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THE EQUIVALENCE BETWEEN THE CONVERGENCE OF ISHIKAWA AND MANN ITERATIVE APPROXIMATIONS FOR AN ASYMPTOTICALLY Φ-HEMICONTRACTIVE MAPPING

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Abstract

Let E be a real Banach space and T be a multivalued asymptotically Φ-hemicontractive mapping. It is shown that the convergence of the Mann iterative set sequence of T is equivalent to the convergence of the Ishikawa iterative set sequence. The results presented in this paper extend the corresponding results of Rhoades and Solutz.

1. Introduction and Preliminaries

In this paper, it is assumed that E is a real Banach space and E^* is the dual space of E. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \| f^* \| \| x \|, \| f^* \| = \| x \|\}$$

is called *normalized duality map* of E, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between elements of E and that of E^*.

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Definition 1.1. Let the domain $D(T)$ of mapping T be nonempty. Then $T : D(T) \subset E \to 2^E$ is *multivalued mapping*.

(1) T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, such that

$$\| u - v \| \leq k_n \| x - y \|, \quad \forall x,\ y \in D(T), \forall n \geq 1,$$

where $u \in T^n x, v \in T^n y$.

(2) T is said to be *asymptotically pseudocontractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, and there exists $j \in J(x - y)$ such that

$$\langle u - v,\ j\rangle \leq k_n \| x - y \|^2, \quad \forall x,\ y \in D(T), \quad \forall n \geq 1,$$

where $u \in T^n x, v \in T^n y$.

(3) T is said to be *asymptotically* Φ*-strongly pseudocontractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, there exist $j \in J(x - y)$ and a real valued function $\Phi : R^+ = [0, \infty) \to R^+$, $\Phi(0) = 0$, $\liminf_{r\to\infty} \Phi(r) > 0$, $h(r) = r\Phi(r)$ nondecreasing such that

$$\langle u - v,\ j\rangle \leq k_n \| x - y \|^2 - \Phi(\| x - y \|) \| x - y \|, \forall x,\ y \in D(T)\ \forall n \geq 1, \quad (1)$$

where $u \in T^n x, v \in T^n y$. If Eq. (1) holds for $u \in T^n x, v \in F(T)$ (the fixed point set of T), then T is called *asymptotically* Φ*-hemicontractive mapping*.

We know that an asymptotically nonexpansive mapping is an asymptotically pseudocontractive. An asymptotically pseudocontractive mapping is an asymptotically Φ-strongly pseudocontractive. But the converse is not true. Setting $n = 1$, $\{k_n\} = \{1\}$, we get the usual definition of Φ-strongly pseudocontractive mapping. Setting $n \geq n_0$, $\{k_n\} = \{1\}$, $\Phi(r) = kr$, $k \in (0, 1)$ and taking T to be a single valued mapping, we get the definition of a strongly successively pseudocontractive mapping ([1, Definition 1(ii), p. 267]).

The asymptotically nonexpansive mappings were introduced by Geobel and Kirk [3] in 1972. Later, in 1991, Schu [8] introduced the asymptotically pseudocontractive mappings. The authors [3, 4] discussed respectively the iterative approximation of fixed point for an asymptotically nonexpansive and asymptotically pseudocontractive mapping in Hilbert and a uniformly convex Banach space. Zhang [12] worked on the convergence of modified Mann and Ishikawa iterative sequence with errors for single valued asymptotically nonexpansive mappings in Banach spaces. The results of [12] extend and improve the corresponding results of [3] and [8]. In [11], the author studied the approximation of the new Ishikawa iteration with errors for multivalued asymptotically Φ-hemicontractive mappings.

In this paper, we discuss the convergence of Mann iteration set sequence which is equivalent to the convergence of Ishikawa iteration set sequence for an asymptotically Φ-hemicontractive mapping in a real Banach space.

Definition 1.2. Let $D(T)$ of the mapping T be nonempty. Then $T : D(T) \subset E \to 2^E$ is an asymptotically Φ-strongly pseudocontractive mapping. For given $z_0 \in D(T), O_0 = \{z_0\}$ (singleton), the iterative set sequence is

$$\{O_n\} = \{O_0, O_1, \ldots, O_n, \ldots\}$$

generated by $\forall z_n \in O_n$ $(n = 0, 1, 2, \ldots)$, $\forall z_{n+1} \in O_{n+1}$ scheme

$$z_{n+1} \in (1 - \alpha_n) z_n + \alpha_n T^n z_n, \;\; n = 0, 1, 2, \ldots. \tag{2}$$

