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AN IMPORTANT DIFFERENCE BETWEEN ODD PERFECT AND EVEN PERFECT NUMBERS

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Abstract

The existence of an odd perfect number is an unsolved problem in Number Theory. We give two different proofs of the statement: the largest prime divisor of an odd perfect number g is less than $\sqrt{g}$. This result is a weaker form of a stronger result concerning the largest prime divisor of odd perfect numbers.

1. Introduction

If a, b are positive integers, then the sigma function $\sigma : \mathbb{N} \to \mathbb{N}$ defined by

$$\sigma(a) := \{\text{the sum of the divisors of } a\} = \sum_{d \mid a} d$$

satisfies the equality

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$$\sigma(ab) = \sigma(a)\sigma(b)$$

whenever $\gcd(a, b) = 1$ and $a > 1, b > 1$. A positive integer g is said to be *perfect* if $\sigma(g) = 2g$ and the following statement gives a characterization of all even perfect numbers.

Theorem 1.1. *An even integer g is perfect if and only if* $g = 2^{n-1}(2^n - 1)$ *for some* $n \in \mathbb{N}$ *and* $2^n - 1$ *is prime.*

The reader is referred to [3, 5, 6, 7, 11, 12] for different proofs of Theorem 1.1. Primes of the form $2^m - 1$ are called *Mersenne primes* and if $2^y - 1$ is prime for some positive integer *y*, then *y* must be prime. For any even perfect number $h = 2^{n-1}(2^n - 1)$, the prime $2^n - 1$ is always greater than $\sqrt{h}$.

Are there odd perfect numbers? According to Euler [6], every odd perfect number must satisfy the following statement.

Theorem 1.2. *If g is an odd perfect number, then* $g = p^k n^2$ *for some prime number p, integer k satisfying* $p \equiv k \equiv 1 \pmod 4$ *and* $\gcd(p, n) = 1$.

If $p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ is the prime-decomposition of the odd perfect number *g*, then according to Theorem 1.2 there is a unique $t \in \{1, 2, 3, ..., k\}$ such that

$$g = p_t^{4f+1} \prod_{j=1,\ j \neq t}^{k} p_j^{2s_j}$$

for some $f,\ s_j \in \mathbb{N} \cup \{0\}$. Touchard [11] also showed that an odd perfect number *g* must satisfy the condition $g \equiv 1,\ 9, 13, 25 \pmod{36}$. Furthermore, according to (i) Chein [4], the number of distinct prime factors of *g* must be greater than 8 and Nielsen [10] improved this lower bound to 9, (ii) Ewell [8],

$$\sum_{k=1}^{(g-1)/2} \sigma(2k-1)\sigma(2g-(2k-1)) \equiv 1 \pmod 2.$$

In our paper [1], the following statement restricts the size of the largest prime divisor of an odd perfect number *g*.

Theorem 1.3. *The largest prime divisor of an odd perfect number g is less than* $(3g)^{1/3}$.

Theorem 1.3 implies that the largest prime divisor of an odd perfect number *g* is less than $\sqrt{g}$. We prove this claim in two different ways in the next section. It is important for the reader to notice that Theorem 1.3 gives an important distinction between odd and even perfect numbers. The largest prime divisor of an even perfect number *g* is always greater than $\sqrt{g}$. However, according to Theorem 1.3, the largest prime divisor of an odd perfect number *h* is always less than $\sqrt{h}$. This is the first time we have obtained such an important distinction that depends on the perfect number *g*.

Let $\mathbb{P}$ denote the set of prime numbers and for any perfect integer *u*, the set of positive divisors of *u* is $D(u)$. Furthermore, define

$$D^{+}(u) = \{d \in \mathbb{N} : d \mid u,\ d > \sqrt{u}\}$$

and

$$D^{-}(u) = \{d \in \mathbb{N} : d \mid u,\ d < \sqrt{u}\}.$$

Since no perfect number is a perfect square, we should have $D(u) = D^{+}(u) \cup D^{-}(u)$ for any perfect number *u*.

2. Main Results

Consider the following lemma.

