Far East Journal of Dynamical Systems
Volume 21, Number 1, 2013, Pages 1-32
Published Online: June 2013
Available online at http://pphmj.com/journals/fjds.htm
Published by Pushpa Publishing House, Allahabad, INDIA

DYNAMICS AND STABILITY OF A NEW CLASS OF NONLINEAR INTEGRATED MODELS WITH RESILIENCE MECHANISMS

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Abstract

This paper attempts to develop an analytic tool for exploring whether dynamic processes in different complex adaptive systems share a common evolutionary mechanism over time. A nonlinear autoregressive integrated (NLARI) model is derived from a class of nonlinear systems with resilience mechanisms by using Newton's second law to stochastic systems. Whether NLARI effectively models a class of dynamic processes in complex adaptive systems with common evolutionary mechanism depends on whether it can characterize and predict major dynamics of time series data from these systems. This study finds that a relative restoring coefficient controls bifurcation and stability of NLARI's deterministic system. Unexpectedly, unstable oscillations may be because the relative restoring coefficient is too large or too small due to a weak resilience,

2010 Mathematics Subject Classification: 34D08, 34D20, 39Axx.

Keywords and phrases: time series model, nonlinear dynamics, exact solutions of difference equation, stability, Lyapunov exponents.

Communicated by K. K. Azad

Received February 12, 2013

while a stable fixed point and stable period-2 cycle only occur in an approximate value of the relative restoring coefficient.

1. Introduction

Complex adaptive systems are characterized by a high degree of adaptive capacity, giving them resilience in the face of perturbation. A number of studies focus on agent-based and complex network-based modeling paradigms under the definition that a complex adaptive system is a dynamic network of many agents [1, 2]. Much attention has also been paid to signatures of complex system behavior over time for waves, rhythms, oscillations, 1/f noise (see [3-6]), chaotic resonance [7, 8] and nonlinear dynamics (e.g., [9, 10]). Surprisingly, little research has been devoted to modeling dynamic processes that occur in different complex adaptive systems but could share a common evolutionary origin based on the same physical principle.

Apparent resilience mechanisms in living organisms, as a crucial characteristic of complex adaptive systems, are associated with homeostatic function of the body that reverts the disturbed system back to normal [11]. However, the homeostatic control is challenged by findings on nonstationarity [12] and nonlinearity or perhaps chaos in biological time series like heart rate series [13]. Hence, identifying possible patterns and control mechanisms of a dynamic process in complex adaptive systems should be performed in a wide range of dynamics including stable fixed point (corresponding to biological homeostasis), nonlinear stable and unstable oscillations.

Recently, a new class of nonlinear autoregressive integrated (NLARI) models was derived from economic systems with delayed resilience mechanisms using Newton’s second law to stochastic systems [14]. When the resilience mechanisms lose, NLARI is a nonstationary unit root process (for definition, see Section 2). NLARI has a line time trend component equal to the mean of the process generated by this model when disturbance mean is not zero. The fact that NLARI has a unit root and a trend component explains the entanglement of the two dynamics in macroeconomic time series and thus defuses a long standing debate about the nature of macroeconomic fluctuations (see [15-19]). Simulations study shows that NLARI can reproduce a wide range of dynamics from nonstationary unit roots to periodic and aperiodic oscillations [14], but their control mechanisms are unclear.

This paper considers NLARI to describe the time evolution of a dynamic process in complex adaptive systems. The NLARI model introduced by He [14] has no time delay in a resistance. Herein, NLARI is extended to include a delay in the resistance and it is given by

$$\begin{aligned} Y_t &= \omega + 2Y_{t-1} - Y_{t-2} - \alpha(Y_{t-\kappa_1} - Y_{t-\kappa_1-1}) \\ &\quad - \beta \frac{Y_{t-\kappa_2} - \mu_{t-\kappa_2}}{\exp[(Y_{t-\kappa_2} - \mu_{t-\kappa_2})^2]} + \varepsilon_t, \end{aligned} \tag{1}$$

where $\mu_t = E(Y_t \mid Y_0, Y_{-1}, ..., Y_{1-\kappa_2})$ and it has been proved that

$$\mu_t = Y_0 + \frac{\omega}{\alpha} t$$

(for proof in $\kappa_1 = 0,$ see [14]); $\alpha > 0,$ $\beta > 0,$ and ω are constants, α is the resistance coefficient, β is the restoring coefficient, κ_1 and κ_2 are time delays in the resistance and restoration, respectively; and $\{\varepsilon_t\}$ is a Gaussian white noise process with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = \sigma^2.$

We investigate dynamics and stability of NLARI's deterministic system (1) for lower order delays. Section 2 describes the derivation of NLARI. Section 3 gives two exact solutions of NLARI's deterministic system. Section 4 analyzes stability control mechanism of the solutions. Simulations are conducted to confirm theoretical results and to compare deterministic and stochastic dynamics near critical values in Section 5. A final section briefly discusses and concludes.

2. Derivation of Models

Consider that an object at a position *y* encounters a resistance force *f*, a restoring force g and a random force ε. Newton's second law says that *a net force results in a change of momentum*: $F = d(m\dot{y})/dt$ of which $F = m\ddot{y}$ is just a special case only if mass m is constant. Without loss of generality, let $m = 1$ because the mass has only a scaling effect; thus,

$$F = f + g + \varepsilon = \ddot{y}.$$

Resistance force is regarded as a function of velocity. When the velocity is relatively slow, the resistance can be approximated as $f = -\alpha\dot{y}$ for $\alpha > 0$ like viscous resistance or frictional force by Stokes' law [20]. Restoring force is a variable force that gives rise to an equilibrium in a physical system. In a spring, this restoring force is directly proportional to the distance that the spring is displaced from the equilibrium position [21]. Herein, the restoring force is defined as a function of the deviation from the mean $\mu = E(y)$, denoted by $g(y - \mu)$. The function g should be absolutely integrable on $\mathbb{R}$ to avoid an explosive solution as argued by [22]. We further assume $xg(x) < 0$ for $x = y - \mu \neq 0$ to reflect the nature that a restoring force tends to bring the system back down- or up-toward equilibrium after the system has been up or down perturbed away from the equilibrium, respectively. Here, we adopt the function $g(x) = -\beta x \exp(-x^2)$ for $\beta > 0$, which satisfies these required conditions.

Consider that τ_1 and τ_2 are time delays in the resistance and restoration, respectively. Then the motion equation is specified by

$$\ddot{y}(t) = -\alpha\dot{y}(t - \tau_1) - \beta\frac{y(t - \tau_2) - \mu(t - \tau_2)}{\exp\{[y(t - \tau_2) - \mu(t - \tau_2)]^2\}} + \varepsilon(t). \qquad (2)$$

Let ι be the interval of the time series and $\kappa_{1,2}$ be the integer of $\tau_{1,2}$ with $\tau_{1,2} \geq \iota$. Denote $Y_t = y(t\iota)$, $\varepsilon_t = \varepsilon(t\iota)$, $\mu_t = E(Y_t \mid Y_0, Y_{-1}, ..., Y_{1-\kappa_2})$. When $\iota = 1$, $\ddot{y}(t) \approx Y_t - 2Y_{t-1} + Y_{t-2}$, $\dot{y}(t - \tau_1) \approx Y_{t-\kappa_1} - Y_{t-\kappa_1-1}$, and $y(t - \tau_2) \approx Y_{t-\kappa_2}$. Let $\omega = E(\varepsilon_t)$ and $\varepsilon_t = \varepsilon_t - \omega$. Substituting them into equation (2) yields equation (1).

This paper focuses on the resistance delays of $\kappa_1 = 0$ and $\kappa_1 = 1$. Equation (1) can be written as

$$Y_t = \frac{\omega}{1+\alpha} + \frac{2+\alpha}{1+\alpha} Y_{t-1} - \frac{1}{1+\alpha} Y_{t-2} - \frac{\beta}{1+\alpha} \frac{Y_{t-\kappa_2} - \mu_{t-\kappa_2}}{\exp[(Y_{t-\kappa_2} - \mu_{t-\kappa_2})^2]} + \frac{\varepsilon_t}{1+\alpha}$$

for $\kappa_1 = 0$ and

$$Y_t = \omega + (2-\alpha)Y_{t-1} + (\alpha - 1)Y_{t-2} - \beta \frac{Y_{t-\kappa_2} - \mu_{t-\kappa_2}}{\exp[(Y_{t-\kappa_2} - \mu_{t-\kappa_2})^2]} + \varepsilon_t$$

for $\kappa_1 = 1$.