This iteration is known as the multivalued modified Mann iterative set sequence. The multivalued modified Ishikawa iterative set sequence is defined as follows: for given $x_0 \in D(T), H_0 = \{x_0\}$ (singleton), the iterative set sequence is

$$\{H_n\} = \{H_0, H_1, \ldots, H_n, \ldots\}$$

generated by $\forall x_n \in H_n$ $(n = 0, 1, 2, \ldots)$, $\forall x_{n+1} \in H_{n+1}$ scheme

$$\begin{aligned} &x_{n+1} \in (1 - \alpha_n) x_n + \alpha_n T^n y_n, \\ &y_n \in (1 - \beta_n) x_n + \beta_n T^n x_n, \quad \forall x_0 \in K, \;\; n = 0, 1, 2, \ldots, \end{aligned} \tag{3}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ are two real numbers sequences satisfying the following conditions:

(i) $0 \leq \alpha_n, \beta_n \leq 1,$ for $n = 0, 1, 2, \ldots,$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty, \alpha_n \to 0, \beta_n \to 0 \ (n \to \infty).$

Definition 1.3 [11]**.** Let $O, O_n \subset E, n \in N.$ We say that $\{O_n\}$ converges strongly to O if

$$W^{+}(O_n, O) = \sup_{y \in O_n} \inf_{x \in O} \| y - x \| \to 0 \ (n \to \infty).$$

It is denoted by $O_n \to O \ (n \to \infty).$

Example 1. Let $S = \{x : 0 \leq x \leq 1\} \cup \{x : 2 \leq x \leq 3\}, S_n = \left\{x : 2 - \frac{1}{n} \leq x \leq 3\right\}, n \in N.$ Then we have $S_n \to S \ (n \to \infty).$

Remark 1. Let $S = \{x : 2 \leq x \leq 3\}$ in Example 1. Then also we have $S_n \to S \ (n \to \infty).$ Therefore, the set S is not unique.

Definition 1.4. Let E be a real Banach space and let $CB(E)$ be the family of all nonempty closed bounded subsets of E. A multivalued mapping $T : E \to CB(E)$ is called *successively uniformly continuous* if $\forall \varepsilon > 0,$ then there exist $\delta > 0$ and $n_0 \in N,$ such that $H(T^n x, T^n y) < \varepsilon$ whenever $\| x - y \| < \delta$ and $n \geq n_0,$ where H is the Hausdorff distance in $CB(E),$ i.e.,

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\}, \quad \forall A, B \subset CB(E).$$

It is clear that a uniformly Lipschitzian mapping is much stronger than a successively uniformly continuous mapping.

Lemma 1.5 [9]**.** *Let* $\{\sigma_n\}, \{\mu_n\}, \{t_n\}$ *be nonnegative real sequences. Let* $\exists n_0 \geq 0$ *such that*

$$\sigma_{n+1} \leq (1 - t_n)\sigma_n + \mu_n, \ \textit{for} \ \forall n \geq n_0,$$

where $0 \leq t_n \leq 1, \sum_{n=1}^{\infty} t_n = \infty, \mu_n = o(t_n)$. *Then* $\sigma_n \to 0\,(n \to \infty)$.

Lemma 1.6 [1]. *If* E *is a real Banach space, then the following relation is true*:

$$\| x + y \|^2 \leq \| x \|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E, \quad \forall j(x + y) \in J(x + y).$$

2. Main Results

Theorem 2.1. *Let* E *be a real Banach space and let* $CB(E)$ *be the family of nonempty closed bounded subsets of* E. *Let* $T : E \to CB(E)$ *be successively uniformly continuous asymptotically* Φ*-hemicontractive mapping with* $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$. *If* $z_0 = x_0 \in E$, *then the following two assertions are equivalent*:

(1) *Multivalued modified Mann iterative set sequence* $\{O_n\}$ *generated by* $\{z_n\}$ *converges strongly to* $\{x^*\} \subset F(T)$;

(2) *Multivalued modified Ishikawa iterative set sequence* $\{H_n\}$ *generated by* $\{x_n\}$ *converges strongly to* $\{x^*\} \subset F(T)$.

