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STABLE SINGULARITIES OF CO-RANK ONE QUASI HOMOGENEOUS MAP GERMS FROM $(\mathbb{C}^{n+1}, 0)$ TO $(\mathbb{C}^n, 0)$, $n = 2, 3$

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Abstract

In this article, we investigate the geometry of quasi homogeneous co-rank one finitely determined map germs from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}^n, 0)$ with $n = 2, 3.$ We give a complete description, in terms of the weights and degrees, of the invariants that are associated to all stable singularities which appear in the discriminant of such map germs. The first class of invariants which we study are the isolated singularities, called 0-stable singularities because they are the 0-dimensional singularities. First, we give a formula to compute the number of A_n points which appear in any stable deformation of a quasi homogeneous co-rank one map germ from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}^n, 0)$ with $n = 2, 3.$ To get such a formula, we apply the Hilbert's syzygy theorem to determine the graded free resolution given by the syzygy modules of the associated iterated Jacobian ideal. Then we show how to obtain the other 0-stable singularities, these isolated singularities are formed by multiple points and here we use the relation among them and the Fitting ideals of the discriminant. For $n = 2,$ there exists only the germ of double points set and for $n = 3$ there are the triple points, named points $A_{1,1,1}$ and the normal crossing between a germ of a cuspidal edge and a germ of a plane, named $A_{2,1}$. For $n = 3,$ there appear also the one-dimensional singularities, which are of two types: germs of cuspidal edges or germs of double points curves. For these singularities, we show how to compute the polar multiplicities and also the local Euler obstruction at the origin in terms of the weights and degrees.

1. Introduction

The study of singularities of differentiable maps was initiated by Whitney who showed that the singularities which appear in any stable map from the plane to the plane are the cusps and double points. The singularities of the stable maps, called *stable singularities*, are also very important in the study of the non-stable maps, in special, for the class of the finitely determined maps, since they have the interesting property that for these maps, they are preserved for any stable deformation.

For the particular singularities which are isolated, called 0-*stable singularities*, the type and also the number of such singularities are very relevant because they contain information about the local geometric behavior of such maps. For instance, we cite the work of Gaffney and Mond in [4] where it is shown how to determine algebraically the number of cusps and double points of finitely determined map germs from the plane to the plane, and moreover, it is shown, in fact, that these numbers are topological invariants in families of such germs. We also remember here the works of Mond on germs of surfaces in $\mathbb{C}^3$ where it is shown that the stable singularities are the cross-caps and the triple points and how to compute these numbers algebraically, see [15] and [16].

For the non-isolated singularities, there are some numbers which are associated to them, in special the polar multiplicities and the local Euler obstruction are very relevant to show information about the local geometric behavior of such maps.

So the determination of formulae to compute these numbers became relevant and in this case, the class of quasi homogeneous maps is known as some of great interest, since for this class there are several results showing how to compute these numbers in terms of weights and degrees.

An analytic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0),\ f = (f_1, ..., f_p)$ is *quasi-homogeneous* of type $(\omega_1, ..., \omega_n; d_1, ..., d_p)$ if there are positive numbers $\omega_1, \omega_2, ..., \omega_n$ in $\mathbb{Q}$ called *weights*, and $d_1, ..., d_p$, called *degrees* such that $f_i(\lambda^{\omega_1} x_1, ..., \lambda^{\omega_n} x_n) = \lambda^{d_i} f_i$ for all $i = 1, ..., p,\ x \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$.

For the class of finitely determined quasi homogeneous map germs from the plane to the plane, or when $n = p = 2,$ Gaffney and Mond [5] obtained formulae to compute the number of cusps and double-folds that involve only weights and degrees. For the case $n = 2$ and $p = 3,$ Mond in [17] derived formulae for the number of the triple points and cross-caps, in terms of the weights and degrees of quasi homogeneous map germs in these dimensions. Marar et al. investigated in [13] the 0-stable singularities for the case when the source dimension and target dimension coincide, i.e., $n = p$. First, they described algebraically all 0-stable singularities that appear in any co-rank one stable map germ, showing that they are the $A_{\mathcal{P}}$ singularities, for any partition $\mathcal{P}$ of n, according to Arnold's notation. Then they showed how to compute the number of such $A_{\mathcal{P}}$ singularities for quasi homogeneous map germs in terms of the weights and degrees. The key tool to obtain such formulae is an application of the theorem of Bezout for zero-dimensional complete intersections.

Our first object of interest in this article is the description of the 0-stable singularities that appear in the discriminant of a stable deformation of weighted homogeneous finitely determined map germ from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}^n, 0)$ with $n = 2$ or 3 also in the co-rank one case.

The main difference from the case of map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^n, 0)$ is that the corresponding defining ideals of the 0-dimensional singularities are not complete intersections. Therefore, we cannot apply the theorem of Bezout.