The above two equations have the following compact form:

$$Y_t = \theta_0 + (1+\theta_1)Y_{t-1} - \theta_1 Y_{t-2} + \theta_2 \frac{Y_{t-\kappa_2} - \mu_{t-\kappa_2}}{\exp[(Y_{t-\kappa_2} - \mu_{t-\kappa_2})^2]} + v_t, \tag{3}$$

where

$$\theta_0 = \begin{cases} \dfrac{\omega}{1+\alpha}, \\ \omega, \end{cases} \quad \theta_1 = \begin{cases} \dfrac{1}{1+\alpha}, \\ 1-\alpha, \end{cases} \quad \theta_2 = \begin{cases} -\dfrac{\beta}{1+\alpha}, \\ -\beta, \end{cases} \quad v_t = \begin{cases} \dfrac{\varepsilon_t}{1+\alpha} & \text{if } \kappa_1 = 0, \\ \varepsilon_t & \text{if } \kappa_1 = 1. \end{cases}$$

We show that NLARI has a time trend component and a unit root component. Using a similar method to $\kappa_1 = 0$ in [14], we can prove that for $\kappa_1 = 1$,

$$\mu_t = Y_0 + \frac{\omega}{\alpha} t$$

and the expansion of equation (3)

$$Y_t = Y_0 + \frac{\omega}{\alpha} t + \sum_{i=1}^{t} \sum_{j=0}^{i-1} \theta_1^j v_{i-j}$$

$$+ \theta_2 \sum_{i=1}^{t} \sum_{j=0}^{i-1} \theta_1^j \frac{Y_{i-j-\kappa_2} - Y_0 - \frac{\omega}{\alpha}(i - j - \kappa_2)}{\exp\{[Y_{i-j-\kappa_2} - Y_0 - \frac{\omega}{\alpha}(i-j-\kappa_2)]^2\}}. \tag{4}$$

When $\beta = 0,$ equation (3) is the linear autoregressive model

$$Y_t = \theta_0 + (1+\theta_1)Y_{t-1} - \theta_1 Y_{t-2} + v_t,$$

whose characteristic equation $\lambda^2 - (1-\theta_1)\lambda - \theta_1 = 0$ has a unit root $\lambda_1 = 1$ and a root satisfying $|\lambda_2| < 1$ for $\alpha < 2,$ thus, it is a unit root process or integrated process in economics. Hence, equation (1) is called the *nonlinear autoregressive integrated model*. A unit root process is nonstationary because its standard deviation changes over time.

Equation (3) has a more compact structure by removing the process mean μ_t (equal to a trend if $\omega \neq 0$). Substituting $\mu_t = Y_0 + (\omega/\alpha)t$ into equation (3) yields

$$\begin{aligned} Y_t - \mu_t &= \theta_0 + (1+\theta_1)Y_{t-1} - \theta_1 Y_{t-2} \\ &\quad + \theta_2 \frac{Y_{t-\kappa_2} - \mu_{t-\kappa_2}}{\exp[(Y_{t-\kappa_2} - \mu_{t-\kappa_2})^2]} + v_t - Y_0 - \frac{\omega}{\alpha}t \\ &= (1+\theta_1)(Y_{t-1} - \mu_{t-1}) - \theta_1(Y_{t-2} - \mu_{t-2}) \\ &\quad + \theta_2 \frac{Y_{t-\kappa_2} - \mu_{t-\kappa_2}}{\exp[(Y_{t-\kappa_2} - \mu_{t-\kappa_2})^2]} + v_t \end{aligned}$$

provided by

$$\theta_0 - Y_0 - \frac{\omega}{\alpha}t + (1+\theta_1)\mu_{t-1} - \theta_1\mu_{t-2} = 0.$$

Letting $X_t = Y_t - \mu_t,$ equation (3) can be rewritten as

$$X_t = (1+\theta_1)X_{t-1} - \theta_1 X_{t-2} + \theta_2 \frac{X_{t-\kappa_2}}{\exp(X_{t-\kappa_2}^2)} + v_t. \tag{5}$$

In what follows, we discuss the segmented trend NLARI model (5).

3. Solutions of Deterministic System

When the standard deviation of disturbance v_t or ε_t is small, equation

(5) approximates to the following deterministic system:

$$x_t = (1+\theta_1)x_{t-1} - \theta_1 x_{t-2} + \theta_2 \frac{x_{t-\kappa_2}}{\exp(x_{t-\kappa_2}^2)}, \tag{6}$$

where

$$\theta_1 = \begin{cases} \dfrac{1}{1+\alpha}, \\ 1-\alpha, \end{cases} \quad \theta_2 = \begin{cases} -\dfrac{\beta}{1+\alpha}, & \text{if } \kappa_1 = 0, \\ -\beta, & \text{if } \kappa_1 = 1. \end{cases}$$

Next, we give analytical solutions of deterministic system (6).

Proposition 1. *Let*

$$\gamma = \begin{cases} \dfrac{\beta}{4+2\alpha} & \textit{if } \kappa_1 = 0, \\ \dfrac{\beta}{4-2\alpha} & \textit{if } \kappa_1 = 1, \ \alpha < 2. \end{cases} \tag{7}$$

(i) *For any order delay* κ_2, *equation* (6) *has a unique fixed point solution* $x_{1t}^* = 0$ *when* $\gamma > 0$.

(ii) *For any odd-order delay* κ_2, *equation* (6) *has a period*-2 *cycle solution* $x_{2t}^* = (-1)^t \sqrt{\ln \gamma}$ *and the periodic solution is unique when* $1 < \gamma < \sqrt{e}$.

(iii) *For any even-order delay* κ_2, *equation* (6) *has no nontrivial period*-2 *cycle solution*.

Proof. For $\kappa_1 = 0$, we easily confirm that $x_{1t}^* = 0$ is a fixed point solution of equation (6). If there is another fixed point solution $x_t = x_1$ for all $t \in \mathbb{Z}^+$, then

$$x_1 = (1+\theta_1)x_1 - \theta_1 x_1 + \theta_2 \frac{x_1}{\exp(x_1^2)}$$

which leads to $\theta_2 x_1 \exp(-x_1^2) = 0$. Hence $x_1 = 0$ by $\beta > 0$. This implies that the system has a unique fixed point $x_{1t}^* = 0$.

When $\kappa_2 = 2s - 1$ for $s \in \mathbb{Z}^+$, substituting $x_{2t}^* = (-1)^t \sqrt{\ln \gamma}$ into the right-hand side of equation (6) yields

$$
\begin{aligned}
&(1+\theta_t)x_{t-1} - \theta_1 x_{t-2} + \theta_2 \frac{x_{t-2s+1}}{\exp(x_{t-2s+1}^2)} \\
&= (-1)^t \sqrt{\ln \gamma}\,[-(1+\theta_1) - \theta_1 + 2(1+\theta_1)] \\
&= (-1)^t \sqrt{\ln \gamma},
\end{aligned}
$$

where the second line is given from $\theta_2 = -2\gamma(1+\theta_1)$. Therefore, $x_{2t}^* = (-1)^t \sqrt{\ln \gamma}$ is a period-2 cycle solution for equation (6) when $\gamma > 1$. If there is another nontrivial period-2 cycle solution, then $x_t = x_{t-2} = x_{t-2s} = x_1$ and $x_{t+1} = x_{t-1} = x_{t-2s+1} = x_2 \neq x_1$. Note that $\theta_1 > 0$ for $\kappa_1 = 0$ and $\theta_1 > -1$ for $\kappa_1 = 1$. Using equation (6) leads to $x_1 = x_2 - 2\gamma x_2 \exp(-x_2^2)$ and $x_2 = x_1 - 2\gamma x_1 \exp(-x_1^2)$ provided by $\theta_2 = -2\gamma(1+\theta_1) \neq 0$. The two equations imply that

$$
e^{x_1^2} + e^{x_1^2(1-2\gamma e^{-x_1^2})^2} = 2\gamma.
$$

Set $y = x_1^2$. Consider the function of (γ, y) on $\mathbb{R}^2$ as

$$
F(\gamma, y) = e^y + e^{y(1-2\gamma e^{-y})^2} - 2\gamma.
$$

The partial derivatives

$$
F_y(\gamma, y) = e^y + e^{y(1-2\gamma e^{-y})^2}(1 - 2\gamma e^{-y})(1 - 2\gamma e^{-y} + 4\gamma y e^{-y})
$$

are continuous in the region of its definition. If

$$
F(\gamma, y) = e^y + e^{y(1-2\gamma e^{-y})^2} - 2\gamma = 0,
$$

then

$$
\begin{aligned}
F_y(\gamma, y) &= e^y + (2\gamma - e^y)(1 - 2\gamma e^{-y})(1 - 2\gamma e^{-y} + 4\gamma y e^{-y}) \\
&= e^y - e^{-2y}(e^y - 2\gamma)^2(e^y - 2\gamma + 4\gamma y).
\end{aligned}
$$

