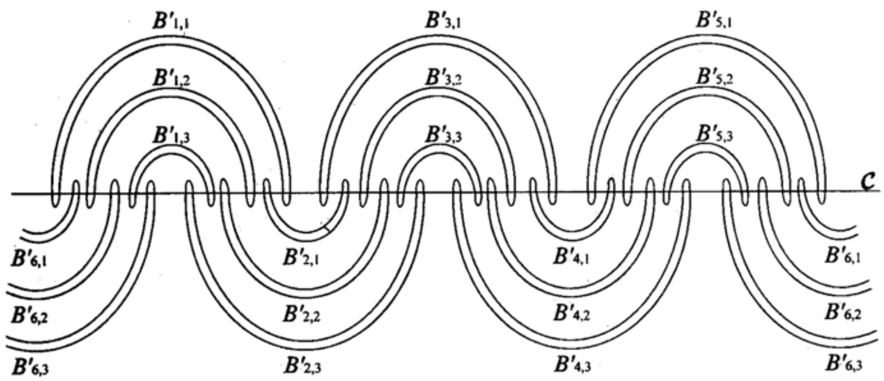


Figure 5. The curve $\mathcal{C}$ drawn as a horizontal line and the modified tangles $B'_{i,j}$

2. Covering Diagrams: The Periodic Case

Let $M_{n,m} = M_{n,m}(p_j/q_j; r_j/s_j)$ be the periodic generalized Takahashi manifold, so the link $K_{n,m} = K_{n,m}(p_j/q_j; r_j/s_j)$ is also n-periodic. To describe $M_{n,m}$ as n-fold cyclic branched coverings we use an algebraic argument from [17]. Let $\mathcal{O}(K_{n,m}, 2)$ be the orbifold whose underlying space is $\mathbb{S}^3$ and whose singular set is the link $K_{n,m}$ with branching index 2 on each component of it. From the presentation of the n-periodic link $K_{n,m}$ given in Figure 2, where $p_{i,j} = p_j$, $q_{i,j} = q_j$, $r_{i,j} = r_j$ and $s_{i,j} = s_j$, we see that the orbifold $\mathcal{O}(K_{n,m}, 2)$ has a symmetry ρ of order n. Furthermore, the singular set of orbifolds and the symmetry axis are disjoint. Therefore, the quotient space $\mathcal{O}(K_{n,m}, 2)/\langle\rho\rangle$ of $\mathcal{O}(K_{n,m}, 2)$ under this symmetry action is the orbifold whose underlying space is $\mathbb{S}^3$, and whose singular set is the link $L_m = L_m(p_j/q_j; r_j/s_j)$ pictured in Figure 6 with branching indices n and 2.

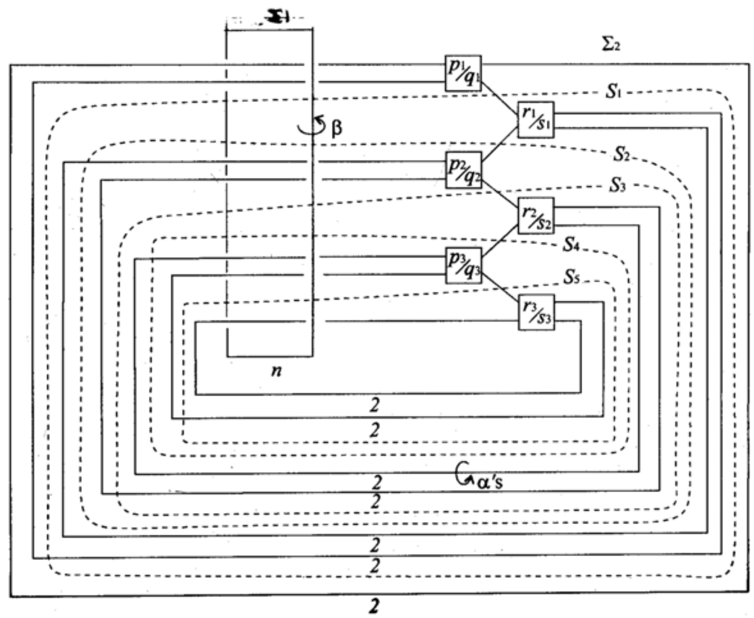


Figure 6. The link $L_m(p_j/q_j;\, r_j/s_j)$ and the orbifold $O(L_m, 2, n)$ (case $m = 3$)