First, we study the mono germs, named A_n singularities, they form the key tool to find the other singularities since there are several relationships between the A_n points and the other. We show that the number of A_n points can be obtained as the complex dimension (as vector space) of some Cohen-Macaulay algebras of type $\frac{\mathcal{O}_n}{J}$. To compute such dimension, first, we apply the Hilbert's syzygy theorem to obtain the free resolution given by the syzygy modules of the ideal J as described in [6, Section 2.5]. Then we find a convenient filtration in $\mathcal{O}_n$ from the weights given by the weighted homogenous germ to get the Hilbert polynomial of the associated graded resolution. The valuation of this polynomial in 1 gives the number.

To describe the 0-singularities which are multi germs, we apply the results of Mond and Pellikaan in [18] where it is shown how the Fitting ideals associated to the discriminant are related to the isolated singularities of map germs.

Last, but not the least, we apply these results to study the 1-dimensional stable singularities which appear in the discriminant. We show how to compute the polar multiplicities of the singular curves of the discriminant only in terms of the weights and degree of the map germ. Moreover, we also apply these results to show how to compute the Euler obstruction at the origin of such singular curves in the discriminant.

2. Stable Singularities

In this section, we describe all stable singularities which appear in the discriminant of any finitely determined map germ $f \in \mathcal{O}(n+1, n)$ with $n = 2, 3.$ As these dimensions are in the range of the nice dimensions of Mather, we know that they form a finite set of strata in the image by *f* of the critical set of *f*, called *discriminant* of *f* and denoted $\Delta(f) = f(\Sigma(f)).$

In general, for any pair of dimensions $(n, p),$ the description of the stable types can be done in terms of sub-schemes of multiple points of a germ *f*, as we can see in [9] for the case $n = p$ or in [8] for the case $n < p.$ We remark that these constructions are done from the original construction of the set of multiple points for any finitely determined map germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^p, 0),$ with $n \geq p,$ as described in details by Goryunov in [7].

We describe here these singularities for the particular cases that we are interested.

For map germs from $\mathbb{C}^3$ to $\mathbb{C}^2$, the discriminant $\Delta(f)$ is a curve in $\mathbb{C}^2$, possible with isolated singularities (the 0-stable singularities) of two types, the double points set denoted $A_{1,1}$ and the cusps set, denoted A_2.

For map germs from $\mathbb{C}^4$ to $\mathbb{C}^3$, the discriminant of *f* is a germ of surface in $\mathbb{C}^3$. In this case, we can have in the discriminant of *f* the isolated stable singularities, called 0-*stable singularities*, and also 1-*dimensional stables singularities*, the 1-*stable singularities*.

Here the 0-stable singularities are of three types, the triple points set, denoted $A_{1,1,1}$, the swallowtails set, denoted A_3 and the intersection of a germ of cuspidal edge with a germ of plane, denoted $A_{2,1}$. The 1-stable singularities are of two types: the double points curve, denoted $A_{1,1}$ and the cuspidal edge, denoted A_2.

Figure 1 shows the geometric models of the isolated stable singularities A_3, $A_{(2,1)}$ and $A_{(1,1,1)}$.

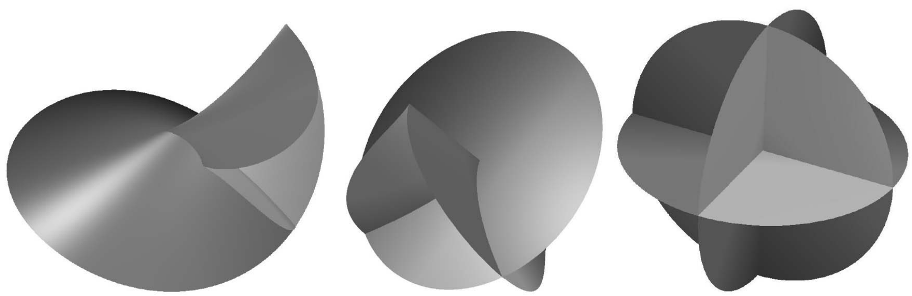

Figure 1. Geometric models: A_3, $A_{(2,1)}$ and $A_{(1,1,1)}$, respectively.

3. Iterated Jacobian Ideals of Points A_n

In this section, we show how to compute algebraically the isolated stable singularities, which appear as a mono germ, in the discriminant of any finitely determined co-rank one map germ from $\mathbb{C}^{n+1}$ to $\mathbb{C}^n$ with $n = 2, 3$.

These mono germs, or points A_n are described in terms of the iterated Jacobian ideals, these ideals were defined by Morin in [19], and the relationship among such numbers and the iterated Jacobian was shown in [2].