Let us now introduce the function

$$h(y) = \frac{e^y}{2(1-2y)}.$$

We have

$$\dot{h}(y) = \frac{3-2y}{2(1-2y)^2}e^y$$

$$= \frac{e^y}{2(1-2y)} + \frac{e^y}{(1-2y)^2}.$$

Letting $\dot{h}(y_0) = 0$, we have $y_0 = 3/2$. Then

$$\ddot{h}\left(\frac{3}{2}\right) = \frac{e^y}{2(1-2y)} + 2\frac{e^y}{(1-2y)^2} + 4\frac{e^y}{(1-2y)^3}\Big|_{y_0=3/2} = -\frac{e\sqrt{e}}{4} < 0.$$

It follows that

$$\frac{e^y}{2(1-2y)} \le h_{\max}(y) = h\left(\frac{3}{2}\right) = -\frac{e\sqrt{e}}{4} < \gamma.$$

For $\gamma < 1/2$, we have

$$e^y - 2\gamma + 4\gamma y < 0$$

which leads to $F_y(\gamma, y) > 0$. Note that $y = \ln\gamma$ is a solution of $F(\gamma, y) = 0$. According to the implicit function theorem, there exist open neighborhoods $A_0 \subset (1, \sqrt{e})$ of γ_0 and $B_0 \subset (0, 1/2)$ of y_0 such that, for all $y \in B_0$, the equation $F(\gamma, y) = 0$ has a unique solution $y = \ln\gamma \in B_0$. Since $y = x_1^2$, thus, $x_1 = (-1)^t\sqrt{\ln\gamma}$ is the unique period-2 cycle solution for equation (6) when $\gamma \in (1, \sqrt{e})$.

When $\kappa_2 = 2s$, if equation (6) has a nontrivial period-2 cycle solution, then

$$x_1 = x_2 + 2\gamma\frac{x_2}{\exp(x_2^2)},$$

$$x_2 = x_1 + 2\gamma \frac{x_1}{\exp(x_1^2)},$$

which result in

$$\frac{x_1}{\exp(x_1^2)} + \frac{x_2}{\exp(x_2^2)} = 0 \tag{8}$$

for $\gamma > 0$. If $x_1 = 0$, then equation (8) implies that $x_2 = 0$ so that $x_1 = x_2 = 0$, which is inconsistent with the assumption of another nontrivial period-2 cycle solution. Therefore, $x_1 \neq 0$. Substituting $x_2 = x_1 + 2\gamma x_1 \exp(-x_1^2)$ into the numerator x_2 of the second term of equation (8) yields

$$\exp(x_1^2) + \exp(x_2^2) = -2\gamma < 0$$

because $x_1 \neq 0$ and $\gamma > 0$. This implies that the assumption of the existence of a nontrivial period-2 cycle solution is not true for an even number κ_2. Similarly, we can verify that the result holds for $\kappa_1 = 1$. □

Using Proposition 1 immediately leads to the following proposition.

Proposition 2. *The deterministic system of NLARI* (3) *has a unique trend solution* $y_{1t}^* = Y_0 + (\omega/\alpha)t$ *for any order delay* κ_2 *and a solution* $y_{2t}^* = Y_0 + (\omega/\alpha)t + (-1)^t\sqrt{\ln\gamma}$ *for any odd-order delay* κ_2 *and* $\gamma \in (1, \sqrt{e})$, *but no nontrivial period*-2 *cycle solution for any even-order delay* κ_2 *after detrending* $Y_0 + (\omega/\alpha)t$, *where* γ *is defined in* (7).

4. Dynamic Analysis

We turn now to the stability of two solutions of equation (6). Note that $\theta_2 = -2(1-\theta_1)\gamma$ for γ defined in (7). Let

$$\varphi(Z_{t-1}) = (1+\theta_1)X_{t-1} - \theta_1 X_{t-2} - 2(1+\theta_1)\gamma \frac{X_{t-\kappa_2}}{\exp(X_{t-\kappa_2}^2)}.$$

Denote

$$Z_t = \begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix}, \quad \Phi(Z_{t-1}) = \begin{pmatrix} \varphi(Z_{t-1}) \\ X_{t-1} \end{pmatrix}.$$

Then equation (5) can be written as

$$Z_t = \Phi(Z_{t-1}) + u_t, \tag{9}$$

where

$$\theta_1 = \begin{cases} \dfrac{1}{1+\alpha}, \\ 1-\alpha, \end{cases} \quad \gamma = \begin{cases} \dfrac{\beta}{4+2\alpha}, \\ \dfrac{\beta}{4-2\alpha}, \end{cases} \quad u_t = \begin{cases} \dfrac{1}{1+\alpha}(\varepsilon_t, 0)' & \text{if } \kappa_1 = 0, \\ (\varepsilon_t, 0)' & \text{if } \kappa_1 = 1. \end{cases}$$

The deterministic system of equation (9) is given as

$$z_t = \Phi(z_{t-1}), \tag{10}$$

where $z_t = (x_t, x_{t-1})'$.

Proposition 3. *For equation* (10) *with* $\kappa_2 = 1$ *and* γ *defined in* (7), *the following results hold*:

(i) *A unique null-fixed point undergoes a transition from stable state when* $0 < \gamma < 1$ *to unstable state when* $\gamma > 1$.

(ii) *A branch of additional period*-2 *cycle* $(-1)^t \sqrt{\ln \gamma}$ *emerges when* $\gamma > 1$ *and the period*-2 *cycle is unique and stable when* $1 < \gamma < \sqrt{e}$.

(iii) *The period*-2 *cycle loses stability when* $\gamma > \sqrt{e}$.

Proof. Consider $z_t = (x_t, x_{t-1})'$ as a disturbance to the solution $z_{jt}^* = (x_{jt}^*, x_{j(t-1)}^*)'$ $(j = 1, 2)$ for the deterministic system (10) and initiated at a small disturbance $z_0 = (x_0, x_{-1})'$. We have the approximate expression $z_t = J_j z_{t-1}$ for small disturbances where the Jacobian matrix

$$J_j = \begin{bmatrix} (1+\theta_1) - 2(1+\theta_1)\gamma \dfrac{1 - 2x_j^{*2}(t-1)}{\exp(x_j^{*2}(t-1))} & -\theta_1 \\ 1 & 0 \end{bmatrix}$$

for $j = 1, 2$. The characteristic equations are given by

$$\lambda^2 - (1+\theta_1)(1-2\gamma)\lambda + \theta_1 = 0 \tag{11}$$

for $x_{1t}^* = 0$ and

$$\lambda^2 - (1+\theta_1)(4\ln\gamma - 1)\lambda + \theta_1 = 0 \tag{12}$$

for $x_{2t}^* = (-1)^t\sqrt{\ln\gamma}$. Due to $|\theta_1| < 1$, we can prove that the eigenvalues λ_{1j} $(j = 1, 2)$ of equation (11) satisfy

$$\begin{cases} |\lambda_{11}| < 1, |\lambda_{12}| < 1 \text{ if } \gamma < 1 \\ |\lambda_{11}| < 1, \lambda_{12} = -1 \text{ if } \gamma = 1 \\ |\lambda_{11}| < 1, \lambda_{12} < -1 \text{ if } \gamma > 1 \end{cases} \tag{13}$$

for $x_{1t}^* = 0$ (for the proof, see Appendix A) and that the eigenvalues λ_{2j} $(j = 1, 2)$ of equation (12) satisfy

$$\begin{cases} |\lambda_{21}| < 1, |\lambda_{22}| < 1 \text{ if } \gamma < \sqrt{e} \\ \lambda_{21} = 1, \quad |\lambda_{22}| < 1 \text{ if } \gamma = \sqrt{e} \\ \lambda_{21} > 1, \quad |\lambda_{12}| < 1 \text{ if } \gamma > \sqrt{e} \end{cases} \tag{14}$$

for $x_{2t}^* = (-1)^t\sqrt{\ln\gamma}$ (for the proof, see Appendix B).

