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MINIMIZATION OVER FIXED POINTS SETS

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Abstract

In this paper, we consider a variational inequality problem $\mathrm{VI}(F, \mathrm{Fix}\, T)$, where F, T are mappings from a real Hilbert space into itself and Fix T is the fixed points set of a demicontractive mapping T. We propose an iterative algorithm given as:

$$x_{n+1} := (1-\omega)v_n + \omega T v_n, \quad v_n := x_n - \alpha_n F(x_n),$$

to find approximate solutions. Under some assumptions on F, T, ω and (α_n), we establish the strong convergence of the sequence (x_n) generated by this schema to the solution of $\mathrm{VI}(F, \mathrm{Fix}\, T)$. An application to convex minimization is provided.

1. Introduction

In this paper, we are interested in the following variational inequality

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problem denoted $\mathrm{VI}(F, \mathrm{Fix}\, T)$:

$$\text{find } x^* \in \mathrm{Fix}\, T \text{ such that } \langle v - x^*, F(x^*)\rangle \geq 0 \quad \forall v \in \mathrm{Fix}\, T, \tag{1}$$

where $F : H \to H$ is a self-mapping on a real Hilbert space H endowed with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$, $T : H \to H$ is a self-mapping on H with a nonempty, closed and convex fixed points set denoted by $\mathrm{Fix}\, T := \{x \in H : Tx = x\}$. Variational inequality problems were introduced first by Hartman and Stampacchia [1] with the goal of computing stationary points for nonlinear programs or finding the best approximation to the initial data. They provide a broad unifying setting for the study of optimization, fixed points problems and equilibrium problems and they have important applications in economics, engineering, operations research, etc.

Numerous algorithms have been proposed to solve variational inequality problems. These algorithms are based on different techniques and they proved strong convergence theorems to the solution, see [2, 7, 8, 13, 12]. In particular, for solving $\mathrm{VI}(\nabla f, \mathrm{Fix}\, T)$, one can apply the extragradient method using the metric projection since a point $x^* \in \mathrm{Fix}\, T$ is a solution of (1) if and only if it is a fixed point of the mapping $P_{\mathrm{Fix}\, T}(I - \alpha F)$, where $\alpha > 0$, I is the identity mapping and $P_{\mathrm{Fix}\, T}$ is the metric projection of H onto $\mathrm{Fix}\, T$. Other algorithms have been developed as the hybrid steepest descent method (see [9-11]), that is, to construct iterations by the scheme:

$$v_{n+1} = T(v_n) - \alpha_{n+1} \nabla f(T(v_n)).$$

Recently, some authors have considered new iterative algorithms to approximate the set of fixed points of a nonexpansive (resp. quasi-nonexpansive, resp. demicontractive) mapping and the set of solutions of the variational inequality. Their results extend and improve many results in the literature. For details, see [5, 6, 4, 3] and the references therein.

Motivated by the recent works, in this paper, we introduce a new hybrid iterative algorithm for solving (1) and we prove a strong convergence theorem. Such a work is inspired by a remark given in the article of Maingé [6]. Maingé has considered the problem of minimizing $f : H \to \mathbb{R}$, a convex function over $\mathrm{Fix}\, T$, that is,

$$\text{find } x^* \in \mathrm{Fix}\, T \text{ such that } f(x^*) = \inf_{x \in \mathrm{Fix}\, T} f(x) \tag{2}$$

and he has proposed an alternative method which is defined by: given $x_0 \in H$ and for any $n \in \mathbb{N}$,

$$\begin{cases} x_{n+1} := (1-\omega)v_n + \omega T v_n, \\ v_n := x_n - \alpha_n \nabla f(x_n). \end{cases} \tag{3}$$

Under appropriate assumptions on f, ω, (α_n) and T, Maingé has given a strong convergence theorem of the sequence (x_n) generated by (3) to the solution x^* of problem (2) which is characterized by

$$\langle v - x^*, \nabla f(x^*) \rangle \geq 0 \quad \forall v \in \mathrm{Fix}\, T.$$

Our paper is concerned by solving the general variational inequality problem (1) with a demi-closed and demicontractive mapping T, using an analogous approach to that in [6]. For this aim, the study is devoted to the asymptotic convergence of the sequence (x_n) generated by the following iteration:

$$x_{n+1} := (1-\omega)v_n + \omega T v_n, \quad v_n := x_n - \alpha_n F(x_n) \quad \forall n \in \mathbb{N}, \tag{4}$$

where x_0 is an initial guess in H, $(\alpha_n) \subset (0, 1)$ and $\omega \in (0, 1)$.