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ and $I \subset \mathcal{O}_n$ be a finite co-length ideal generated by the system $g_1, ..., g_r$. For each $t \in$

$\{1, \ldots, n\}$, we define the *Jacobian extension of the rank t* for the pair (f, I), by $\Delta_t(f, I) := I + I_t(d(f_1, \ldots, f_p, g_1, \ldots, g_r))$, where

$$I_t(d(f_1, \ldots, f_p, g_1, \ldots, g_r))$$

denotes the ideal generated by minors of size $t \times t$ from the Jacobian matrix of the $f_1, \ldots, f_p, g_1, \ldots, g_r(d(f_1, \ldots, f_p, g_1, \ldots, g_r))$.

If $i = (i_1, \ldots, i_k)$ is a Boardman number, then we inductively define the *iterated Jacobian ideal* for *i*, $J_i(f)$ in the following manner:

$$J_i(f) = \begin{cases} \Delta_{n-i_1+1}(f, \{0\}) & \text{if } k = 1, \\ \Delta_{n-i_k+1}(f, J_{i_1,\ldots,i_{k-1}}(f)) & \text{if } k > 1. \end{cases}$$

We see in [2] that if the map germ is finitely determined and the ring $\dfrac{\mathcal{O}_n}{J_i(f)}$ is Cohen-Macaulay, then the complex dimension (as vector space) of $\dfrac{\mathcal{O}_n}{J_i(f)}$ gives the number of some isolated singularities, denoted $c_i(f)$, in the discriminant of *f*. In the cases which we are working in this article, these isolated singularities are the A_n singularities.

For finitely determined co-rank one map germs from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^2, 0)$, we are looking for the number of points A_2, the iterated Jacobian ideal associated to these points is the ideal $J_{2,1}(f)$ which we describe below.

Assume that $f(x, y, z)$ is written in the form $f(x, y, z) = (x, g(x, y, z))$, first, we obtain the ideal $J_2(f)$ which is generated by the minors of order two of the Jacobian matrix of *f* and one has $J_2(f) = I_2(d(f)) = \langle g_y, g_z \rangle$, where g_w denotes the partial derivative of *g* with respect to the variable $w \in \{x, y, z\}$. Applying the recursive process of Morin, we obtain the ideal $J_{2,1}(f) = J_2(f) + I_3(d(f, g_y, g_z)) = \langle g_y, g_z, M \rangle$, where *M* denotes the determinant of the matrix of order 2 given by the second partial derivatives of g with respect to the variables (y, z). From the results shown in [2], we obtain $\#A_2 = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{J_{2,1}(f)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\langle g_y, g_z, M\rangle}$.

For finitely determined co-rank one map germs from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^3, 0)$, the iterated Jacobian ideal associated to the points A_3 is the ideal $J_{2,1,1}(f)$. Write f as $f(x, y, u, v) = (x, y, g(x, y, u, v))$, the Jacobian matrix of f is the 3×4 matrix $d(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ g_x & g_y & g_u & g_v \end{pmatrix}$. To obtain the iterated Jacobian ideal $J_{2,1,1}$, first, we get $J_2(f) = I_3(d(f)) = \langle g_u, g_v\rangle$ and $J_{2,1} = J_2(f) + I_4(d(f, g_u, g_v)) = \langle g_u, g_v, H\rangle$, where $H := g_{uu}g_{vv} - (g_{uv})^2$ and

$$\begin{aligned} J_{2,1,1}(f) &= J_{2,1} + I_4(d(f, g_u, g_v, H)) \\ &= \langle g_u, g_v, H, g_{uu}H_v - g_{uv}H_u, g_{uv}H_v - g_{vv}H_u\rangle. \end{aligned}$$

From Corollary 4.4. of [2], one has

$$\begin{aligned} \#A_3 &= \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{J_{2,1,1}(f)} \\ &= \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{\langle g_u, g_v, H, g_{uu}H_v - g_{uv}H_u, g_{uv}H_v - g_{vv}H_u\rangle}. \end{aligned}$$

We remark that if the map germ f in $\mathcal{O}_{(4,3)}$ is a suspension of a map germ from $\mathbb{C}^3$ to $\mathbb{C}^3$, then it can be written as $f(x, y, u, v) = (x, y, g(x, y, u) + v^2)$, therefore, the iterated Jacobian ideal $J_{2,1,1}$ is generated by $\{g_u, g_{uu}, g_{uuu}, v\}$ and the ring $\frac{\mathcal{O}_4}{J_{2,1,1}}$ is isomorphic to the ring $\frac{\mathcal{O}_3}{\langle g_u, g_{uu}, g_{uuu}\rangle}$. Therefore, for suspensions in these dimensions, we have $\#A_3 = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\langle g_u, g_{uu}, g_{uuu}\rangle}$.

4. Points A_n and Free Resolutions

The first step in the method to compute the complex dimension of any ring $\frac{\mathcal{O}_n}{I}$ for any finite co-length ideal I in $\mathcal{O}_n$ is to obtain a projective resolution of the ring $\frac{\mathcal{O}_n}{I}$. The existence of such resolution is shown in the Hilbert's syzygy theorem, which we recover here in a more general set-up.