Without losing generality, we let z_t represent the eigenvector of J_j. Then $z_t = J_j z_{t-1} = \lambda_{ij} z_{t-1}$ which results in

$$\frac{\| z_t \|}{\| z_0 \|} = \frac{\| z_t \|}{\| z_{t-1} \|}\frac{\| z_{t-1} \|}{\| z_{t-2} \|} \cdots \frac{\| z_1 \|}{\| z_0 \|} = |\lambda_{ij}|^t$$

for $i, j = 1, 2$. Therefore, the Lyapunov exponent is given by

$$LE_{ij} = \frac{1}{t}\ln\left(\frac{\| z_t \|}{\| z_0 \|}\right) = \ln|\lambda_{ij}|$$

for $i, j = 1, 2$. Using equations (13) and (14) yields

$$LE_{11}\begin{cases} < 0 \text{ if } \gamma < 1, \\ < 0 \text{ if } \gamma = 1, \\ < 0 \text{ if } \gamma > 1, \end{cases} \quad LE_{12}\begin{cases} < 0 \text{ if } \gamma < 1 \\ = 0 \text{ if } \gamma = 1 \\ > 0 \text{ if } \gamma > 1 \end{cases}$$

for $x_{1t}^* = 0$ and

$$LE_{21}\begin{cases} < 0 \text{ if } \gamma < \sqrt{e}, \\ = 0 \text{ if } \gamma = \sqrt{e}, \\ > 0 \text{ if } \gamma > \sqrt{e}, \end{cases} \quad LE_{22}\begin{cases} < 0 \text{ if } \gamma < \sqrt{e} \\ < 0 \text{ if } \gamma = \sqrt{e} \\ < 0 \text{ if } \gamma > \sqrt{e} \end{cases}$$

for $x_{2t}^* = (-1)^t\sqrt{\ln\gamma}$. Hence, the deterministic system (10) or (6) gives rise to a stable fixed point when $0 < \gamma < 1$, an unstable fixed point when $\gamma > 1$, a unique stable period-2 cycle when $1 < \gamma < \sqrt{e}$, and an unstable period-2 cycle when $\gamma > \sqrt{e}$ [23]. We have thus proved Proposition 3. □

Proposition 4. *For equation* (6) *with* $(\kappa_1, \kappa_2) = (0, 2)$, *the unique fixed point* $x_t^* = 0$ *undergoes a transition from a stable state when* $0 < \gamma < 1$ *to an unstable state when* $\gamma > 1$ *where* $\gamma = \beta/\alpha$.

Proof. Let z_t be a disturbance to the solution $z^* = (0, 0)'$. For small disturbances initiated at $z_0 = (x_0, x_{-1})'$, we have the expression $z_t = Jz_{t-1}$ with the Jacobian matrix J at the origin

$$J = \begin{bmatrix} 1 + \theta_1 & -\theta_1 + \theta_2 \\ 1 & 0 \end{bmatrix}$$

whose characteristic equation is given by

$$\lambda^2 - (1 + \theta_1)\lambda + \theta_1 - \theta_2 = 0. \tag{15}$$

We can prove that the eigenvalues λ_{1j} of equation (15) satisfy

$$\begin{cases} |\lambda_{1j}| < 1 \text{ if } \beta < \alpha \\ |\lambda_{1j}| = 1 \text{ if } \beta = \alpha \\ |\lambda_{1j}| > 1 \text{ if } \beta > \alpha \end{cases} \tag{16}$$

for $j = 1, 2$ (for the proof, see Appendix C). Similar to the proof of Proposition 3, we can obtain that

$$LE_1 = LE_2 \begin{cases} < 0 \text{ if } \beta < \alpha \\ = 0 \text{ if } \beta = \alpha \\ > 0 \text{ if } \beta > \alpha \end{cases}$$

for $x_t^* = 0$. Therefore, the deterministic system (10) or (6) gives rise to a stable fixed point for $\beta < \alpha$ and an unstable fixed point if $\beta > \alpha$, which completes the proof. □

Proposition 5. *For equation* (6) *with* $(\kappa_1, \kappa_2) = (1, 2)$,

(I) *If* $\beta > \alpha^2/4$, *then we have the following*:

(i) *The unique fixed point* $x_t^* = 0$ *is stable when* $0 < \gamma < 1$.

(ii) *The unique fixed point* $x_t^* = 0$ *loses stability when* $\gamma > 1$, *where* $\gamma = \beta/\alpha$.

(II) *If* $\beta \le \alpha^2/4$, *then we have the following*:

(i) *When* $\alpha < 2$, *the unique fixed point* $x_t^* = 0$ *is stable*.

(ii) *When* $\alpha > 4$, *the unique fixed point* $x_t^* = 0$ *is unstable*.

(iii) *When* $2 < \alpha < 4$, *the unique fixed point* $x_t^* = 0$ *undergoes a transition from an unstable state when* $0 < \gamma < 1$ *to a stable state when* $\gamma > 1$, *where* $\gamma = \beta/(2\alpha - 4)$.

Proof. Let z_t be a disturbance to the solution $z^* = (0, 0)$. For small disturbances initiated at $z_0 = (x_0, x_{-1})'$, we have the approximate expression $z_t = Jz_{t-1}$ with the Jacobian matrix J at the origin

$$J = \begin{bmatrix} 2-\alpha & \alpha-\beta-1 \\ 1 & 0 \end{bmatrix}$$

and the characteristic equation

$$\lambda^2 - (2-\alpha)\lambda - (\alpha-\beta-1) = 0. \tag{17}$$

When $\beta > \alpha^2/4$, we can prove that the eigenvalues of equation (17) satisfy

$$\begin{cases} |\lambda_{1j}| < 1 \text{ if } \gamma < 1, \\ |\lambda_{1j}| = 1 \text{ if } \gamma = 1, \\ |\lambda_{1j}| > 1 \text{ if } \gamma > 1, \end{cases} \tag{18}$$

where $\gamma = \beta/\alpha$; when $\beta \leq \alpha^2/4$, the eigenvalues of equation (17) satisfy

$$|\lambda_{1j}| < 1 \tag{19}$$

for $0 < \alpha < 2$,

$$\begin{cases} |\lambda_{11}| < 1, |\lambda_{12}| < 1 \text{ if } \gamma < 1 \\ |\lambda_{11}| < 1, \lambda_{12} = -1 \text{ if } \gamma = 1 \\ |\lambda_{11}| < 1, \lambda_{12} < -1 \text{ if } \gamma > 1 \end{cases} \tag{20}$$

for $2 < \alpha < 4$, and

$$\begin{cases} \lambda_{11} < -1, \lambda_{12} < -1 \text{ if } \gamma < 1 \\ \lambda_{11} = -1, \lambda_{12} < -1 \text{ if } \gamma = 1 \\ |\lambda_{11}| < 1, \lambda_{12} < -1 \text{ if } \gamma > 1 \end{cases} \tag{21}$$

for $\alpha > 4$, where $\gamma = \beta/(2\alpha - 4)$ (for the proofs of (18) to (21), see Appendix D).

Set $LE_1 = \ln|\lambda_{11}|$ and $LE_2 = \ln|\lambda_{12}|$. Using equation (18) yields

$$LE_1 = LE_2 \begin{cases} < 0 \text{ if } \gamma < 1, \\ = 0 \text{ if } \gamma = 1, \\ > 0 \text{ if } \gamma > 1. \end{cases} \tag{22}$$

From equations (19)-(21), we have the following results:

$$LE_1 = LE_2 < 0 \tag{23}$$

for $0 < \alpha < 2$,

$$LE_1 \begin{cases} < 0 \text{ if } \gamma < 1, \\ < 0 \text{ if } \gamma = 1, \\ < 0 \text{ if } \gamma > 1, \end{cases} \quad LE_2 \begin{cases} < 0 \text{ if } \gamma < 1 \\ = 0 \text{ if } \gamma = 1 \\ > 0 \text{ if } \gamma > 1 \end{cases} \tag{24}$$

for $2 < \alpha < 4$, and

$$LE_1 \begin{cases} > 0 \text{ if } \gamma < 1, \\ = 0 \text{ if } \gamma = 1, \\ < 0 \text{ if } \gamma > 1, \end{cases} \quad LE_2 \begin{cases} > 0 \text{ if } \gamma < 1 \\ > 0 \text{ if } \gamma = 1 \\ > 0 \text{ if } \gamma > 1 \end{cases} \tag{25}$$

for $\alpha > 4$. Proposition 5 immediately follows from equations (22)-(25). □

Propositions 3 to 5 show that the parameter γ controls the bifurcation and stability of NLARI's deterministic system. We give the following:

Definition 1. The control parameter

$$\gamma = \begin{cases} \dfrac{\beta}{4 + 2\alpha} & \text{if } (\kappa_1, \kappa_2) = (0, 1) \\ \dfrac{\beta}{\alpha} & \text{if } (\kappa_1, \kappa_2) = (0, 2) \\ \dfrac{\beta}{4 - 2\alpha} & \text{if } (\kappa_1, \kappa_2) = (1, 1),\ \alpha < 2 \\ \dfrac{\beta}{\alpha} & \text{if } (\kappa_1, \kappa_2) = (1, 2),\ \beta > \dfrac{\alpha^2}{4} \\ \dfrac{\beta}{2\alpha - 4} & \text{if } (\kappa_1, \kappa_2) = (1, 2),\ \alpha > 2,\ \beta \le \dfrac{\alpha^2}{4} \end{cases} \tag{26}$$

is called the *relative restoring coefficient*.