Recall that T is said to be *demicontractive* if there exists some constant $k \in (0, 1)$ such that

$$\| q - Tx \|^2 \leq \| q - x \|^2 + k \| x - Tx \|^2 \quad \forall x \in H, \quad \forall q \in \mathrm{Fix}\, T,$$

or equivalently (see [8]),

$$\langle x - Tx, x - q \rangle \geq \frac{1-k}{2} \| x - Tx \|^2 \quad \forall x \in H, \quad \forall q \in \mathrm{Fix}\, T.$$

Note that the class of demicontractive maps has been introduced independently by Hicks and Kubicek [2] and Maruster [7]. This class strictly includes the class of strictly pseudocontractive maps which are the maps $T : H \to H$ such that there exists $k \in (0, 1)$ so that

$$\| Tx - Ty \|^2 \leq \| x - y \|^2 + k\| (I - T)x - (I - T)y \|^2 \quad \forall x,\ y \in H$$

And, consequently, it includes the important class of quasi-nonexpansive maps, that is

$$\| q - Tx \| \leq \| q - x \| \quad \forall x \in H, \quad \forall q \in \operatorname{Fix} T.$$

Furthermore, it is clear that the class of quasi-nonexpansive maps contains largely the class of nonexpansive maps which are demi-closed. Namely, a map $T : H \to H$ is said to be *demi-closed* if its graph $\mathrm{Gr}(T)$ is sequentially closed in the product of the weak topology on H with the norm topology on H, that is, for any sequence $(x_n) \subset H$, we have:

$$x_n \rightharpoonup x \text{ and } T(x_n) \to y \Rightarrow y = T(x). \tag{DC}$$

We recall that $F : H \to H$ is a bounded (resp. Lipschitz-continuous, resp. strongly monotone) operator if for some $m \geq 0$ (resp. $L \geq 0$, resp. $\mu > 0$), one has

$$\| F(x) \| \leq m \quad \forall x \in H, \tag{B}$$

$$(\text{resp.} \| F(x) - F(y) \| \leq L\| x - y \| \quad \forall x,\ y \in H), \tag{LC}$$

$$(\text{resp.} \langle F(x) - F(y),\ x - y \rangle \geq \mu \| x - y \|^2 \quad \forall x,\ y \in H). \tag{SM}$$

2. Some Technical Lemmas

In this section, we give some useful lemmas. Let us first note that algorithm (1) can be written as:

$$v_{n+1} = T_\omega(v_n) - \alpha_n F(T_\omega(v_n))$$

with $T_\omega := (1 - \omega)I + \omega T$ and I is the identity mapping on H. As throughout this work, the mapping T is k-demicontractive with $\operatorname{Fix} T \neq \emptyset$, one can check easily that T_ω is quasi-nonexpansive for any $\omega \in (0, 1-k]$. So, in this case, Fix T is a closed convex subset of H as the fixed points set of a quasi-nonexpansive mapping, that is, $\operatorname{Fix} T = \operatorname{Fix} T_\omega$ (see [10]).

Lemma 1. *Let $T : H \to H$ be a k-demicontractive mapping on H. Assume furthermore that properties* (*B*), (*LC*) *are satisfied, the sequence* (α_n) *is non-increasing with* $\alpha_n \in \left(0, \frac{1}{2L}\right)$ (*where* $L \neq 0$) *and that* $\omega \in \left(0, \frac{1-k}{2}\right]$. *Then for any* $q \in \operatorname{Fix} T$, *the sequence* (x_n) *generated by* (4) *satisfies for any* $n \in \mathbb{N}$,

$$-\alpha_n \langle x_n - q, F(x_n)\rangle \geq \beta_{n+1} - \beta_n + \frac{1}{4}(1 - 2L\alpha_n)\| x_{n+1} - x_n \|^2 - \gamma_n m^2,$$

where $\beta_n := \frac{1}{2}\| x_n - q \|^2$, $\gamma_n := \frac{\alpha_n^2}{1 - 2L\alpha_0}$ *and* $m := \sup_{x \in H} \| F(x) \|$.