Proof. If the multivalued modified Ishikawa iterative set sequence $\{H_n\}$ generated by $\{x_n\}$ converges strongly to $\{x^*\}$, then setting $\beta_n = 0$, $n = 0, 1, 2, \ldots,$ we get the convergence of modified Mann iterative set sequence $\{O_n\}$. Conversely, we shall prove that (1) $\Rightarrow$ (2). In [11], we have proved that the new Mann iterative set sequence and the new Ishikawa iterative set sequence converge strongly to the fixed point set of T. Let $c_n \in T^n z_n$. Then (2) can be rewritten as

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n c_n, \quad n = 0, 1, 2, \ldots. \tag{4}$$

Let $u_n \in T^n y_n$, $v_n \in T^n x_n$. Then (3) can be rewritten as

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n u_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n v_n, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{5}$$

First of all, for $n \geq 1$, we have $T^n(E) \subset T(E) \subset CB(E)$. Therefore the sequences $\{c_n\}$, $\{u_n\}$, $\{v_n\}$ are bounded. We denote

$$M_1 = \sup_{n\geq 0}\{\| c_n - x^* \| + \| u_n - x^* \| + \| v_n - x^* \|\} + \| x_0 - x^* \|,$$

$$M = \max_{n\geq 0}\{M_1, \| z_0 - x^* \|\}.$$

This implies that

$$\begin{aligned}\| z_1 - x^* \| &= \| (1-\alpha_0)(z_0 - x^*) + \alpha_0(c_0 - x^*) \| \\ &\leq (1-\alpha_0)\| z_0 - x^* \| + \alpha_0\| c_0 - x^* \| \leq M.\end{aligned}$$

Inductively, we obtain

$$\| z_n - x^* \| \leq M \quad \text{for } \forall n \geq 0$$

as well as

$$\| x_n - x^* \| \leq M, \quad \| y_n - x^* \| \leq M.$$

Since $u_n \in T^n y_n \subset CB(E)$, $c_n \in T^n z_n \subset CB(E)$, similar to Nadler [5], we can prove that there exist $\omega_{n+1} \in T^n x_{n+1}$, $\lambda_{n+1} \in T^n z_{n+1}$ such that

$$\| u_n - \omega_{n+1} \| \leq \left(1 + \frac{1}{n}\right) H(T^n y_n, T^n x_{n+1}), \tag{6}$$

$$\| c_n - \lambda_{n+1} \| \leq \left(1 + \frac{1}{n}\right) H(T^n z_n, T^n z_{n+1}). \tag{7}$$

From (4), (5), and $\alpha_n \to 0, \beta_n \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}\| x_{n+1} - y_n \| &= \| -\alpha_n x_n + \beta_n x_n + \alpha_n u_n - \beta_n v_n \| \\ &\leq \alpha_n(\| x_n \| + \| u_n \|) + \beta_n(\| x_n \| + \| v_n \|) \\ &\leq (\alpha_n + \beta_n)D \to 0, \ (n \to \infty),\end{aligned}$$

$$\begin{aligned}\| z_n - z_{n+1} \| &= \| -\alpha_n z_n + \alpha_n c_n \| \\ &\leq \alpha_n(\| z_n \| + \| c_n \|) \leq \alpha_n D \to 0, \ (n \to \infty),\end{aligned}$$

where $D = \max_{n\geq 0}\{(\| x_n \| + \| u_n \|), (\| z_n \| + \| c_n \|), (\| x_n \| + \| v_n \|)\} < \infty$. Since

T is successively uniformly continuous, for $n > n_0 \in N$, using (6), (7), we have

$$
\begin{aligned}
q_n &= \| u_n - c_n - (\omega_{n+1} - \lambda_{n+1}) \| = \| (u_n - \omega_{n+1}) - (c_n - \lambda_{n+1}) \| \\
&\le \| u_n - \omega_{n+1} \| + \| c_n - \lambda_{n+1} \| \\
&\le \left(1 + \frac{1}{n}\right) H(T^n y_n, T^n x_{n+1}) \\
&\quad + \left(1 + \frac{1}{n}\right) H(T^n z_n, T^n z_{n+1}) \to 0, \ (n \to \infty). \qquad (8)
\end{aligned}
$$

Using (4), (5), (8) and T is an asymptotically Φ-hemicontractive mapping, we obtain

$$
\begin{aligned}
\| x_{n+1} - z_{n+1} \|^2 &\le (1 - \alpha_n) \| x_n - z_n \| \| x_{n+1} - z_{n+1} \| \\
&\quad + \alpha_n \langle u_n - c_n, \ j(x_{n+1} - z_{n+1}) \rangle \\
&= (1 - \alpha_n) \| x_n - z_n \| \| x_{n+1} - z_{n+1} \| \\
&\quad + \alpha_n \langle \omega_{n+1} - \lambda_{n+1}, \ j(x_{n+1} - z_{n+1}) \rangle \\
&\quad + \alpha_n \langle (u_n - c_n) - (\omega_{n+1} - \lambda_{n+1}), \ j(x_{n+1} - z_{n+1}) \rangle \\
&\le (1 - \alpha_n) \| x_n - z_n \| \| x_{n+1} - z_{n+1} \| \\
&\quad + \alpha_n [k_n \| x_{n+1} - z_{n+1} \|^2 \\
&\quad - \Phi(\| x_{n+1} - z_{n+1} \|) \| x_{n+1} - z_{n+1} \|] \\
&\quad + \alpha_n q_n \| x_{n+1} - z_{n+1} \| \qquad (9)
\end{aligned}
$$