5. Simulation Study

Simulations are carried out to confirm whether the relative restoring coefficient (γ) controls the dynamics and stability of NLARI's deterministic system (6) and to compare deterministic and stochastic dynamics near the γ critical values. Equation (5) with $\sigma = 0$ plots the trajectories of deterministic dynamics over time, while the corresponding stochastic dynamics are stochastic realizations of equation (5) with $\sigma = 0.2$. In each case, $\kappa_1 = 0$, $\kappa_2 = 1$, $\alpha = 7/3$, the initial values are $X_{-1} = -0.1$ and $X_0 = 0.15$.

5.1. A fixed point gains stability

Figure 1(a) shows that after disturbed by the non-null initial values, the deterministic dynamics at the critical value $\gamma = 0$ converges to a non-null constant, while all the deterministic dynamics near the critical value $\gamma = 0.01$ to 0.1 revert to zero. From Figure 1(b), we see that the corresponding stochastic dynamics are differentiated into two completely different types: deviation far from the null mean for $\gamma \leq 0.05$ and fluctuations around the null mean for $\gamma > 0.05$. Consequently, simulation study confirms the theoretical result that NLARI's deterministic system represents an unstable fixed point corresponding to a nonstationary unit root process in stochastic systems when $\gamma = 0$, but the fixed point gains stability when $\gamma > 0$.

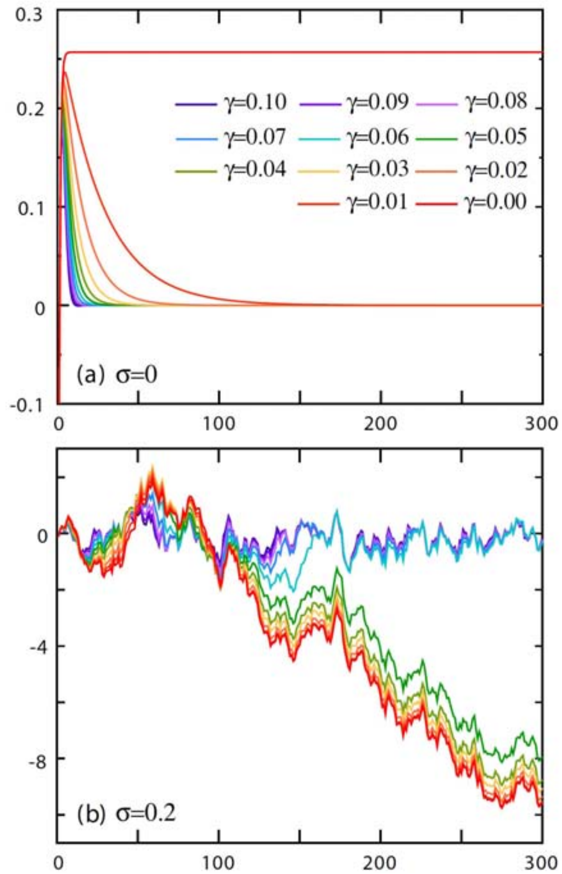

Figure 1. NLARI's deterministic $(\sigma = 0)$ and stochastic dynamics $(\sigma = 0.2)$ near the critical value $\gamma = 0$. NLARI is a nonstationary unit root process when $\gamma = 0$ and a unique fixed point of NLARI's deterministic system gains stability when $\gamma > 0$, where σ is the standard deviation of disturbances and γ is the relative restoring coefficient.

5.2. A branch of period cycle from a fixed point

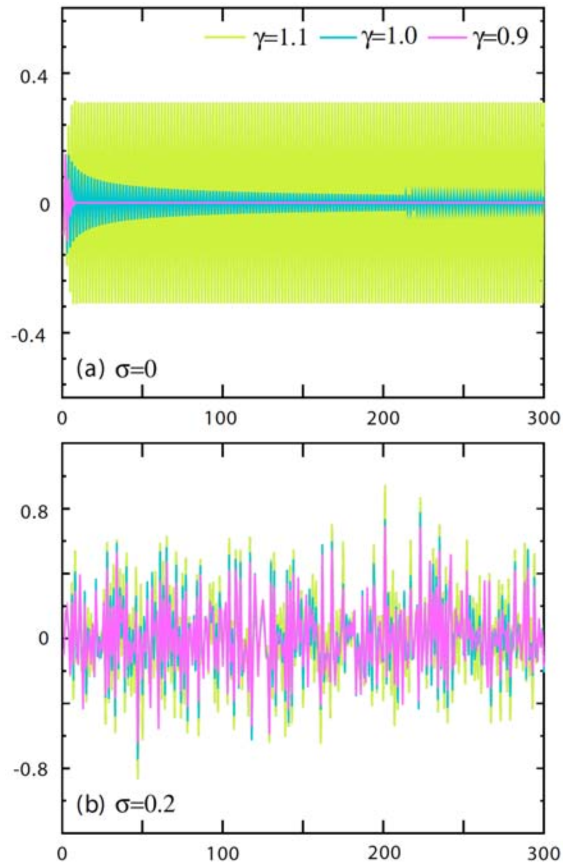

Figure 2. NLARI's deterministic $(\sigma = 0)$ and stochastic dynamics $(\sigma = 0.2)$ near the bifurcation $\gamma = 1.$ A unique fixed point of NLARI's deterministic system loses stability and a branch of an additional unique stable period-2 cycle $(-1)^t \sqrt{\ln \gamma}$ emerges when $\gamma > 1,$ where σ is the standard deviation of disturbances and γ is the relative restoring coefficient.

Figure 2(a) indicates that NLARI's deterministic dynamics near the critical bifurcation value $\gamma = 1 - 0.1$ converges to zero, but exhibits periodic oscillations at the critical bifurcation value $\gamma = 1$ and a regular period-2 cycle with ± 0.31 amplitudes near the critical bifurcation value $\gamma = 1 + 0.1.$

Note that $(-1)^t\sqrt{\ln 1.1} \approx \pm 0.31$. In contrast, Figure 2(b) shows that the corresponding stochastic system produces sudden erratic bursts at and near the critical bifurcation value $\gamma = 1 \pm 0.1$. These results confirm the theoretical result that a unique period-2 cycle $(-1)^t\sqrt{\ln\gamma}$ bifurcates from a unique stable fixed point when $\gamma > 1$.

5.3. Period-2 cycle loses stability

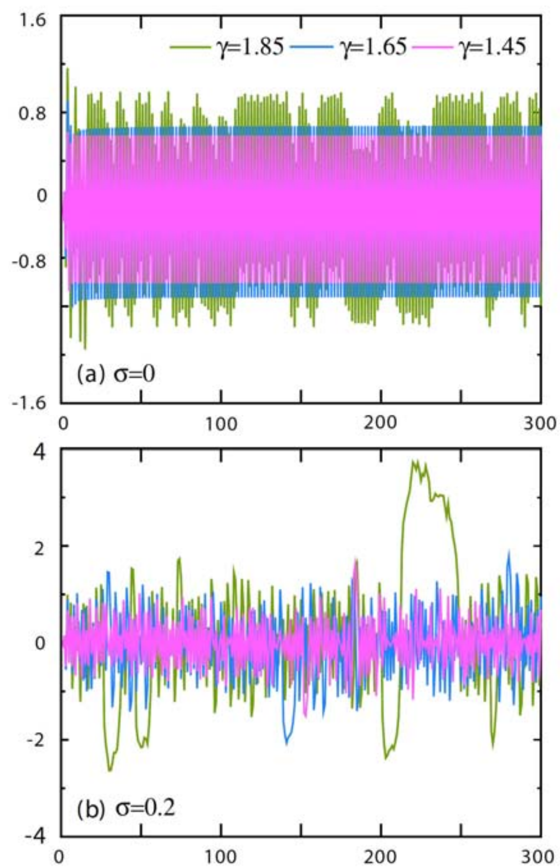

Figure 3. NLARI's deterministic (σ = 0) and stochastic dynamics (σ = 0.2) near the critical value $\gamma = \sqrt{e}$. A unique period-2 cycle $(-1)^t\sqrt{\ln\gamma}$ of NLARI's deterministic system loses stability when $\gamma > \sqrt{e}$, where σ is the standard deviation of disturbances and γ is the relative restoring coefficient.