Proof. Given an arbitrary $q \in \operatorname{Fix} T$, by (4), we have

$$\begin{aligned} \| x_{n+1} - q \|^2 &\leq \| v_n - q \|^2 - \omega(1 - k - \omega)\| v_n - Tv_n \|^2 \\ &\leq \| v_n - q \|^2 - \omega^{-1}(1 - k - \omega)\| x_{n+1} - v_n \|^2. \end{aligned}$$

Hence,

$$\| x_{n+1} - q \|^2 \leq \| v_n - q \|^2 - \| x_{n+1} - v_n \|^2 \tag{5}$$

since $\omega \in \left(0, \frac{1-k}{2}\right]$.

Let us now estimate each term in the right-hand side of the previous inequality. We have

$$\| v_n - q \|^2 = \| x_n - q \|^2 - 2\alpha_n \langle x_n - q, F(x_n)\rangle + \alpha_n^2 \| F(x_n) \|^2, \tag{6}$$

$$\begin{aligned}\| x_{n+1} - v_n \|^2 &= \| x_{n+1} - x_n \|^2 + 2\alpha_n \langle x_{n+1} - x_n,\ F(x_n) \rangle + \alpha_n^2 \| F(x_n) \|^2 \\ &= \| x_{n+1} - x_n \|^2 + 2\alpha_n \langle x_{n+1} - x_n,\ F(x_n) - F(x_{n+1}) \rangle \\ &\quad + 2\alpha_n \langle x_{n+1} - x_n,\ F(x_{n+1}) \rangle + \alpha_n^2 \| F(x_n) \|^2.\end{aligned}$$

On the other hand, as F is m-bounded and L-Lipschitzian,

$$\langle x_{n+1} - x_n,\ F(x_{n+1}) \rangle \geq -m \| x_{n+1} - x_n \|,$$

$$\langle x_{n+1} - x_n,\ F(x_n) - F(x_{n+1}) \rangle \geq -L \| x_{n+1} - x_n \|^2.$$

We obtain then

$$\begin{aligned}&\| x_{n+1} - v_n \|^2 \\ &\geq (1 - 2L\alpha_n) \| x_{n+1} - x_n \|^2 - 2\alpha_n m \| x_{n+1} - x_n \| + \alpha_n^2 \| F(x_n) \|^2,\end{aligned} \tag{7}$$

and from (5), (6) and (7), we get

$$\begin{aligned}\| x_{n+1} - q \|^2 \leq \| x_n - q \|^2 &- 2\alpha_n \langle x_n - q,\ F(x_n) \rangle \\ &- (1 - 2L\alpha_n) \| x_{n+1} - x_n \|^2 + 2\alpha_n m \| x_{n+1} - x_n \|.\end{aligned}$$

Thus,

$$\begin{aligned}-\alpha_n \langle x_n - q,\ F(x_n) \rangle &\geq \frac{1}{2} \| x_{n+1} - q \|^2 - \frac{1}{2} \| x_n - q \|^2 \\ &\quad + \frac{1}{2}(1 - 2L\alpha_n) \| x_{n+1} - x_n \|^2 - \alpha_n m \| x_{n+1} - x_n \| \\ &\geq \frac{1}{2} \| x_{n+1} - q \|^2 - \frac{1}{2} \| x_n - q \|^2 \\ &\quad + \frac{1}{4}(1 - 2L\alpha_n) \| x_{n+1} - x_n \|^2 - \frac{\alpha_n^2}{1 - 2L\alpha_n} m^2\end{aligned}$$

since (thanks to the canonical form)

$$\frac{1}{4}(1 - 2L\alpha_n) \| x_{n+1} - x_n \|^2 - \alpha_n m \| x_{n+1} - x_n \| \geq -\frac{\alpha_n^2}{1 - 2L\alpha_n} m^2.$$

Therefore, setting $\beta_n := \frac{1}{2}\| x_n - q \|^2$, $\gamma_n := \frac{\alpha_n^2}{1 - 2L\alpha_0}$ for all $n \in \mathbb{N}$, and assuming the sequence (α_n) non-increasing, we get

$$-\alpha_n \langle x_n - q, F(x_n) \rangle \geq \beta_{n+1} - \beta_n + \frac{1}{4}(1 - 2L\alpha_n)\| x_{n+1} - x_n \|^2 - \gamma_n m^2.$$

This leads to the conclusion. □

Observe that for the boundedness condition (B), we need only to assume that the sequence $(F(x_n))$ is bounded which is satisfied when (x_n) generated by (4) is bounded and F is Lipschitz-continuous.