because $\alpha_n \to 0$, $q_n \to 0$, $k_n \to 1 \, (n \to \infty)$, then $\exists N_0 > n_0$ such that $1 - \alpha_n(q_n + 2k_n) > 0$ for $n \ge N_0$. Without loss of generality, let $1 - \alpha_n(q_n + 2k_n) > 0$ for $\forall n > n_0$. Note that

$$
\begin{aligned}
&(1 - \alpha_n) \| x_n - z_n \| \| x_{n+1} - z_{n+1} \| \\
&\le \frac{1}{2} [(1 - \alpha_n)^2 \| x_n - z_n \|^2 + \| x_{n+1} - z_{n+1} \|^2] \qquad (10)
\end{aligned}
$$

$$
\| x_{n+1} - z_{n+1} \| \le \frac{1}{2} (1 + \| x_{n+1} - z_{n+1} \|^2). \qquad (11)
$$

Substituting (10), (11) into (9) we obtain

$$\| x_{n+1} - z_{n+1} \|^2 \leq \frac{(1-\alpha_n)^2}{1-\alpha_n(q_n + 2k_n)} \| x_n - z_n \|^2$$

$$- \frac{2\alpha_n}{1-\alpha_n(q_n+2k_n)} \Phi(\| x_{n+1} - z_{n+1} \|) \| x_{n+1} - z_{n+1} \|$$

$$+ \frac{\alpha_n q_n}{1-\alpha_n(q_n+2k_n)}, \text{ for } n > n_0. \tag{12}$$

It readily follows from (12) that $\| x_{n+1} - z_{n+1} \| \to 0$ as $n \to \infty$. However, for completeness, we present the details. We discuss the following cases which cover all the possibilities:

1. Suppose $\inf_{n \geq 0} \| x_{n+1} - z_{n+1} \| > 0$, due to the assumption on the function Φ, then there exists $r > 0$ such that

$$r < \frac{\Phi(\| x_{n+1} - z_{n+1} \|)}{\| x_{n+1} - z_{n+1} \|}, \text{ for } \forall n > n_0. \tag{13}$$

Using (13), then (12) yields

$$\| x_{n+1} - z_{n+1} \|^2 \leq \frac{(1-\alpha_n)^2}{1-\alpha_n(q_n+2k_n)+2\alpha_n r} \| x_n - z_n \|^2$$

$$+ \frac{\alpha_n q_n}{1-\alpha_n(q_n+2k_n)+2\alpha_n r} \tag{14}$$

because $\alpha_n \to 0,\, q_n \to 0,\, k_n \to 1\, (n \to \infty)$, then there exists $n_1 > n_0$ such that

$$(1-\alpha_n)^2 - \left(1 - \frac{r}{2}\alpha_n\right)[1-\alpha_n(q_n+2k_n)+2\alpha_n r]$$

$$= \alpha_n\left[2(k_n - 1) + \alpha_n + q_n + r^2\alpha_n - \frac{3}{2}r\right] \leq 0 \text{ for } \forall n \geq n_1$$

and therefore, we get

$$\frac{(1-\alpha_n)^2}{1-\alpha_n(q_n+2k_n)+2\alpha_n r} \leq 1 - \frac{r}{2}\alpha_n, \text{ for } \forall n \geq n_1. \tag{15}$$

Using (15), then (14) yields

$$\| x_{n+1} - z_{n+1} \|^2 \leq \left(1 - \frac{r}{2}\alpha_n\right)\| x_n - z_n \|^2$$
$$+ \frac{\alpha_n q_n}{1 - \alpha_n(q_n + 2k_n) + 2\alpha_n r} \text{ for } \forall n \geq n_1.$$

Using Lemma 1.5, let $\mu_n = \frac{\alpha_n q_n}{1 - \alpha_n(q_n + 2k_n) + 2\alpha_n r}$. Then we obtain $\| x_n - z_n \| \to 0\,(n \to \infty)$, which is a contradiction. This means that