Theorem 4.1 [6, Theorem 2.4.11]**.** *Let $(A, \mathfrak{m})$ be a local Noetherian ring and M be a finitely generated A-module. M has a minimal free resolution*

$$0 \to F_m \overset{\alpha_m}{\to} F_{m-1} \overset{\alpha_{m-1}}{\to} \cdots \overset{\alpha_1}{\to} F_0 \to M \to 0$$

of length $m \leq n$, where F_i are free R-modules.

The construction of such free resolution is based in the description of morphisms α_i between the modules F_i and F_{i-1}. These morphisms form the main tool to fix the right filtration in the ring R which we need to get the formulae for the number of A_n points. In Appendix of this article, we show how to construct these morphisms in an example.

The next step is to define an appropriate graduation for each member F_k in the exact sequence of the free resolution in such a way that the morphisms are of degree zero. From this graded exact sequence, we get the Poincaré series (or the Hilbert Samuel polynomial) and the evaluation of this polynomial at 1 gives us the desired formula.

Next, we show how to obtain such resolution and the formulae for the cases which we are interested.

4.1. Points A_2 for map germs in $\mathcal{O}_{3,2}$

First, we show how to compute the number of points $\# A_2$ of any finitely determined co-rank one weighted homogeneous map germ from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^2, 0)$.

Let $f(x, u, v) = (x, g(x, u, v))$ be such a germ. Then we have $\#A_2 = \dim_{\mathbb{C}} \dfrac{\mathcal{O}_3}{J_{2,1}(f)}$ with $J_{2,1}(f) = \langle g_u, g_v, g_{uu} \cdot g_{vv} - (g_{uv})^2 \rangle$.

Since $J_{2,1}(f)$ has a minimal system of generators with three elements in $\mathcal{O}_3$, the minimal free resolution of $\dim_{\mathbb{C}} \dfrac{\mathcal{O}_3}{J_{2,1}(f)}$ is well known and is given by the following exact sequence:

$$0 \to \mathcal{O}_3 \xrightarrow{\alpha_3} \bigoplus_{i=1}^{3} \mathcal{O}_3 \xrightarrow{\alpha_2} \bigoplus_{i=1}^{3} \mathcal{O}_3 \xrightarrow{\alpha_1} \mathcal{O}_3 \xrightarrow{\pi} \frac{\mathcal{O}_3}{J_{2,1}(f)} \to 0.$$

Here π is the natural projection, the morphism α_1 is defined by the (1×3)-matrix associated to $\ker(\pi)$, or $\alpha_1 = (g_u, g_v, g_{uu} \cdot g_{vv} - (g_{uv})^2)$. The morphism α_2 is defined by the 3×3-matrix associated to $\ker(\alpha_1)$ and α_3 is defined by the corresponding (3×1)-matrix of $\ker(\alpha_2)$.

To obtain the corresponding sequence of graded modules with zero degree morphisms, we suppose that $g(x, u, v)$ is quasi-homogeneous of type $(\omega_1, \omega_2, \omega_3; d)$. From the weights and the degree of the germ g, we have that each generator of the ideal $J_{2,1}(f)$ is weighted homogeneous and therefore we can write this resolution as follows:

$$0 \to \mathcal{O}_3[-C] \to \bigoplus_{i=1}^{3} \mathcal{O}_3[-B_i] \to \bigoplus_{i=1}^{3} \mathcal{O}_3[-A_i] \to \mathcal{O}_3 \to \frac{\mathcal{O}_3}{J_{2,1}(f)} \to 0,$$

where $A_1 = d - \omega_2$, $A_2 = d - \omega_3$, $A_3 = 2(d - \omega_2 - \omega_3)$, $B_1 = 2d - \omega_2 - \omega_3$, $B_2 = 3d - 3\omega_2 - 2\omega_3$, $B_3 = 3d - 2\omega_2 - 3\omega_3$ and $C = 4d - 3\omega_2 - 3\omega_3$.

Now, we remember that for each $\mathcal{O}_3[-r]$, the associated Poincaré series is defined as:

$$P_{\mathcal{O}_3[-r]}(t) = \frac{t^r}{(1 - t^{\omega_1})(1 - t^{\omega_2})(1 - t^{\omega_3})}.$$

Since the alternating sum of the Poincaré series (or the Hilbert polynomial) $P_M(t)$ of an exact sequence of graded modules with degree zero morphisms is equal to 0, we conclude that the Poincaré series of $\frac{\mathcal{O}_3}{J_{2,1}(f)}$ is the alternate sum of the Poincaré series of each $\mathcal{O}_3[-r]$ in the sequence, hence

$$P_{\frac{\mathcal{O}_3}{J_{2,1}(f)}}(t) = \frac{1 - \left(\sum_{i=1}^{3} t^{A_i}\right) + \left(\sum_{i=1}^{3} t^{B_i}\right) - (t^C)}{(1 - t^{\omega_1})(1 - t^{\omega_2})(1 - t^{\omega_3})}.$$