The branching index is 2 on the components which are images of those of $K_{n,m}$, and the branching index is n on the unknotted component. The link L_m can be regarded as the union $\Sigma_1 \cup \Sigma_2$, where Σ_1 is the trivial knot (with branching index n), and Σ_2 is precisely the link $K_{1,m}$ (with branching index 2 on its components). From Figure 6 we see that S_h are decomposing 2-spheres so $K_{1,m}$ is the connected sum of $2m$ 2-bridge links $\mathbf{b}(p_j, q_j)$ and $\mathbf{b}(r_j, s_j)$, $j = 1, \ldots, m$. Let $\mathcal{O}(L_m, 2, n)$ denote the quotient orbifold $\mathcal{O}(K_{n,m}, 2)/\langle\rho\rangle$. We have the following covering diagram

$$M_{n,m} \xrightarrow{\;2\;} \mathcal{O}(K_{n,m}, 2) \xrightarrow{\;n\;} \mathcal{O}(L_m, 2, n)$$

and a sequence of normal subgroups

$$G_{n,m} = \pi_1(M_{n,m}) \triangleleft H_2 = \pi_1^{orb}(\mathcal{O}(K_{n,m}, 2)) \triangleleft \Omega_{2,n} = \pi_1^{orb}(\mathcal{O}(L_m, 2, n)),$$

where $[\Omega_{2,n} : H_2] = n$ and $[H_2 : G_{n,m}] = 2$. Let α's be the loops around the components of Σ_2 and β be the loop around the unknot Σ_1 as indicated in Figure 6. Let us consider the group

$$\mathbb{Z}_n \oplus \mathbb{Z}_2 = \langle a : a^n = 1 \rangle \oplus \langle b : b^2 = 1 \rangle$$

and the epimorphism $\gamma : \Omega_{2,n} \to \mathbb{Z}_n \oplus \mathbb{Z}_2$ which sends the loop β around the trivial knot Σ_1 to the generator a, and loops α's around the components of the link Σ_2 to the generator b. By construction of the n-fold covering

$$\mathcal{O}(K_{n,m}, 2) \xrightarrow{\ n\ } \mathcal{O}(L_m, 2, n)$$

the loop $\beta \in \Omega_{2,n}$ lifts to a trivial loop in H_2, and the loops α's, $\alpha \in \Omega_{2,n}$, lift to loops in H_2 which generate cyclic subgroups of order 2. Thus it follows that

$$H_2 = \pi_1^{orb}(\mathcal{O}(K_{n,m}, 2)) = \gamma^{-1}(\langle b : b^2 = 1 \rangle) = \gamma^{-1}(\mathbb{Z}_2).$$

For the $2n$-fold covering

$$M_{n,m} \xrightarrow{\ 2n\ } \mathcal{O}(L_m, 2, n)$$

both the loops around Σ_1 and Σ_2 from $\Omega_{2,n}$ lift to trivial loops in $G_{n,m} = \pi_1(M_{n,m})$, hence $G_{n,m} = Ker\,\gamma$, and $\Omega_{2,n}/G_{n,m} \cong \mathbb{Z}_n \oplus \mathbb{Z}_2$.

Let $\Gamma_n = \Gamma_n(p_j/q_j;\, r_j/s_j)$ be the subgroup of $\Omega_{2,n}$ given by

$$\Gamma_n = \gamma^{-1}(\langle a : a^n = 1 \rangle) = \gamma^{-1}(\mathbb{Z}_n).$$

Then we get a sequence of normal subgroups

$$G_{n,m} \triangleleft \Gamma_n \triangleleft \Omega_{2,n},$$

where $[\Omega_{2,n} : \Gamma_n] = 2$ and $[\Gamma_n : G_{n,m}] = n$. Let $X_{n,m} = X_{n,m}(p_j/q_j;\, r_j/s_j)$ be the universal covering of $M_{n,m}$, i.e., $X_{n,m}/G_{n,m} \cong M_{n,m}$. Thus we get the orbifold $X_{n,m}/\Gamma_n$ and the covering diagram

$$M_{n,m} \xrightarrow{\ n\ } X_{n,m}/\Gamma_n \xrightarrow{\ 2\ } \mathcal{O}(L_m, 2, n).$$