2. $\inf_{n \geq 0} \| x_{n+1} - z_{n+1} \| = 0$, then $\exists \{x_{n_j+1} - z_{n_j+1}\} \subset \{x_n - z_n\}$ such that $\| x_{n_j+1} - z_{n_j+1} \| \to 0\,(j \to \infty)$. Due to $k_n \to 1, \alpha_n \to 0, q_n \to 0\,(n \to \infty)$, then for $\forall \varepsilon \in (0, 1)$ there always exists large enough positive integer n_{j0} such that

$$\left.\begin{aligned} &\| x_{n_{j0}+1} - z_{n_{j0}+1} \| < \varepsilon \\ &\alpha_n < \frac{1}{4}\Phi(\varepsilon),\ q_n < \frac{1}{4}\Phi(\varepsilon)\varepsilon,\ k_n - 1 < \frac{1}{4}\Phi(\varepsilon) \end{aligned}\right\} \tag{16}$$

for $n \geq n_{j0}$. Our next step is to show that

$$\| x_{n_{j0}+i} - z_{n_{j0}+i} \| \leq \varepsilon, \text{ for } \forall i \geq 1. \tag{17}$$

In fact, as $i = 1$, we know the conclusion holds from (16). As $i = 2$, assume the conclusion does not hold, then we have

$$\| x_{n_{j0}+2} - z_{n_{j0}+2} \| > \varepsilon. \tag{18}$$

Using nondecreasing nature of $h(r)$, we have $\Phi(\| x_{n_{j0}+2} - z_{n_{j0}+2} \|)\| x_{n_{j0}+2} - z_{n_{j0}+2} \| \geq \Phi(\varepsilon)\varepsilon$. Let $h_n = \frac{1}{1 - \alpha_n(q_n + 2k_n)}$. Then the first term of (12) becomes

$$\frac{(1 - \alpha_n)^2}{1 - \alpha_n(q_n + 2k_n)}\| x_n - z_n \|^2 = \| x_n - z_n \|^2$$
$$+ h_n\alpha_n[2(k_n - 1) + \alpha_n + q_n]\| x_n - z_n \|^2.$$

Using (12) and (16), we obtain

$$\| x_{n_{j_0}+2} - z_{n_{j_0}+2} \|^2 \leq \| x_{n_{j_0}+1} - z_{n_{j_0}+1} \|^2$$

$$+ h_{n_{j_0}+1}\alpha_{n_{j_0}+1}\Big\{[2(k_{n_{j_0}+1} - 1) + \alpha_{n_{j_0}+1}$$

$$+ q_{n_{j_0}+1}]\| x_{n_{j_0}+1} - z_{n_{j_0}+1} \|^2 - 2\Phi(\varepsilon)\varepsilon + q_{n_{j_0}+1}\Big\}$$

$$\leq \varepsilon^2 + h_{n_{j_0}+1}\alpha_{n_{j_0}+1}\Big\{[2 \cdot \frac{1}{4}\Phi(\varepsilon) + \frac{1}{4}\Phi(\varepsilon) + \frac{1}{4}\Phi(\varepsilon)\varepsilon]\varepsilon^2$$

$$- 2\Phi(\varepsilon)\varepsilon + \frac{1}{4}\Phi(\varepsilon)\varepsilon\Big\}$$

$$\leq \varepsilon^2 + h_{n_{j_0}+1}\alpha_{n_{j_0}+1}\Big\{\frac{5}{4}\Phi(\varepsilon)\varepsilon - 2\Phi(\varepsilon)\varepsilon\Big\} < \varepsilon^2$$

which is a contradiction with (18). Hence $\| x_{n_{j_0}+2} - z_{n_{j_0}+2} \| < \varepsilon$ holds and inductively we get (17). This implies that

$$\lim_{n\to\infty} \| x_n - z_n \| = 0. \tag{19}$$

Suppose that the modified Mann iterative set sequence $\{O_n\}$ generated by $\{z_n\}$ converges strongly to $x^* \in F(T)$, that is $O_n \to \{x^*\}(n \to \infty)$, i.e.,

$$\sup_{y\in O_n} \inf_{x^*\in F(T)} \| y - x^* \| = \sup_{y\in O_n} \| y - x^* \| \to 0\,(n \to \infty).$$