From Figure 3(a), we see that NLARI's deterministic dynamics is a regular period-2 cycle with amplitudes $\pm\sqrt{\ln 1.65} \approx \pm 0.71$, $\pm\sqrt{\ln 1.45} \approx \pm 0.61$ at and near the bifurcation value $\gamma = 1.65\ (\approx \sqrt{e})$, $1.65 - 0.2$, respectively, and irregular oscillations when $\gamma = 1.65 + 0.2$. However, Figure 3(b) displays that the corresponding stochastic system exhibits larger irregular fluctuations than the deterministic dynamics at and near the critical bifurcation value $\gamma = 1.65 \pm 0.2$. These results are consistent with the theoretical result that a unique stable period-2 cycle $(-1)^t\sqrt{\ln\gamma}$ loses stability when $\gamma > \sqrt{e}$.

5.4. Unit root dynamics to unstable period-2 cycle

In Figure 4(a), we observe that NLARI's deterministic dynamics reverts to zero inside the stable fixed point range $\gamma = 0.01, 0.5 \in (0, 1)$ but deviates from zero at the critical value $\gamma = 0$ after being disturbed, and moreover, the deterministic system exhibits a regular period-2 cycle with amplitudes $\pm\sqrt{\ln 1.5} \approx \pm 0.64$ inside the stable period cycle range $\gamma = 1.5 \in (1, \sqrt{e})$ and irregular oscillations inside the unstable period cycle range $\gamma = 2.25 \in (\sqrt{e}, +\infty)$. Figure 4(b) shows that the corresponding stochastic dynamics has an obviously larger amplitude when $\gamma = 0, 0.01$ than those when $\gamma = 0.5, 1.5$ in which the dynamics for $\gamma = 0$ as a unit root process is strikingly similar to noise-disturbed unstable period-2 cycle for $\gamma = 2.25$.

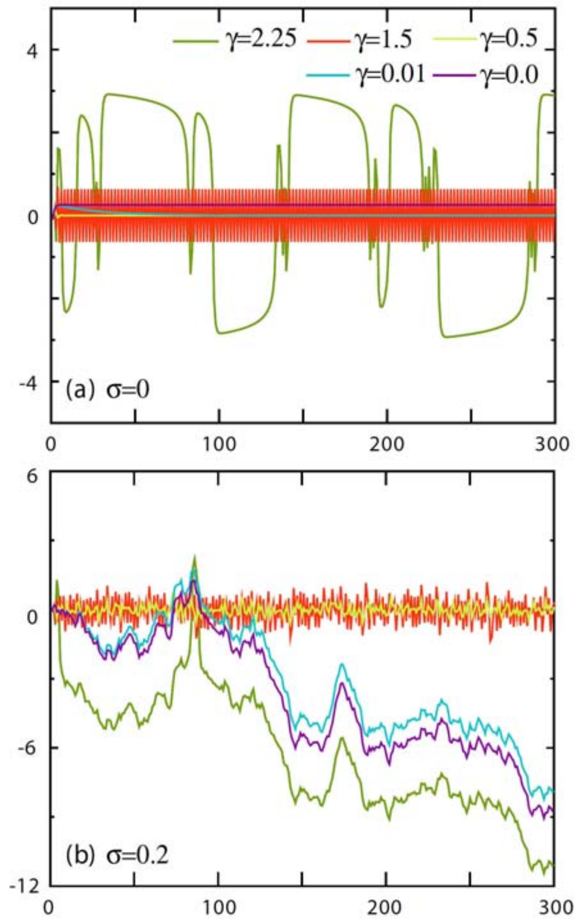

Figure 4. NLARI's deterministic ($\sigma = 0$) and stochastic dynamics ($\sigma = 0.2$). NLARI can capture a unit root process when $\gamma = 0$, a stable fixed point when $0 < \gamma < 1$ ($\gamma = 0.5$), a stable period-2 cycle when $1 < \gamma < \sqrt{e}$ ($\gamma = 1.5$), and an unstable period-2 cycle when $\gamma > \sqrt{e}$ ($\gamma = 2.25$). The noise-disturbed unstable period-2 cycle displays very similar dynamics to a unit root process. The parameter σ is the standard deviation of disturbances and γ is the relative restoring coefficient.

6. Conclusion and Discussions

In this paper, we have demonstrated that the relative restoring coefficient (γ) controls dynamic patterns and stability of NLARI for lower order delays. When $\gamma = 0$, NLARI is a nonstationary unit root process in economics. For the segmented trend NLARI's deterministic system with $\kappa_1 = 0$ or 1 and $\kappa_2 = 1$, a unique fixed point undergoes a transition from stable to unstable dynamics and a branch of an additional unique stable period-2 cycle $(-1)^t \sqrt{\ln \gamma}$ emerges when $\gamma > 1$, but the period cycle loses stability when $\gamma > \sqrt{e}$. Simulation results confirm these theological results and show noise-disturbed deterministic dynamics near the critical values.

There is no universally accepted mathematical definition of chaos. In economics, the behavior of dynamic systems that are highly sensitive to initial conditions and exhibit irregular oscillations is often regarded as chaotic dynamics. In this sense, we say that NLARI's deterministic system *possibly exhibits chaos* when $\gamma > \sqrt{e}$.

It is difficult to derive a control parameter of NLARI in a higher order delay because this problem refers to solving a high order polynomial characteristic equation. In a separated paper, simulation study shows that a higher delay not only affects NLARI's dynamic patterns and stability, but also induces a long memory or long-range dependence. The long-range dependence is associated with fractals which appear in many complex adaptive systems such as heart rate variability.

References

[1] M. M. Waldrop, Complexity: The Emerging Science at the Edge of Order and Chaos, Simon & Schuster Paperbacks, New York, London, Toronto, Sydney, 1992.

[2] V. Gontar, The dynamics of living and thinking systems, biological networks, and the laws of physics, Discrete Dyn. Nat. Soc. 2004(1) (2004), 101-111.

[3] A. L. Goldberger, L. A. N. Amaral, J. M. Hausdorff, P. Ch. Ivanov, C.-K. Peng and H. E. Stanley, Fractal dynamics in physiology: alterations with disease and aging, Proc. Nat. Acad. Sci. (USA) 99 (Suppl 1) (2002), 2466-2472.

[4] J. S. Perkiomaki, T. H. Makikallio and H. V. Huikuri, Fractal and complexity measures of heart rate variability, Clin. Exp. Hypertens. 27 (2005), 149-158.

[5] P. Ch. Ivanov, M. G. Rosenblum, L. A. N. Amaral, Z. Struzik, S. Havlin, A. L. Goldberger and H. E. Stanley, Multifractality in human heartbeat dynamics, Nature 399 (1999), 461-465.

[6] P. Ch. Ivanov, L. A. N. Amaral, A. L. Goldberger, S. Havlin, M. G. Rosenblum, H. E. Stanley and Z. Struzik, From 1/f noise to multifractal cascades in heartbeat dynamics, Chaos 11 (2001), 641-652.

[7] A. L. Goldberger, D. R. Rigney and B. J. West, Chaos and fractals in human physiology, Sci. Am. 262 (1990), 43-49.

[8] C. S. Poon and C. K. Merrill, Decrease of cardiac chaos in congestive heart failure, Nature 389 (1997), 492-495.

[9] J. K. Kanters, N. H. Holstein-Rathlou and E. Agner, Lack of evidence for low-dimensional chaos in heart rate variability, J. Cardiovasc. Electrophysiol. 5 (1994), 591-601.

[10] M. Costa, I. R. Pimentel, T. Santiago, P. Sarreira, J. Melo and E. Ducla-Soares, No evidence of chaos in the heart rate variability of normal and cardiac transplant human subjects, J. Cardiovasc. Electrophysiol. 10 (1999), 1350-1357.

[11] W. B. Cannon, Organization for physiological homeostasis, Physiol. Rev. 9 (1929), 399-431.