Therefore,

$$\# A_2 = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{J_{2,1}(f)} = P_{\frac{\mathcal{O}_3}{J_{2,1}(f)}}(1) = \lim_{t \to 1} P_{\frac{\mathcal{O}_3}{J_{2,1}(f)}}(t)$$

and

$$\# A_2 = \frac{1}{\omega_1\omega_2\omega_3} \cdot \{P_3 d^3 + P_2 d^2 + P_1 d + P_0\} = \frac{P_3 d^3 + P_2 d^2 + P_1 d + P_0}{\omega_1\omega_2\omega_3},$$

where $P_3 = 2,\ P_2 = -4\omega_2 - 4\omega_3,\ P_1 = 2\omega_2^2 + 6\omega_2.\omega_3 + 2\omega_3^2,\ P_0 = -2\omega_2^2\omega_3 - 2\omega_2\omega_3^2$.

Example. Let $F_k : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0) : F_k(x, u, v) := (x, xu + uv^2 + u^k)$ for $k \in \mathbb{N}$ with $k > 3$. Then the map germ F_k is finitely determined and quasi-homogeneous of type $\left(k - 1, \frac{k-1}{2}, 1; k - 1, k\right)$. From the above result, we obtain $\# A_2(F_k) = k + 1$.

4.2. A_3 points of suspensions in $\mathcal{O}_{4,3}$

To compute the number of points A_3 for co-rank one map germs from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^3, 0)$, we need to split our calculation in two different situations.

One is the case that the map germ is a suspension from a map germ from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^3, 0)$ and the other is the general case, where the map germ is not a suspension.

The main difference is the minimal number of generators of the ideal $J_{2,1,1}$, this gives rise to different minimal free resolutions of the ring $\frac{\mathcal{O}_4}{J_{2,1,1}}$.

For suspensions, the zero set of the associated iterated Jacobian ideal $J_{2,1,1}$ is a complete intersection formed by isolated points in $\mathbb{C}^4$. When the germ is not a suspension, any system of generators of $J_{2,1,1}$ has at least 5 elements, therefore the zero set is not a complete intersection. To obtain the corresponding minimal resolution we applied the algorithm given in [6, p. 165].

To clarify this difference, first, we describe the resolution for suspensions. In Subsection 4.3, we describe the resolution for map germs which are not suspensions.

Let $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$ be a suspension of a finitely determined corank one map germ $g : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ written as $f(x, y, u, v) = (x, y, g(x, y, u) + v^2)$. Then the associated iterated Jacobian ideal is $J_{2,1,1}(f) = \langle g_u, g_{uu}, g_{uuu}, v \rangle$.

If g is quasi-homogeneous of type $(\omega_1, \omega_2, \omega_3; d)$, then f is also quasi-homogeneous with respect to the weights $(\omega_1, \omega_2, \omega_3, d/2)$. Now, we call $g_1 := g_u$, $g_2 := g_{uu}$, $g_3 := g_{uuu}$ and $g_4 := v$. Observe that each generator of $J_{2,1,1}$ is a weighted homogeneous polynomial of degrees $d_1 := d - \omega_3$, $d_2 := d - 2\omega_3$, $d_3 := d - 3\omega_3$, $d_4 := \frac{d}{2}$, respectively.

The zero set of the ideals $J_{2,1,1}(f)$ is a complete intersection of isolated points $\mathbb{C}^4$ and the free resolution of $\frac{\mathcal{O}_4}{J_{2,1,1}(f)}$ is:

$$0 \to \mathcal{O}_4 \xrightarrow{\alpha_4} \bigoplus_{i=1}^{4} \mathcal{O}_4 \xrightarrow{\alpha_3} \bigoplus_{i=1}^{6} \mathcal{O}_4 \xrightarrow{\alpha_2} \bigoplus_{i=1}^{4} \mathcal{O}_4 \xrightarrow{\alpha_1} \mathcal{O}_4 \xrightarrow{\pi} \frac{\mathcal{O}_4}{J_{2,1,1}(f)} \to 0.$$

The graded resolution associated to sequence above is

$$0 \to \mathcal{O}_4[-D] \to \bigoplus_{i=1}^{4} \mathcal{O}_4[-C_i] \to \bigoplus_{i=1}^{6} \mathcal{O}_4[-B_i]$$

$$\to \bigoplus_{i=1}^{4} \mathcal{O}_4[-A_i] \to \mathcal{O}_4 \to \frac{\mathcal{O}_4}{J_{2,1,1}(f)} \to 0,$$

where

$$D = \frac{7}{2}d - 6\omega_3,\ A_1 = \frac{d}{2},\ A_2 = d - 3\omega_3,\ A_3 = d - \omega_3,\ A_4 = d - 2\omega_3,$$

$$B_1 = 2d - 3\omega_3,\quad B_2 = 2d - 5\omega_3,\quad B_3 = 2d - 4\omega_3,\quad B_4 = \frac{3}{2}d - 2\omega_3,$$

$$B_5 = \frac{3}{2}d - \omega_3,\quad B_6 = \frac{3}{2}d - 3\omega_3,\quad C_1 = 3d - 6\omega_3,$$

$$C_2 = \frac{5}{2}d - 3\omega_3,\ C_3 = \frac{5}{2}d - 5\omega_3 \text{ and } C_4 = \frac{5}{2}d - 4\omega_3.$$