In this case the second covering is cyclic and it is branched over the component Σ_2 with branching index 2 of the singular set of $\mathcal{O}(L_m, 2, n)$. But the component Σ_2 is precisely the connected sum of $2m$ 2-bridge links $\mathbf{b}(p_j, q_j)$ and $\mathbf{b}(r_j, s_j)$, $1 \leq j \leq m$. So the underlying space of the orbifold $X_{n,m}/\Gamma_n$ is topologically the connected sum $L(p_1, q_1) \# L(r_1, s_1) \# \cdots \# L(p_m, q_m) \# L(r_m, s_m)$ (including possible summands homeomorphic to either $\mathbb{S}^1 \times \mathbb{S}^2 = L(0, 1)$ or $\mathbb{S}^3 = L(1, q)$). By construction of the 2-fold covering

$$X_{n,m}/\Gamma_n \xrightarrow{\ 2\ } \mathcal{O}(L_m, 2, n)$$

the loops α's around Σ_2 lift to trivial loops in Γ_n, and the loop β around Σ_1 lifts to a loop in Γ_n which generates a cyclic group of order n. Because the singularity index is equal to n, let $\mathcal{O}_{n,m} = \mathcal{O}_{n,m}(p_j/q_j;\, r_j/s_j)$ denote the orbifold $X_{n,m}/\Gamma_n$ whose underlying space is topologically the connected sum

$$L(p_1, q_1) \# L(r_1, s_1) \# \cdots \# L(p_m, q_m) \# L(r_m, s_m)$$

and whose singular set is a knot K with branching index n. Moreover, the knot K does not depend on n. So we have obtained a different proof of the following results given in [14] and [17] for periodic Takahashi manifolds and in [15] for periodic generalized Takahashi manifolds.

Theorem 2.1. *With the above notations, the following commutative diagram holds for each periodic generalized Takahashi manifold*

$$\begin{tikzcd}
& M_{n,m}(p_j/q_j;\, r_j/s_j) \arrow[dl, "2"'] \arrow[dr, "n"] & \\
\mathcal{O}(K_{n,m}(p_j/q_j;\, r_j/s_j), 2) \arrow[dr, "n"'] & & \mathcal{O}_{n,m}(p_j/q_j;\, r_j/s_j) \arrow[dl, "2"] \\
& \mathcal{O}(L_m(p_j/q_j;\, r_j/s_j), 2, n) &
\end{tikzcd}$$

Corollary 2.2. *The periodic generalized Takahashi manifold $M_{n,m}(p_j/q_j;\, r_j/s_j)$ is the $\mathbb{Z}_n \oplus \mathbb{Z}_2$-covering of the orbifold $\mathcal{O}(L_m(p_j/q_j;\, r_j/s_j), 2, n)$.*

Corollary 2.3. *The periodic generalized Takahashi manifold* $M_{n,m}(p_j/q_j;\, r_j/s_j)$ *is the n-fold covering of the connected sum of lens spaces*

$$L(p_1, q_1) \# L(r_1, s_1) \# \cdots \# L(p_m, q_m) \# L(r_m, s_m)$$

branched over a knot which does not depend on n.

3. Further Results

First we give a short alternative proof of the following result given in [14] (case $m = 1$) and successively extended in [15] for any m (we use the Conway notation for 2-bridge knots from [5]). As remarked in Section 1, this shows that the family of periodic generalized Takahashi manifolds contains all cyclic coverings of 2-bridge knots.

Theorem 3.1. *For any* q_j *and* $s_j \in \mathbb{Z}$, $j = 1, \ldots, m$, *and* $n > 1$, *the periodic generalized Takahashi manifold* $M_{n,m}(1/q_j;\, 1/s_j)$ *is homeomorphic to the n-fold cyclic covering of the 3-sphere branched over the 2-bridge knot corresponding to Conway parameters* $[-2q_1, 2s_1, \ldots, -2q_m, 2s_m]$. *In particular,* $M_{n,1}(1/q;\, 1/s)$ *is the n-fold cyclic covering of the 2-bridge knot of genus one* $\mathbf{b}(|4qs - 1|, 2q)$.

Proof. By Corollary 2.3 the manifold $M_{n,m}(1/q_j;\, 1/s_j)$ is the n-fold cyclic covering of the 3-sphere branched over a knot K which does not depend on n. So it suffices to consider the case $n = 2$ to know the knot type of K. By Theorem 1.1 the manifold $M_{2,m} = M_{2,m}(1/q_j;\, 1/s_j)$ is the 2-fold covering of the 3-sphere branched along $K_{2,m} = K_{2,m}(1/q_j;\, 1/s_j)$ (see Figure 7 for $m = 2$). As one can easily seen from Figure 8, $K_{2,m}$ is equivalent by isotopy to the 2-bridge knot $\mathbf{b}(\alpha, \beta)$, where

$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{-a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \; \cfrac{1}{-a_{2m}}}}}$$

with $a_{2j} = a'_{2j} + a''_{2j} = -2s_j$ and $a_{2j-1} = a'_{2j-1} + a''_{2j-1} = -2q_j$, $1 \leq j \leq m$ (see [18]). So $M_{2,m}$ is the lens space $L(\alpha, \beta)$, and K is the 2-bridge knot $\mathbf{b}(\alpha, \beta)$ (recall that a lens space admits a unique representation as a 2-fold covering of the 3-sphere).