[12] U. R. Acharya, P. K. Joseph, N. Kannathal, C. M. Lim and J. S. Suri, Heart rate variability: a review, Med. Biol. Eng. Comput. 44 (2006), 1031-1051.

[13] A. L. Goldberger, Is the normal heartbeat chaotic or homeostatic? News Physiol. Sci. 6 (1991), 87-91.

[14] Z. L. He, A new class of nonlinear integrated models, Far East J. Theor. Stat. 23 (2007), 31-50.

[15] C. R. Nelson and C. I. Plosser, Trend and random walks in macro-economic time series, J. Monetary Econ. 10 (1982), 139-162.

[16] P. Perron, The great crash, the oil price shock, and the unit root hypothesis, Econometrica 57 (1989), 1361-1401.

[17] D. N. DeJong, J. C. Nankervis, N. E. Savin and C. H. Whiteman, Integration versus trend stationary in time series, Econometrica 60 (1992), 423-433.

[18] D. Kwiatkowski, P. C. B. Phillips, P. Schmidt and Y. Shin, Testing the null hypothesis of stationarity against the alternative of a unit root: how sure are we that economic time series have a unit root? J. Econometrics 54 (1992), 159-178.

[19] M. J. Chambers, Fractional integration, trend stationarity and difference stationarity evidence from some U.K. macroeconomic time series, Econ. Lett. 50 (1996), 19-24.

[20] G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, Cambridge, 1967, p. 233.

[21] R. D. Knight, Physics for Scientists and Engineers: A Strategic Approach, Addison-Wesley, San Francisco, 2008

[22] C. W. J. Granger and A. P. Anderson, An Introduction to Bilinear Time Series Models, Vandenhoeck & Ruprecht, Göttingen, 1978, p. 8.

[23] B.-L. Hao, Starting with Parabolas: An Introduction to Chaotic Dynamics, Chinese, Shanghai Scientific and Technological Education Publishing House, 1993.

Appendix A

Proof of equation (13). The characteristic roots of equation (11) are given by

$$\lambda_{1j} = \frac{(1+\theta_1)(1-2\gamma) \pm \sqrt{(1+\theta_1)^2(1-2\gamma)^2 - 4\theta_1}}{2} \text{ for } j = 1,\ 2. \tag{A1}$$

Recall that

$$\theta_1 = \begin{cases} \dfrac{1}{1+\alpha} & \text{for } \kappa_1 = 0, \\ 1-\alpha & \text{for } \kappa_1 = 1,\ \alpha < 2. \end{cases} \tag{A2}$$

Denote $\Delta = (1+\theta_1)^2(1-2\gamma)^2 - 4\theta_1$. From (A2), we see that

$$|\,\theta_1\,| < 1. \tag{A3}$$

(i) $|\,\lambda_{1j}\,| < 1$ for $\gamma < 1$ and $\alpha < 2$ if $\kappa_1 = 1$:

For $\Delta < 0$, we have

$$\lambda_{1j}^2 = \frac{1}{4}[(1+\theta_1)^2(1-2\gamma)^2 - (1+\theta_1)^2(1-2\gamma)^2 + 4\theta_1] = \theta_1$$

which leads to $|\,\lambda_{1j}\,| < 1$ by (A3).

For $\Delta \geq 0$, if we can prove that

$$\pm\sqrt{(1+\theta_1)^2(1-2\gamma)^2 - 4\theta_1} < 2 \pm (1+\theta_1)(1-2\gamma), \tag{A4}$$

then $|\,\lambda_{1j}\,| < 1$ holds. When $0 < \gamma < 1$ and $\alpha < 2$ for $\kappa_1 = 1$, we have

$$-4\theta_1 < 4 \pm 4(1+\theta_1)(1-2\gamma)$$

which leads to

$$(1+\theta_1)^2(1-2\gamma)^2 - 4\theta_1 < 4 \pm 4(1+\theta_1)(1-2\gamma) + (1+\theta_1)^2(1-2\gamma)^2 \tag{A5}$$

and

$$2 \pm (1+\theta_1)(1-2\gamma) > 1-\theta_1 > 0 \tag{A6}$$

provided by (A2). Because $\Delta \geq 0$, (A5) and (A6), we get (A4) and thus $|\lambda_{1j}| < 1$.

(ii) $\lambda_{11} = -\theta_1 \in (-1, 1)$ and $\lambda_{12} = -1$ if $\gamma = 1$:

According to (A1), we have

$$\lambda_{1j} = \frac{-(1+\theta_1) \pm \sqrt{(1+\theta_1)^2 - 4\theta_1}}{2} = \frac{-(1+\theta_1) \pm (1-\theta_1)}{2}$$

which leads to $\lambda_{11} = -\theta_1 \in (-1, 1)$ due to (A3) and $\lambda_{12} = -1$.

(iii) $|\lambda_{11}| < 1$ and $\lambda_{12} < -1$ if $\gamma > 1$:

Note that $1-2\gamma < -1$ for $\gamma > 1$. Then we can write

$$\Delta = (1+\theta_1)^2(1-2\gamma)^2 - 4\theta_1 > (1+\theta_1)^2 - 4\theta_1 = (1-\theta_1)^2 > 0$$

by (A2). Because

$$(1+\theta_1)^2(1-2\gamma)^2 - 4\theta_1 < 4 + (1+\theta_1)^2(1-2\gamma)^2 - 4(1+\theta_1)(1-2\gamma) \tag{A7}$$

provided by $1+\theta_1 > 0$ and $\gamma > 0$, thus,

$$\pm\sqrt{(1+\theta_1)^2(1-2\gamma)^2 - 4\theta_1} < 2 - (1+\theta_1)(1-2\gamma) \tag{A8}$$

due to $2 > (1+\theta_1)(1-2\gamma)$. Accordingly, $\lambda_{11} < 1$. Note that

$$(1+\theta_1)^2(1-2\gamma)^2 - 4\theta_1 > [2 + (1+\theta_1)(1-2\gamma)]^2$$

provided by $\gamma > 1$. Then

$$\sqrt{(1+\theta_1)^2(1-2\gamma)^2 - 4\theta_1} > \pm[2 + (1+\theta_1)(1-2\gamma)], \tag{A9}$$

from which $\lambda_{11} > -1$ holds. It follows that $|\lambda_{11}| < 1$. Using (A9) yields $\lambda_{12} < -1$. We have obtained the result (13). □

Appendix B

Proof of equation (14). The characteristic roots of equation (12) are given by

$$\lambda_{2j} = \frac{(1+\theta_1)(4\ln\gamma - 1) \pm \sqrt{(1+\theta_1)^2(4\ln\gamma - 1)^2 - 4\theta_1}}{2} \tag{B1}$$

for $j = 1, 2.$ Denote $\Delta = (1+\theta_1)^2(4\ln\gamma - 1)^2 - 4\theta_1.$

(i) $|\lambda_{2j}| < 1$ if $1 < \gamma < \sqrt{e}$:

Due to $1 < \gamma < \sqrt{e},$ we have

$$-1 < 4\ln\gamma - 1 < 1 \tag{B2}$$

which produces $\pm(1+\theta_1)(4\ln\gamma - 1) < 1 + \theta_1 < 2.$

If $\Delta < 0,$ then we have

$$\lambda_{2j}^2 = \frac{1}{4}[(1+\theta_1)^2(4\ln\gamma - 1)^2 - (1+\theta_1)^2(4\ln\gamma - 1)^2 + 4\theta_1] = \theta_1$$

from which $|\lambda_{2j}| < 1$ holds due to (A3).

If $\Delta \geq 0,$ then we wish to prove that

$$\sqrt{(1+\theta_1)^2(4\ln\gamma - 1)^2 - 4\theta_1} < 2 \pm (1+\theta_1)(4\ln\gamma - 1) \tag{B3}$$

from which $|\lambda_{2j}| < 1$ holds. Using (B2) yields

$$2 \pm (1+\theta_1)(4\ln\gamma - 1) > 0. \tag{B4}$$

From (B2), we have

$$-\theta_1 < 1 \pm (1+\theta_1)(4\ln\gamma - 1).$$

Then

$$\begin{aligned}&(1+\theta_1)^2(4\ln\gamma - 1)^2 - 4\theta_1 \\ &< 4 \pm 4(1+\theta_1)(4\ln\gamma - 1) + (1+\theta_1)^2(4\ln\gamma - 1)^2\end{aligned}$$

which leads to (B3) provided by (B4).

(ii) $\lambda_{21} = 1$ and $\lambda_{22} = \theta_1 \in (-1, 1)$ if $\gamma = \sqrt{e}$:

Substituting $\gamma = \sqrt{e}$ into (B1) produces that $\lambda_{21} = 1$ and $\lambda_{22} = \theta_1$ implying that $\lambda_{22} \in (-1, 1)$ due to (B3).