From this grading, we obtain the graded exact sequence with Hilbert polynomial

$$P_{\frac{\mathcal{O}_4}{J_{2,1,1}(f)}}(t) = \frac{1 - \left(\sum_{i=1}^{4} t^{A_i}\right) + \left(\sum_{i=1}^{6} t^{B_i}\right) - \left(\sum_{i=1}^{4} t^{C_i}\right) + t^D}{(1 - t^{\omega_1})(1 - t^{\omega_2})(1 - t^{\omega_3})\left(1 - t^{\frac{d}{2}}\right)}.$$

Therefore,

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_4}{J_{2,1,1}(f)} = P_{\frac{\mathcal{O}_4}{J_{2,1,1}(f)}}(1) = \lim_{t \to 1} P_{\frac{\mathcal{O}_4}{J_{2,1,1}(f)}}(t)$$

$$= \frac{12d^4 - 72\omega_3 d^3 + 132\omega_3 d^2 - 72\omega_3^3 d}{4!\,\omega_1\omega_2\omega_3 \frac{d}{2}}.$$

Thus,

$$\#A_3(f) = \frac{1}{\omega_1\omega_2\omega_3} \cdot \{d^3 + (-6\omega_3)d^2 + (11\omega_3^2)d + (-6\omega_3^3)\}$$

$$= \frac{d^3 - 6\omega_3 d^2 + 11\omega_3^2 d - 6\omega_3^3}{\omega_1\omega_2\omega_3}.$$

As f is a suspension of g, this formula is equal than the formula of the number of points A_3 of the map germ $g : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$, in this case the iterated Jacobian ideal associated to the number of points A_3 of g is $J_{1,1,1}(g)$ with free resolution:

$$0 \to \mathcal{O}_3 \xrightarrow{\alpha_3} \bigoplus_{i=1}^{3} \mathcal{O}_3 \xrightarrow{\alpha_2} \bigoplus_{i=1}^{3} \mathcal{O}_3 \xrightarrow{\alpha_1} \mathcal{O}_3 \xrightarrow{\pi} \frac{\mathcal{O}_3}{J_{(1,1,1)}(g)} \to 0.$$

With graded resolution,

$$0 \to \mathcal{O}_3[-D] \to \bigoplus_{i=1}^{3} \mathcal{O}_3[-L_i] \to \bigoplus_{i=1}^{3} \mathcal{O}_3[-P_i] \to \mathcal{O}_3 \to \frac{\mathcal{O}_3}{J_{(1,1,1)}(f)} \to 0,$$

where $D = 3d - \omega_3$, $L_i = 2d - (6-i)\omega_3$, $\forall i \in \{1, 2, 3\}$, $P_i = d - (4-i)\omega_3$, $\forall i \in \{1, 2, 3\}$.

Therefore,

$$\#A_3(g) = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{J_{(1,1,1)}(g)} = P_{\frac{\mathcal{O}_3}{J_{2,1,1}(g)}}(1) = \frac{d^3 - 6\omega_3 d^2 + 11\omega_3^2 d - 6\omega_3^3}{\omega_1\omega_2\omega_3}.$$

We remember that this formula is shown in [13] as a consequence of the theorem of Bezout.

4.3. A_3 points of non-suspensions in $\mathcal{O}_{4,3}$

Now, we compute the number of points of type A_3 for map germs from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^3, 0)$ which are not suspensions. Write any finitely determined co-rank one map germ $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$ as $f(x, y, u, v) = (x, y, g(x, y, u, v))$.

In this case, the system of generators obtained from the construction of the iterated Jacobian of $J_{2,1,1}(f)$ is

$$\{g_u, g_v, H, g_{uu}H_v - g_{uv}H_u, g_{uv}H_v - g_{vv}H_u\}.$$

If all these generators are non-zero, then from [6, Theorem 2.5.9, p. 162], it is a minimal system of generators of $J_{2,1,1}(f)$. Therefore, the theorem of Hilbert guarantees the existence of a minimal free resolution of the ring $\frac{\mathcal{O}_4}{J_{2,1,1}(f)}$ and applying [6, Algorithm 2.5.16, p. 165], we obtain:

$$0 \to \bigoplus_{i=1}^{2} \mathcal{O}_4 \xrightarrow{\alpha_4} \bigoplus_{i=1}^{7} \mathcal{O}_4 \xrightarrow{\alpha_3} \bigoplus_{i=1}^{9} \mathcal{O}_4 \xrightarrow{\alpha_2} \bigoplus_{i=1}^{5} \mathcal{O}_4 \xrightarrow{\alpha_1} \mathcal{O}_4 \xrightarrow{\pi} \frac{\mathcal{O}_4}{J_{2,1,1}(f)} \to 0.$$