Then, we have $\lim_{n\to\infty} z_n = x^*$. The inequality

$$0 \leq \| x_n - x^* \| \leq \| x_n - z_n \| + \| z_n - x^* \| \tag{20}$$

and (19) imply that $\lim_{n\to\infty} x_n = x^*$. Let δH_n denote the diameter of the set $\{H_n\}$. Then we have

$$\delta H_n = \sup_{x, x'\in H_n} \| x - x' \| \leq 2 \sup_{x\in H_n} \| x - x' \| \to 0\,(n \to \infty)$$

$$W^{+}(H_n, \{x^*\}) = \sup_{x \in H_n} \inf_{x_* \in F} \| x - x^* \|$$

$$= \sup_{x \in H_n} \| x - x^* \|$$

$$\leq \| x_n - x^* \| + \sup_{x \in H_n} \| x_n - x \|$$

$$\leq \| x_n - x^* \| + \delta H_n \to 0 \, (n \to \infty),$$

i.e., $\{H_n\} \to \{x^*\}$. This completes the proof.

Remark 2. Theorem 5 and Theorem 8 of [7] are special cases of Theorem 2.1.

1. The Mann iteration method and the Ishikawa iteration method in [7] are replaced by the more general multivalued modified Mann iterative set sequence and multivalued modified Ishikawa iterative set sequence, respectively.

2. A strongly successively pseudocontractive mapping in [7] is extended to an asymptotically Φ-hemicontractive mapping and uniformly Lipschitzian of mapping T in Theorem 5 or uniformly convex of E^* in Theorem 6 is replaced by successively uniformly continuous.

Corollary 2.2. *Let E be a real Banach space and $D(T) = K$ be a nonempty closed convex subset of E. Let $T : K \to 2^K$ be a successively uniformly continuous asymptotically Φ-hemicontractive mapping with $\{k_n\} \subset [1, \infty)$, $\lim_{n \to \infty} k_n = 1$. Further, it is provided that there exist bounded selections $\{u_n\}, \{v_n\}, \{c_n\}$ with $v_n \in T^n x_n$, $u_n \in T^n y_n$, $c_n \in T^n z_n$. If $z_0 = x_0 \in E$, then the following two assertions are equivalent*:

(1) *Multivalued modified Mann iterative set sequence $\{O_n\}$ generated by $\{z_n\}$ converges strongly to $\{x^*\} \subset F(T)$;*

(2) *Multivalued modified Ishikawa iterative set sequence $\{H_n\}$ generated by $\{x_n\}$ converges strongly to $\{x^*\} \subset F(T)$.*

If the mapping T does not satisfy successively uniform continuity, then we have the following result:

Theorem 2.3. *Let E be a real uniformly smooth Banach space and $T : E \to 2^E$ be an asymptotically Φ-hemicontractive mapping with nonempty closed values and $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$. Further, it is provided that there exist bounded selections $\{u_n\}$, $\{v_n\}$, $\{c_n\}$ with $v_n \in T^n x_n$, $u_n \in T^n y_n$, $c_n \in T^n z_n$. If $z_0 = x_0 \in K$, then the following two assertions are equivalent*:

(1) *The multivalued modified Mann iterative set sequence $\{O_n\}$ generated by $\{z_n\}$ converges strongly to $\{x^*\} \subset F(T)$;*

(2) *The multivalued modified Ishikawa iterative set sequence $\{H_n\}$ generated by $\{x_n\}$ converges strongly to $\{x^*\} \subset F(T)$.*

Proof. Using (4), (5) and Lemma 1.6, since T is asymptotically Φ-hemicontractive, we obtain

$$\begin{aligned}
\| x_{n+1} - z_{n+1} \|^2 &= \| (1 - \alpha_n)(x_n - z_n) + \alpha_n(u_n - c_n) \|^2 \\
&\leq (1 - \alpha_n)^2 \| x_n - z_n \|^2 + 2\alpha_n \langle u_n - c_n, j(x_{n+1} - z_{n+1}) \rangle \\
&= (1 - \alpha_n)^2 \| x_n - z_n \|^2 + 2\alpha_n \langle u_n - c_n, j(y_n - z_n) \rangle \\
&\quad + 2\alpha_n \langle u_n - c_n, j(x_{n+1} - z_{n+1}) - j(y_n - z_n) \rangle \\
&\leq (1 - \alpha_n)^2 \| x_n - z_n \|^2 + 2\alpha_n \{ k_n \| y_n - z_n \|^2 \\
&\quad - \Phi(\| y_n - z_n \|) \| y_n - z_n \| \} \\
&\quad + 2\alpha_n \| u_n - c_n \| \| j(x_{n+1} - z_{n+1}) - j(y_n - z_n) \|. \qquad (21)
\end{aligned}$$