Lemma 2. *Let* $T : H \to H$ *be a k-demicontractive mapping on H. Assume in addition that properties* (B), (LC), (SM) *are satisfied, the sequence* $(\alpha_n) \subset \left(0, \frac{1}{2L}\right)(L \neq 0)$ *and that* $\omega \in \left(0, \frac{1-k}{2}\right]$. *Then for any* $q \in \operatorname{Fix} T$, *the sequence* (x_n) *generated by* (4) *satisfies for any* $n \in \mathbb{N}$,

$$\langle x_n - q, F(x_n) \rangle \geq \frac{1}{1 + \mu\alpha_n}\left(\mu\beta_n - \frac{1}{2\mu} m^2\right),$$

where $\beta_n := \frac{1}{2}\| x_n - q \|^2$ *and* $m := \sup_{x \in H} \| F(x) \|$.

Proof. Take any $q \in \operatorname{Fix} T$, observe that

$$\begin{aligned} \langle x_n - q, F(x_n) \rangle &\geq \langle x_n - q, F(q) \rangle \\ &\geq -\| x_n - q \| \| F(q) \| \\ &\geq -m\| x_n - q \| \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \langle x_n - q, F(x_n) \rangle &= \langle x_n - q, F(x_n) - F(q) \rangle + \langle x_n - q, F(q) \rangle \\ &\geq \mu\| x_n - q \|^2 - m\| x_n - q \|. \end{aligned}$$

Now, setting $\lambda_n := \frac{1}{1+\mu\alpha_n} \in (0, 1)$ (so that $1-\lambda_n = \lambda_n\mu\alpha_n$), we get the following inequalities:

$$
\begin{aligned}
&\langle x_n - q,\ F(x_n)\rangle \\
&= (1-\lambda_n)\langle x_n - q,\ F(x_n)\rangle + \lambda_n\langle x_n - q,\ F(x_n)\rangle \\
&\geq -(1-\lambda_n)m\| x_n - q \| + \lambda_n\mu\| x_n - q \|^2 - m\lambda_n\| x_n - q \| \\
&\geq \lambda_n\mu\| x_n - q \|^2 - m\| x_n - q \| \\
&\geq \frac{1}{2}\lambda_n\mu\| x_n - q \|^2 + \lambda_n\mu\left(\left(\frac{1}{\sqrt{2}}\| x_n - q \| - \frac{m}{\sqrt{2}\mu}\right)^2 - \frac{m^2}{2\mu^2}\right) \\
&\geq \frac{1}{2}\lambda_n\mu\| x_n - q \|^2 - \lambda_n\frac{m^2}{2\mu}.
\end{aligned}
$$

Consider (β_n) as defined in Lemma 2, we conclude that

$$
\langle x_n - q,\ F(x_n)\rangle \geq \frac{1}{1+\mu\alpha_n}\left(\mu\beta_n - \frac{m^2}{2\mu}\right). \qquad \square
$$

We obtain then the following proposition which ensures the boundedness of the iterations (x_n):

Proposition 3. *Under the assumptions of the previous lemma, the sequence* (x_n) *generated by* (4) *is bounded.*

Proof. It suffices to prove that (β_n) is bounded. Indeed, from Lemmas 1 and 2, we obtain

$$
\begin{aligned}
\beta_{n+1} - \beta_n &\leq -\alpha_n\langle x_n - q,\ F(x_n)\rangle + \gamma_n m^2 \\
&\leq -\alpha_n\lambda_n\mu\beta_n + \alpha_n\lambda_n\frac{m^2}{2\mu} + \gamma_n m^2.
\end{aligned}
$$

Noting that $\gamma_n = \frac{\alpha_n}{1-2_{\alpha_0}}\alpha_n \le \frac{\alpha_0}{1-2_{\alpha_0}}\alpha_n$ and that the sequence (λ_n) is increasing with $\lambda_n \in (0, 1)$, we obtain the following estimation:

$$\beta_{n+1} - \beta_n \le -\sigma\alpha_n\beta_n + \alpha_n C_0,$$

where $\sigma := \lambda_0\mu$ and $C_0 := \left(\frac{1}{2\mu} + \frac{\alpha_0}{1-2_{\alpha_0}}\right)m^2$.