The morphism α_1 is given by $\alpha_1 = \ker(\pi)$, where π denotes the natural projection and for $i \geq 2$, each morphism α_i of this sequence is defined inductively by the matrix corresponding of the set all syzygies of the submodule associated to the matrix which defines the morphism α_{i-1}. See the definition of the syzygies in Appendix of this article, where we describe this construction for one example.

To obtain the corresponding graded resolution, we suppose that g is quasi-homogeneous of type $(\omega_1, \omega_2, \omega_3, \omega_4; d)$. Then f is quasi-homogeneous of type $(\omega_1, \omega_2, \omega_3, \omega_4; \omega_1, \omega_2, d)$.

Call $g_1 := g_u$, $g_2 := g_v$, $g_3 := H$, $g_4 := g_{uu}H_v - g_{uv}H_u$ and $g_5 := g_{uv}H_v - g_{vv}H_u$.

Therefore, each generator g_i of $J_{2,1,1}$ is weighted homogeneous with degrees $d_1 := d - \omega_3$, $d_2 := d - \omega_4$, $d_3 := 2d - 2\omega_3 - 2\omega_4$, $d_4 := 3d - 4\omega_3 - 3\omega_4$ and $d_5 := 3d - 3\omega_3 - 4\omega_4$, respectively.

Hence we can write this resolution as an exact sequence of graded modules with degree zero morphisms as follows:

$$0 \to \bigoplus_{i=1}^{2} \mathcal{O}_4[-D_i] \to \bigoplus_{i=1}^{7} \mathcal{O}_4[-C_i] \to \bigoplus_{i=1}^{9} \mathcal{O}_4[-B_i]$$

$$\to \bigoplus_{i=1}^{5} \mathcal{O}_4[-A_i] \to \mathcal{O}_4 \to \frac{\mathcal{O}_4}{J_{2,1,1}(f)} \to 0,$$

here $Ai = d_i$, $\forall i \in \{1, ..., 5\}$, $B_1 = 5d - 7\omega_3 - 5\omega_4$, $B_2 = 5d - 5\omega_3 - 7\omega_4$, $B_3 = 4d - 3\omega_3 - 5\omega_4$, $B_4 = 3d - 2\omega_3 - 3\omega_4$, $B_5 = 4d - 4\omega_3 - 4\omega_4$, $B_6 = 2d - \omega_3 - \omega_4$, $B_7 = 4d - 4\omega_3 - 4\omega_4$, $B_8 = 3d - 3\omega_3 - 2\omega_4$, $B_9 = 4d - 5\omega_3 - 5\omega_4$, $C_1 = 5d - 5\omega_3 - 4\omega_4$, $C_2 = 4d - 3\omega_3 - 3\omega_4$, $C_3 = 5d - 4\omega_3 - 4\omega_4$, $C_4 = 6d - 5\omega_3 - 8\omega_4$, $C_5 = 6d - 6\omega_3 - 7\omega_4$, $C_6 = 6d - 7\omega_3 - 6\omega_4$, $C_7 = 6d - 8\omega_3 - 5\omega_4$ and $D_1 = 7d - 6\omega_3 - 8\omega_4$, $D_2 = 7d - 8\omega_3 - 6\omega_4$.

Now, we compute the Hilbert polynomial of $\frac{\mathcal{O}_4}{J_{2,1,1}(f)}$, given as the alternate sum of the other Hilbert polynomials:

$$P_{\frac{\mathcal{O}_4}{J_{2,1,1}(f)}}(t) = \frac{1 - \left(\sum_{i=1}^{5} t^{A_i}\right) + \left(\sum_{i=1}^{9} t^{B_i}\right) - \left(\sum_{i=1}^{7} t^{C_i}\right) + \left(\sum_{i=1}^{2} t^{D_i}\right)}{(1 - t^{\omega_1})(1 - t^{\omega_2})(1 - t^{\omega_3})(1 - t^{\omega_4})}.$$

Therefore,

$$\#A_3 = \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{J_{2,1,1}(f)} = P_{\frac{\mathcal{O}_4}{J_{2,1,1}(f)}}(1) = \lim_{t \to 1} P_{\frac{\mathcal{O}_4}{J_{2,1,1}(f)}}(t)$$

$$= \frac{P_4 d^4 + P_3 d^3 + P_2 d^2 + P_1 d + P_0}{\omega_1 \omega_2 \omega_3 \omega_4},$$

with $P_4 = 16$, $P_3 = -3 - 55\omega_4 - 55\omega_3$, $P_2 = \frac{15}{2}\omega_3 + 63\omega_3^2 + 63\omega_4^2 + 139\omega_3\omega_4$