Let M_1, M be as in Theorem 2.1. Then

$$\| u_n - c_n \| \leq M_1$$

and we have

$$\begin{aligned}\|(x_{n+1}-z_{n+1})-(y_n-z_n)\| &= \|(x_{n+1}-y_n)-(z_{n+1}-z_n)\| \\ &= \|-\alpha_n x_n+\beta_n x_n+\alpha_n u_n-\beta_n v_n-\alpha_n z_n+\alpha_n c_n\| \\ &\le \alpha_n(\|x_n\|+\|u_n\|+\|z_n\|+\|c_n\|) \\ &\quad+\beta_n(\|x_n\|+\|v_n\|) \\ &\le (\alpha_n+\beta_n)D \to 0\ (n\to\infty),\end{aligned}$$

where $D=\max_{n\ge 0}\{(\|x_n\|+\|u_n\|+\|z_n\|+\|c_n\|), (\|x_n\|+\|v_n\|)\}<\infty$. Since E is a real uniformly smooth Banach space, j is a single map and uniformly continuous on every bounded set. So we have

$$j(x_{n+1}-z_{n+1})-j(y_n-z_n)\to 0 \text{ as } n\to\infty.$$

Denote

$$\sigma_n=2\alpha_n M_1\|j(x_{n+1}-z_{n+1})-j(y_n-z_n)\|. \tag{22}$$

Again, using (4) and Lemma 1.6, we get

$$\begin{aligned}\|y_n-z_n\|^2 &= \|(1-\beta_n)(x_n-z_n)+\beta_n(v_n-z_n)\|^2 \\ &\le (1-\beta_n)^2\|x_n-z_n\|^2+2\beta_n\langle v_n-z_n, j(y_n-z_n)\rangle.\end{aligned} \tag{23}$$

For the second term of (23), we have

$$\begin{aligned}2\beta_n\langle v_n-z_n, j(y_n-z_n)\rangle &\le 2\beta_n\|v_n-z_n\|\|y_n-z_n\| \\ &\le 2\beta_n[\|v_n-x^*\|+\|z_n-x^*\|][\|y_n-x^*\|+\|z_n-x^*\|] \\ &\le 8\beta_n M^2.\end{aligned} \tag{24}$$

Substituting (24) into (23), we get

$$\|y_n-z_n\|^2\le\|x_n-z_n\|^2+8\beta_n M^2. \tag{25}$$

Substituting (25) into (21), we obtain

$$\| x_{n+1} - z_{n+1} \|^2 \le [1 + 2\alpha_n (k_n - 1) + \alpha_n^2] \| x_n - z_n \|^2$$

$$- 2\alpha_n \Phi(\| y_n - z_n \|) \| y_n - z_n \| + \sigma_n + 16\alpha_n \beta_n k_n M^2$$

$$\le \| x_n - z_n \|^2 - 2\alpha_n \Phi(\| y_n - z_n \|) \| y_n - z_n \| + \alpha_n \theta_n, \quad (26)$$

where $\theta_n = [2(k_n - 1) + \alpha_n + 16\beta_n k_n] M^2 + 2M_1 \| j(x_{n+1} - z_{n+1}) - j(y_n - z_n) \| \to 0 \, (n \to \infty)$. Suppose $\inf_n \| y_n - z_n \| = \delta > 0$, then for $\forall n \ge 0$, $\| y_n - z_n \| \ge \delta$, $\Phi(\| y_n - z_n \|) \| y_n - z_n \| \ge \Phi(\delta)\delta > 0$. From $\theta_n \to 0 \, (n \to \infty)$, we have $\exists n_1 > 0$ such that $\theta_n \le \Phi(\delta)\delta$ for $n \ge n_1$. Using (26), we get

$$\| x_{n+1} - z_{n+1} \|^2 \le \| x_n - z_n \|^2 - \alpha_n \Phi(\delta)\delta, \text{ for } n \ge n_1$$

and thus

$$\Phi(\delta)\delta \sum_{n=n_1}^{\infty} \alpha_n \le \sum_{n=n_1}^{\infty} (\| x_n - z_n \|^2 - \| x_{n+1} - z_{n+1} \|^2) < \infty$$

which is a contradiction with $\sum_{n=n_1}^{\infty} \alpha_n = \infty$, i.e., $\delta = 0$. This means that $\exists \{y_{n_j} - z_{n_j}\} \subset \{y_n - z_n\}$ such that $\| y_{n_j} - z_{n_j} \| \to 0 \, (j \to \infty)$. Using (5) we obtain

$$(1 - \beta_{n_j}) \| x_{n_j} - z_{n_j} \| \le \| y_{n_j} - z_{n_j} \| + \beta_{n_j} \| v_{n_j} - z_{n_j} \|$$