On the other hand, using the fact that for $t_k := \sum_{i=0}^{k}\alpha_i$ and $\delta_k := e^{\sigma t_k}$ for all $k \in \mathbb{N}$,

$$1 - \sigma\alpha_{n+1} \le e^{-\sigma\alpha_{n+1}} = \frac{\delta_n}{\delta_{n+1}},$$

one can easily check that for any n,

$$\begin{aligned}\delta_{n+1}\beta_{n+1} - \delta_n\beta_n &\le \delta_{n+1}(\beta_{n+1} - \beta_n + \sigma\alpha_{n+1}\beta_n)\\ &\le \delta_{n+1}(\sigma(\alpha_{n+1} - \alpha_n)\beta_n + \alpha_n C_0)\\ &\le \delta_{n+1}\alpha_n C_0\end{aligned}$$

since (α_n) is a non-increasing sequence. Consequently,

$$\delta_{n+1}\beta_{n+1} - \delta_0\beta_0 \le C_0\sum_{k=0}^{n}\alpha_k\delta_{k+1}$$

and thus

$$\beta_{n+1} \le e^{-\sigma t_{n+1}}\delta_0\beta_0 + C_0 e^{-\sigma t_{n+1}}\sum_{k=0}^{n}\alpha_k e^{\sigma t_{k+1}}. \tag{8}$$

In addition, as $e^a - e^b \ge (a-b)e^b$ for $a \ge b$, thus by taking $a := \sigma t_k$ and

$b := \sigma t_{k-1} = \sigma t_{k+1} - \sigma(\alpha_{k+1} + \alpha_k)(k \geq 1),$ we get the following inequalities:

$$\sigma\alpha_k e^{-2\sigma} e^{\sigma t_{k+1}} \leq \sigma\alpha_k e^{-\sigma(\alpha_{k+1}+\alpha_k)} e^{\sigma t_{k+1}} \leq e^{\sigma t_k} - e^{\sigma t_{k-1}}.$$

The inequality (8) then becomes

$$\beta_{n+1} \leq e^{-\sigma t_{n+1}}\delta_0\beta_0 + C_0\left(\alpha_0 e^{\sigma(t_1 - t_{n+1})} + \frac{e^{2\sigma}}{\sigma}\left(e^{\sigma(t_n - t_{n+1})} - e^{\sigma(t_0 - t_{n+1})}\right)\right)$$

$$\leq e^{-\sigma t_{n+1}}\delta_0\beta_0 + C_0\left(\alpha_0 e^{\sigma t_1} + \frac{e^{2\sigma}}{\sigma} e^{-\sigma\alpha_{n+1}}\right)$$

$$\leq \delta_0\beta_0 + \frac{e^{2\sigma}}{\sigma} C_0(\alpha_0\sigma + 1)$$

and the conclusion follows. □

To provide the strong convergence theorem we need to get the following lemma:

Lemma 4. *Let* $T : H \to H$ *be a k-demicontractive and demi-closed mapping and let* $F : H \to H$ *be an L-Lipschitzian mapping. Suppose that* (α_n) *converges to* 0 *and assume furthermore that the sequence* (x_n) *generated by* (4) *is bounded and satisfies*

$$\| x_{n+1} - x_n \| \to 0.$$

Then any weak cluster point of (x_n) *is a fixed point of T and we have*

$$\liminf_{n\to\infty} \langle x_n - x^*, F(x^*)\rangle \geq 0, \tag{9}$$

where x^* *is the solution of* (1).