(iii) $\lambda_{21} > 1$ and $|\lambda_{22}| < 1$ if $\gamma > \sqrt{e}$:

When $\gamma > \sqrt{e}$, we have $4 \ln \gamma - 1 > 1$ and then

$$\Delta = (1 + \theta_1)^2 (4 \ln \gamma - 1)^2 - 4\theta_1 > (1 + \theta_1)^2 - 4\theta_1 = (1 - \theta_1)^2$$

which implies that $\Delta > 0$. Note that

$$(1 + \theta_1)[1 - (4 \ln \gamma - 1)] < 0$$

because $4 \ln \gamma - 1 > 1$. Therefore,

$$(1 + \theta_1)^2 (4 \ln \gamma - 1)^2 - 4\theta_1 > [2 - (1 + \theta_1)(4 \ln \gamma - 1)]^2$$

resulting in

$$\sqrt{(1 + \theta_1)^2 (4 \ln \gamma - 1)^2 - 4\theta_1} > \pm[2 - (1 + \theta_1)(4 \ln \gamma - 1)] \qquad \text{(B5)}$$

from which $\lambda_{21} > 1$ holds. According to (B5), we have

$$-\sqrt{(1 + \theta_1)^2 (4 \ln \gamma - 1)^2 - 4\theta_1} < 2 - (1 + \theta_1)(4 \ln \gamma - 1)$$

and hence $\lambda_{22} < 1$. Since $\alpha > 0$ and $\gamma > \sqrt{e}$, thus,

$$-4\theta_1 < 4 + 4(1 + \theta_1)(4 \ln \gamma - 1)$$

and then

$$\begin{aligned}&(1 + \theta_1)^2 (4 \ln \gamma - 1)^2 - 4\theta_1 \\ &< 4 + 4(1 + \theta_1)(4 \ln \gamma - 1) + (1 + \theta_1)^2 (4 \ln \gamma - 1)^2.\end{aligned}$$

Note that $2 + (1 + \theta_1)(4 \ln \gamma - 1) > 0$. Hence we have

$$\sqrt{(1 + \theta_1)^2 (4 \ln \gamma - 1)^2 - 4\theta_1} < 2 + (1 + \theta_1)(4 \ln \gamma - 1)$$

or

$$-\sqrt{(1+\theta_1)^2(4\ln\gamma-1)^2-4\theta_1} > -[2+(1+\theta_1)(4\ln\gamma-1)]. \qquad \text{(B6)}$$

Using (B6) yields $\lambda_{22} > -1$. It follows that $|\lambda_{22}| < 1$. Consequently, we have obtained the result (14). □

Appendix C

Proof of equation (16). The characteristic roots of equation (15) are given by

$$\lambda_{1j} = \frac{1}{2}\left[1+\frac{1}{1+\alpha} \pm \sqrt{\left(1+\frac{1}{1+\alpha}\right)^2 - 4\frac{1+\beta}{1+\alpha}}\right] \qquad \text{(C1)}$$

for $j = 1, 2$. Denote

$$\Delta = \left(1+\frac{1}{1+\alpha}\right)^2 - 4\frac{1+\beta}{1+\alpha}.$$

(i) For $\beta \le \frac{1}{4}\alpha^2/(1+\alpha)$, we have

$$\beta < \alpha \qquad \text{(C2)}$$

and

$$4\frac{1+\beta}{1+\alpha} \le \frac{\alpha^2}{(1+\alpha)^2} + \frac{4}{1+\alpha}$$

which implies that

$$\Delta \ge \left(1+\frac{1}{1+\alpha}\right)^2 - \left[\frac{\alpha^2}{(1+\alpha)^2} + \frac{4}{1+\alpha}\right] = 0.$$

Note that

$$(2+\alpha)^2 - 4(1+\alpha)(1+\beta) < \alpha^2$$

due to $\beta > 0$. Then we can write

$$\pm\sqrt{\left(1+\frac{1}{1+\alpha}\right)^2 - 4\frac{1+\beta}{1+\alpha}} < \frac{\alpha}{1+\alpha} = 2 - \left(1+\frac{1}{1+\alpha}\right)$$

which leads to $\lambda_{1j} < 1$. Obviously,

$$\pm\sqrt{\left(1+\frac{1}{1+\alpha}\right)^2 - 4\frac{1+\beta}{1+\alpha}} < 1+\frac{1}{1+\alpha}$$

from which $\lambda_{1j} > 0$ holds. It follows that $0 < \lambda_{1j} < 1$.

(ii) For $\beta > \frac{1}{4}\alpha^2/(1+\alpha)$, we have $\Delta < 0$ and hence

$$\lambda_{1j}^2 = \frac{1+\beta}{1+\alpha}.$$

Therefore, $|\lambda_{1j}| < 1$ if $\beta < \alpha$. Combining with (C2) yields

$$\begin{cases} |\lambda_{1j}| < 1 \text{ if } \beta < \alpha \\ |\lambda_{1j}| = 1 \text{ if } \beta = \alpha \\ |\lambda_{1j}| > 1 \text{ if } \beta > \alpha \end{cases}$$

which immediately leads to the result (16). □

Appendix D

Proofs of equations (18)-(21). The characteristic roots of equation (17) are given as

$$\lambda_{1j} = \frac{1}{2}\left(2-\alpha \pm \sqrt{(2-\alpha)^2 + 4(\alpha-\beta-1)}\right) \text{ for } j = 1, 2.$$

Denote $\Delta = (2-\alpha)^2 + 4(\alpha-\beta-1)$. Since

$$(2-\alpha)^2 + 4(\alpha-\beta-1) = \alpha^2 - 4\beta,$$

thus,

$$\lambda_{1j} = \frac{1}{2}(2 - \alpha \pm \sqrt{\alpha^2 - 4\beta}) \text{ for } j = 1, 2. \tag{D1}$$

(i) For $\Delta < 0$, i.e., $\beta > \alpha^2/4$, we have

$$\lambda_{1j}^2 = \frac{1}{4}[(2 - \alpha)^2 + (4\beta - \alpha^2)] = 1 - \alpha + \beta.$$

Since $\beta > \alpha^2/4 > \alpha - 2$, thus, $1 - \alpha + \beta > -1$. Note that $1 - \alpha + \beta < 1$ for $\beta < \alpha$. Then $|\lambda_{1j}| < 1$ if $\beta < \alpha$. It is easy to see that $|\lambda_{1j}| = 1$ if $\beta = \alpha$ and $|\lambda_{1j}| > 1$ if $\beta > \alpha$. Define $\gamma = \beta/\alpha$. We immediately obtain the result (18).

(ii) For $\Delta \geq 0$, i.e., $\beta \leq \alpha^2/4$, we can show that for $\alpha < 4$,

$$\begin{cases} |\lambda_{11}| < 1, |\lambda_{12}| < 1 & \text{if } \beta > 2\alpha - 4 \\ \lambda_{11} = 3 - \alpha, \lambda_{12} = -1 & \text{if } \beta = 2\alpha - 4 \\ |\lambda_{11}| < 1, \lambda_{12} < -1 & \text{if } \beta < 2\alpha - 4 \end{cases} \tag{D2}$$

which implies that $|\lambda_{1j}| < 1$ $(j = 1, 2)$ for $0 < \alpha < 2$ and

$$\begin{cases} |\lambda_{11}| < 1, |\lambda_{12}| < 1 \text{ if } \beta > 2\alpha - 4 \\ |\lambda_{11}| < 1, \lambda_{12} = -1 \text{ if } \beta = 2\alpha - 4 \\ |\lambda_{11}| < 1, \lambda_{12} < -1 \text{ if } \beta < 2\alpha - 4 \end{cases} \tag{D3}$$

for $2 < \alpha < 4$. For $\alpha > 4$, we can show that

$$\begin{cases} \lambda_{11} < -1, \lambda_{12} < -1 & \text{if } \beta > 2\alpha - 4 \\ \lambda_{11} = -1, \lambda_{12} = 3 - \alpha < -1 & \text{if } \beta = 2\alpha - 4 \\ |\lambda_{11}| < 1, \lambda_{12} < -1 & \text{if } \beta < 2\alpha - 4. \end{cases} \tag{D4}$$

Define $\gamma = \beta/(2\alpha - 4)$ for $\alpha > 2$. We have proved the results (19) to (21).

Consequently, we have obtained the results (18) to (21). □