Lemma 2.1. *Let g be a perfect number and* $b = \min D^{+}(g)$. *If* $b \in \mathbb{P}$, *then*

(i) $b \mid y,\ \forall y \in D^{+}(g)$, *and*

(ii) $b = \sum_{y \in D^{-}(g)} y$.

Proof. (i) $b \in D^+(g)$ and so we have $\gcd(a, b) = 1$, $\forall a \in D^-(g)$. Therefore, $b \mid y$ for all $y \in D^+(g)$.

(ii) Given that $b \in D^+(g)$, we have $\gcd(b, e) = 1$, where $e = \max D^-(g)$. Therefore, $a \mid e$ for all $a \in D^-(g)$ and

$$\sigma(e) = \sum_{a \in D^-(g)} a.$$

Using (i), $\sum_{y \in D^+(g)} y = kb$ (for some $k \in \mathbb{N}$) and so $b \mid \sum_{x \in D^-(g)} x$. We need to show that if $\sum_{x \in D^-(g)} x = bh$ for some $h \in \mathbb{N}$, then $h = 1$. Suppose $\sum_{x \in D^-(g)} x = hb$ for some $h \in \mathbb{N}$ and $h > 1$, then

$$\sigma(e) = \sum_{x \in D^-(g)} x = hb \geq 2b > 2e$$

since $b > e$. This implies that e is abundant which is not possible since every proper divisor of a perfect number must be deficient and so we have a contradiction. Therefore, $b = \sum_{x \in D^-(g)} x$. □

We now give our first proof of the statement that the largest prime divisor of an odd perfect number is less than the square root of the number.

Theorem 2.2. *Let g be a perfect number. Then $D^+(g)$ contains a prime number if and only if $e = \max D^-(g) = 2^y$ for some $y \in \mathbb{N}$.*

Proof. Suppose $e = 2^y$ for some positive integer y, then $b = \min D^+(g)$ must be prime since no power of 2 is a perfect number. Conversely, suppose b is prime, then

$$b = \sum_{y \in D^-(g)} y.$$

Therefore, suppose that $e = \prod_{i=1}^{n} p_i^{r_i}$ for some positive integers $r_1, r_2, \ldots,$

$r_n (n > 1)$ and distinct primes $p_1, p_2, \ldots, p_n$. We have $\sigma(e) = \sum_{y \in D^-(g)} y$ since b is prime, but

$$\sigma(e) = \prod_{i=1}^{n} \frac{p_i^{r_i+1} - 1}{p_i - 1} = b$$

is not possible if the primes $p_1, p_2, \ldots, p_n$ are distinct, odd and $n > 1$. Therefore, $p_i = 2$ for all $i \ (1 \le i \le n)$ and the result follows. □

A direct consequence of Theorem 2.2 is the following statement.

Theorem 2.3. *The largest prime divisor of an odd perfect number g is less than $\sqrt{g}$.*

We give another proof of Theorem 2.3.

Proof. Given g is an odd perfect number, it must be of the form $g = p^{4k+1} n^2 = p^{4k+1} \prod_{i=1}^{m} q_i^{2t_i}$, where $m \in \mathbb{N}$, $m > 8$, $\gcd(p, n) = 1$, $\prod_{i=1}^{m} q_i^{2t_i}$ is the prime decomposition of n^2 and $p \equiv 1 (\bmod 4)$. The theorem follows easily if $k \ne 0$ since every prime divisor of g is less than $\sqrt{p^{4k+1} n^2} = q_1^{t_1} \cdots q_m^{t_m} \sqrt{p^{4k+1}}$. If $k = 0$, then $q_i < \sqrt{g}$ for all $i \,(1 \le i \le m)$ and since g is a perfect number, we must have

$$2g = \sigma(p) \prod_{i=1}^{m} \sigma(q_i^{2t_i}).$$

Therefore, p must divide $\prod_{i=1}^{m} \sigma(q_i^{2t_i})$ and since $m > 8$, $p(p+1)d = 2g$ for some $d > 2$; completing the proof. □

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