Proof. Let us consider a subsequence $(x_{s(n)})$ of (x_n) such that $x_{s(n)} \rightharpoonup \bar{x},$ where $s : \mathbb{N} \to \mathbb{N}$ is an increasing function. As the property

(LC) is satisfied, the sequence $(F(x_{s(n)}))$ is bounded so we get that

$$v_{s(n)} = x_{s(n)} - \alpha_{s(n)} F(x_{s(n)}) \rightharpoonup \bar{x}$$

since $\alpha_{s(n)} \to 0.$ Furthermore, by (4), we have

$$\begin{aligned} Tv_{s(n)} - v_{s(n)} &= \omega^{-1}(x_{s(n+1)} - v_{s(n)}) \\ &= \omega^{-1}(x_{s(n+1)} - x_{s(n)} + x_{s(n)} - v_{s(n)}) \\ &= \omega^{-1}(x_{s(n+1)} - x_{s(n)}) + \omega^{-1}\alpha_{s(n)} F(x_{s(n)}) \end{aligned}$$

so that $(I - T)v_{s(n)} \to 0.$ Thus $\bar{x} = T\bar{x}$ since T is demi-closed and we conclude that the set of weak cluster point of (x_n) is included in Fix T.

It remains to prove inequality (9). As (x_n) is assumed to be bounded, so is the sequence $(\langle x_n - x^*, F(x^*)\rangle).$ Then there exists a subsequence of (x_n) (denoted $(x_{s(n)})$) which converges weakly to some point $\bar{x} \in \operatorname{Fix} T$ such that

$$\begin{aligned} \liminf_{n\to\infty} \langle x_n - x^*, F(x^*)\rangle &= \lim_{n\to\infty} \langle x_{s(n)} - x^*, F(x^*)\rangle \\ &= \langle \bar{x} - x^*, F(x^*)\rangle \geq 0 \end{aligned}$$

because x^* is the solution of (1). □

3. Convergence Results

Let us give now our main result.

Theorem 5. *Let* $T : H \to H$ *be a k-demicontractive and demi-closed mapping and let* $F : H \to H$ *be an m-bounded, L-Lipschitzian and* μ*-strongly monotone mapping. Suppose furthermore that* $\omega \in \left(0, \frac{1-k}{2}\right]$

and the sequence $(\alpha_n) \subset \left(0, \frac{1}{2L}\right)$ *is non-increasing and converges to* 0 *so that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \ \textit{and} \ \sum_{n=0}^{\infty} \alpha_n^2 < \infty. \tag{10}$$

Then the sequence (x_n) *generated by* (4) *converges strongly to the solution* x^* *of* (1).

Proof. Let us assume first that the sequence $(\| x_n - x^* \|)$ converges and $\| x_{n+1} - x_n \| \to 0.$ It suffices then to prove that $\lambda := \lim_{n\to\infty} \| x_n - x^* \| = 0.$ From the condition (*SM*), we have for any *n*,

$$\langle x_n - x^*, F(x_n) \rangle \geq \mu \| x_n - x^* \|^2 + \langle x_n - x^*, F(x^*) \rangle$$

and taking the lower limit, we get

$$\liminf_{n\to\infty} \langle x_n - x^*, F(x_n) \rangle \geq \mu\lambda^2 \tag{11}$$

since by Lemma 4, $\liminf_{n\to\infty} \langle x_n - x^*, F(x^*) \rangle \geq 0$ (observe that (x_n) is bounded as the sequence $(\| x_n - x^* \|)$ is convergent). If λ is greater than 0, the inequality (11) ensures that we can pick $N \in \mathbb{N}$ so that

$$\langle x_n - x^*, F(x_n) \rangle \geq \frac{1}{2}\mu\lambda^2 \quad \forall n \geq N.$$

Moreover, Lemma 1 yields for $q := x^*$:

$$\begin{aligned} \beta_{n+1} - \beta_n &\leq -\alpha_n \langle x_n - x^*, F(x_n) \rangle + \gamma_n m^2 \\ &\leq -\frac{1}{2}\mu\lambda^2\alpha_n + \gamma_n m^2 \end{aligned}$$

with $\beta_n := \frac{1}{2}\| x_n - x^* \|^2$ which converges and $\gamma_n := \frac{\alpha_n^2}{1 - 2L\alpha_0} \to 0.$

Hence,

$$\frac{1}{2}\mu\lambda^2\sum_{k=n_0}^{n}\alpha_k - m^2\sum_{k=n_0}^{n}\gamma_k \le \beta_{n_0} - \beta_{n+1}$$

which leads to a contradiction by assumptions (10).