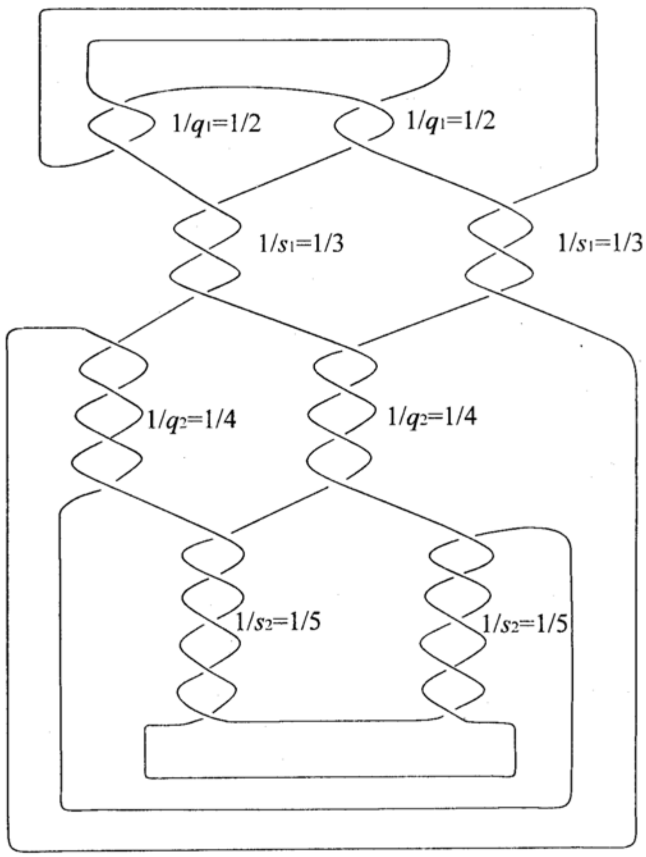


Figure 7. The knot $K_{2,m}(1/q_j;\ 1/s_j)$

(case $m = 2$, $q_1 = 2$, $q_2 = 4$, $s_1 = 3$, $s_2 = 5$)

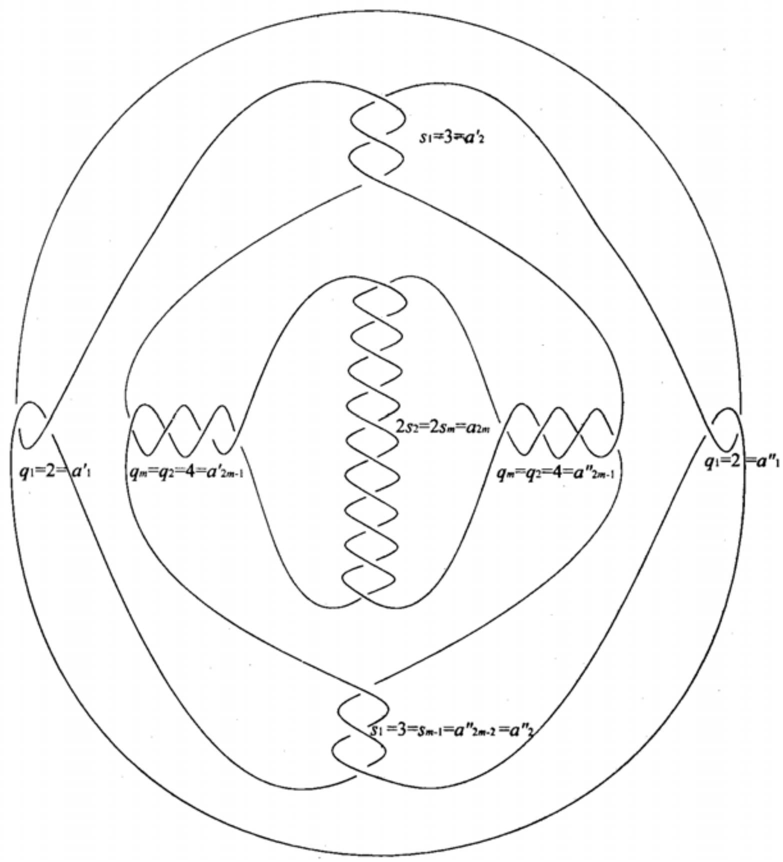


Figure 8. A representation of $K_{2,m}(1/q_j;\, 1/s_j)$ as a 2-bridge knot according to [17]