$$\le \| y_{n_j} - z_{n_j} \| + 2\beta_{n_j} M.$$

Thus $\exists \{x_{n_j} - z_{n_j}\} \subset \{x_n - z_n\}$ such that $\| x_{n_j} - z_{n_j} \| \to 0 \, (j \to \infty)$. Therefore, $\forall \varepsilon \in (0, 1), \exists N_j > 0$ such that $\| x_{n_j} - z_{n_j} \| < \varepsilon$ for $\forall n \ge N_j \ge n_1$. Moreover,

$$\theta_n \le 2\Phi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2}$$

$$d_n = \alpha_n (\| x_n \| + \| z_n \|) + \alpha_n (\| u_n \| + \| c_n \|) \le \frac{\varepsilon}{4}$$

for $n \geq N_j$. Using (5), we get

$$\| x_n - y_n \| \leq \beta_n \| x_n \| + \beta_n \| v_n \| \leq \frac{\varepsilon}{4}.$$

In the next step we shall show that

$$\| x_{n_j+1} - z_{n_j+1} \| \leq \varepsilon.$$

If it is not the case, then we get $\| x_{n_j+1} - z_{n_j+1} \| > \varepsilon$. Using (5), we have

$$\varepsilon < \| x_{n_j+1} - z_{n_j+1} \| \leq \| y_{n_j} - z_{n_j} \| + \| x_{n_j} - y_{n_j} \| + d_{n_j} \leq \| y_{n_j} - z_{n_j} \| + \frac{\varepsilon}{2}$$

and thus

$$\| y_{n_j} - z_{n_j} \| > \frac{\varepsilon}{2}.$$

Because $h(r)$ is nondecreasing,

$$\Phi(\| y_{n_j} - z_{n_j} \|) \| y_{n_j} - z_{n_j} \| \geq \Phi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2}.$$

Using (26), we get

$$\| x_{n_j+1} - z_{n_j+1} \|^2 \leq \| x_{n_j} - z_{n_j} \|^2 - 2\alpha_{n_j}\Phi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2} + 2\alpha_{n_j}\Phi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2} \leq \varepsilon^2$$

which is a contradiction with $\| x_{n_j+1} - z_{n_j+1} \| > \varepsilon$. Inductively, we get

$$\| x_{n_j+m} - z_{n_j+m} \| \leq \varepsilon, \text{ for } m \geq 1,$$

i.e., $\lim_{n \to \infty} \| x_n - z_n \| = 0$. The rest of proof have analogy with that of Theorem 2.1. This completes the proof.

The results above also hold for Mann and Ishikawa iterative sequences with errors, if such sequences admit appropriate conditions.

References

[1] S. S. Chang, Some problems and results in the study of nonlinear analysis, Nonlinear Anal. TMA 30(7) (1997), 4197-4208.

[2] C. E. Chidume, Convergence theorems for strongly pseudocontractive and strongly accretive maps, J. Math. Anal. Appl. 228 (1998), 254-264.

[3] K. Geobel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972), 171-174.

[4] Q. H. Liu, Convergence theorems of the sequence of iterates for asymptotically Demi-contractive and Hemi-contractive mappings, Nonlinear Anal. 26 (1996), 1825-1842.

[5] S. B. Nadler, Jr., Multivalued contraction mappings, Pacific J. Math. 30 (1969), 475-488.

[6] B. E. Rhoades and S. M. Solutz, On the equivalence of Mann and Ishikawa iteration methods, Internat. J. Math. Sci. 2003 (2003), 451-459.

[7] B. E. Rhoades and S. M. Solutz, The equivalence between convergences of Ishikawa and Mann iterations for an asymptotically pseudocontractive map, J. Math. Anal. Appl. 283 (2003), 681-688.

[8] J. Schu, Iterative construction of fixed point asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158 (1991), 407-413.

[9] X. Weng, Fixed point iteration for local strictly pseudocontractive mapping, Proc. Amer. Math. Soc. 113 (1991), 727-731.

[10] Y. G. Xu, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive operators, J. Math. Anal. Appl. 224 (1998), 91-101.

[11] Liping Yang, The new Ishikawa iteration approximation with errors for asymptotically ϕ-hemicontractive mapping, Numer. Math. J. Chinese Univer. 26(1) (2004), 47-52 (in Chinese).

[12] S. S. Zhang, Iterative approximation problem of fixed points for asymptotically nonexpansive mapping in Banach spaces, Acta Math. Appl. Sinica 24(2) (2001), 236-241 (in Chinese).

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