Consequently, (x_n) generated by (4) converges strongly to x^* whenever

$$(\| x_n - x^* \|) \text{ converges and } \lim_{n\to\infty}\| x_{n+1} - x_n \| = 0.$$

It remains to check that these conditions are satisfied. Indeed, according to the previous section the sequence $(\| x_n - x^* \|^2)$ is bounded, one has

$$|\langle x_n - x^*, F(x_n)\rangle| \le m\| x_n - x^* \| \le C$$

for some constant C. So by setting $\beta_n = \frac{1}{2}\| x_n - x^* \|^2$, Lemma 1 yields:

$$\beta_{n+1} - \beta_n + \frac{1}{4}(1 - 2L\alpha_n)\| x_{n+1} - x_n \|^2 - \gamma_n m^2 \le \alpha_n C$$

and, clearly, if (β_n) (i.e., $(\| x_n - x^* \|^2)$) is convergent, $\lim_{n\to\infty}\| x_{n+1} - x_n \|^2 = 0$.

As the sequence (β_n) is bounded, its convergence is immediate when $(\beta_n)_{n\ge n_0}$ is a monotonous sequence from some index n_0. Let us discuss the non-monotonic case, that is, for any n_0 large enough, there exist $p, q \ge n_0$ such that

$$\beta_p \le \beta_{p+1} \text{ and } \beta_{q+1} \le \beta_q.$$

So we can pick a subsequence $(\beta_{s(n)})_{n\ge n_0}$ so that $\beta_{s(n)} \le \beta_{s(n)+1}$ for $n \ge n_0$, where $s : \mathbb{N} \to \mathbb{N}$ is defined as

$$s(m) := \max\{k \in \mathbb{N} : k \le m, \beta_k \le \beta_{k+1}\} \quad \forall m \in \mathbb{N},$$

see [6] for more details. Hence $\lambda := \lim_{n\to\infty}\| x_{s(n)} - x^* \|$ exists with $\lambda \ge 0$

and $\lim_{n\to\infty} \| x_{s(n)+1} - x_{s(n)} \|^2 = 0.$ So if $\lambda \neq 0$ (i.e., $\lambda > 0$), we get

$$\langle x_{s(n)} - x^*, F(x_{s(n)}) \rangle \geq \frac{1}{2}\mu\lambda^2$$

while Lemma 1 yields that

$$\langle x_{s(n)} - x^*, F(x_{s(n)}) \rangle \leq \frac{\alpha_n}{1 - 2L\alpha_n} m^2.$$

Thereby

$$\frac{\alpha_n}{1 - 2L\alpha_n} m^2 \geq \frac{1}{2}\mu\lambda^2$$

which leads to a contradiction, thus $\lambda = 0$.

Moreover, by the definition of $s(n)$, we have either $s(n) = n$ (that is, $\beta_k \leq \beta_{k+1}$ for all $k \leq n$) or $s(n) < n$ and then, in this case $\beta_n \leq \beta_{s(n)+1}$. We conclude that

$$0 \leq \beta_n \leq \max(\beta_{s(n)}, \beta_{s(n)+1}) \text{ for any } n \geq n_0$$

so that $\beta_n \to 0$ and the result is then proved. □

Let us consider now the problem of minimization given by (2) which is equivalent to the variational inequality problem:

$$\text{find } x^* \in \operatorname{Fix} T \text{ such that } \langle v - x^*, \nabla f(x^*) \rangle \geq 0 \quad \forall v \in \operatorname{Fix} T, \tag{12}$$

where the function *f* is convex and Gâteaux differentiable. Consequently, by taking $F := \nabla f : H \to H$, the Gâteaux derivative of *f*, we get:

Corollary 6. *Given* $T : H \to H$, *a k-demicontractive and demi-closed mapping, and* $f : H \to \mathbb{R}$, *a convex and Gâteaux differentiable function. Assume* $\nabla f(\cdot)$ *satisfies* (*B*), (*LC*) *with* $L \neq 0$ *and* (*SM*). *Suppose furthermore that* $\omega \in \left(0, \frac{1-k}{2}\right]$, $(\alpha_n) \subset \left(0, \frac{1}{2L}\right)$ *is non-increasing to* 0 *and such that*

(10) *holds. Then the sequence* (x_n) *generated by*:

$$x_{n+1} := (1-\omega)v_n + \omega T v_n, \quad v_n := x_n - \alpha_n \nabla f(x_n) \quad \forall n \in \mathbb{N}$$

with initial guess $x_0 \in H$, *converges strongly to the solution* x^* *of* (12) *or* (2).

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