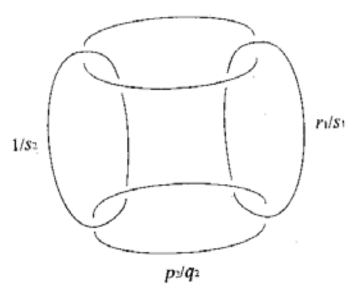
The proof of Theorem 3.1 can be repeated to show that the generalized Takahashi manifold $M_{2,m}(1/q_{1,j}, 1/q_{2,j}; 1/s_{1,j}, 1/s_{2,j})$ is homeomorphic to the lens space $L(a, b)$ precised in the next statement. So we get the following new result.

Theorem 3.2. *For all* $q_{2i-1,j} = q_{1,j}$, $q_{2i,j} = q_{2,j}$, $s_{2i-1,j} = s_{1,j}$, $s_{2i,j} = s_{2,j}$, $i = 1, ..., k$, $j = 1, ..., m$, *and* $n = 2k$, $k > 1$, *the generalized Takahashi manifold* $M_{n,m}(1/q_{i,j}; 1/s_{i,j})$ *is a k-fold cyclic branched covering of the lens space* $L(a, b)$, *where a and b are coprime integers obtained from the continued fraction corresponding to Conway parameters* $[-q_{1,1} - q_{2,1}, s_{1,1} + s_{2,1}, ..., -q_{1,m} - q_{2,m}, s_{1,m} + s_{2,m}]$.

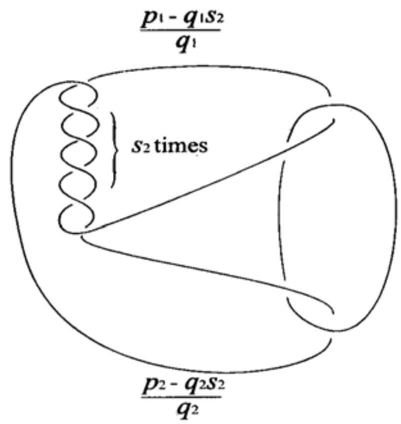
To end the section, let us consider the manifold $M_2(p_1/q_1, p_2/q_2; r_1/s_1, 1/s_2)$ whose surgery description is given in Figure 9(a) (case $m = 1$). Applying $-s_2$ twists about the component on the left side yields a link with surgery coefficients shown in Figure 9(b). By Theorem 10 of [1], we get that our manifold is homeomorphic to the $\mathbb{S}^1$-manifold with Seifert invariants

$$(b; (o, O, f, 0); (p_1, q_1)(p_2, q_2)(s_1 + s_2 r_1, s_1 + (s_2 - 1)r_1)).$$

Here $b = -1$ when $p_1 \neq 0$, $p_2 \neq 0$, and $s_1 + s_2 r_1 \neq 0$. In particular, if $p_1 = p_2 = r_1 = 1$, $q_1 = q_2 = q$ and $s_1 = s_2 = s$, then we get the Seifert manifold with invariants $(O o O : -1(1, q)(1, q)(2s, 2s-1))$ which is homeomorphic to the lens space $L(|4qs - 1|, 2q)$. This gives a further alternative proof of the second statement in Theorem 3.1.



(a)



(b)

Figure 9. Surgery descriptions of the Takahashi manifold $M_2(p_1/q_1,\ p_2/q_2;\ r_1/s_1, 1/s_2)$

Finally, we immediately obtain the following result.

Theorem 3.3. *For all* $p_{2i-1} = p_1,\ \ p_{2i} = p_2,\ \ q_{2i-1} = q_1,\ \ q_{2i} = q_2,$ $r_{2i-1} = r_1,\ r_{2i} = 1,\ s_{2i-1} = s_1,\ s_{2i} = s_2,\ i = 1, \ldots, k,$ *and* $n = 4k,\ k > 1,$ *the Takahashi manifold* $M_n(p_i/q_i;\ r_i/s_i)$ *is a k-fold cyclic branched covering of the* $\mathbb{S}^1$*-manifold whose Seifert invariants are written above.*

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