$+\frac{15}{2}\omega_4$, $P_1 = -24\omega_3^3 - 108\omega_3^2\omega_4 - 24\omega_4^3 - 12\omega_3\omega_4 - 108\omega_3\omega_4^2 - \frac{9}{2}\omega_3^2 - \frac{9}{2}\omega_4^2$

and $P_0 = 24\omega_3^3\omega_4 + 24\omega_3\omega_4^3 + 45\omega_3^2\omega_4^2 + \frac{9}{2}\omega_3^2\omega_4 + \frac{9}{2}\omega_3\omega_4^2$.

Example 4.2. Let F_1 and F_2 be map germs from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^3, 0)$ defined, respectively, by

$$F_1(x, y, u, v) := (x, y, G_1(x, y, u, v))$$

$$\text{with } G_1(x, y, u, v) := yu + xv + uv^2 + u^3,$$

$$F_2(x, y, u, v) := (x, y, G_2(x, y, u, v))$$

$$\text{with } G_2(x, y, u, v) := yu + xv + u^3 + v^3.$$

Note that both the maps are quasi-homogeneous of type $(2, 2, 1, 1; 3)$.

We remark that in these examples the iterated Jacobian ideals $J_{2,1,1}(F_1)$ and $J_{2,1,1}(F_2)$ have the same standard basis $\langle y, x, v^2, uv, u^2\rangle$ which are needed to construct the free resolution.

Applying the result above and substituting $d = 3$, $\omega_1 = 2 = \omega_2$ and $\omega_3 = 1 = \omega_4$, we obtain $\#A_3(F_1) = 3 = \#A_3(F_2)$.

Next, we resume these results in the following:

Theorem 4.3. (1) *For* $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$, $f(x, u, v) = (x, g(x, u, v))$, *we have*

$$\#A_2 = \frac{P_3 d^3 + P_2 d^2 + P_1 d + P_0}{\omega_1\omega_2\omega_3},$$

where $P_3 = 2$, $P_2 = -4\omega_2 - 4\omega_3$, $P_1 = 2\omega_2^2 + 6\omega_2.\omega_3 + 2\omega_3^2$, $P_0 = -2\omega_2^2\omega_3 - 2\omega_2\omega_3^2$.

(2) *If* $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$, $f(x, y, u, v) = (x, y, g(x, y, u) + v^2)$ *is a suspension*, *then*

$$\#A_3 = \frac{d^3 - 6\omega_3 d^2 + 11\omega_3^2 d - 6\omega_3^3}{\omega_1\omega_2\omega_3}.$$

(3) *If* $f : (\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$, $f(x, y, u, v) = (x, y, g(x, y, u, v))$ *is not a suspension*, *then*

$$\#A_3 = \frac{P_4 d^4 + P_3 d^3 + P_2 d^2 + P_1 d + P_0}{\omega_1\omega_2\omega_3\omega_4},$$

where $P_4 = 16$, $P_3 = -3 - 55\omega_4 - 55\omega_3$, $P_2 = \frac{15}{2}\omega_3 + 63\omega_3^2 + 63\omega_4^2 + 139\omega_3\omega_4 + \frac{15}{2}\omega_4$, $P_1 = -24\omega_3^3 - 108\omega_3^2\omega_4 - 24\omega_4^3 - 12\omega_3\omega_4 - 108\omega_3\omega_4^2 - \frac{9}{2}\omega_3^2 - \frac{9}{2}\omega_4^2$ *and* $P_0 = 24\omega_3^3\omega_4 + 24\omega_3\omega_4^3 + 45\omega_3^2\omega_4^2 + \frac{9}{2}\omega_3^2\omega_4 + \frac{9}{2}\omega_3\omega_4^2$.

4.4. A_n **points in** $O_{n+\ell,\,n}$ **with** $\ell > 1$

In the general case of finitely determined map germs from $\mathbb{C}^{n+\ell}$ to $\mathbb{C}^n$ with $\ell > 1$ and $n \geq 3$, even in the co-rank one case, the associated zero set of the iterated Jacobian ideal $J_{\ell+1,\ldots,1}$ is not Cohen-Macaulay, so we cannot compute the number $\#A_n$ as the dimension of the ring $\frac{\mathcal{O}_{n+\ell}}{J_{\ell+1,\ldots,1}}$.

For instance, in the case of co-rank one map germs from $\mathbb{C}^5$ to $\mathbb{C}^4$, the number of points A_4 is different from $\dim_{\mathbb{C}} \frac{\mathcal{O}_5}{J_{2,1,1,1}}$, since we see in [3, Corollary 2.2] that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_5}{J_{2,1,1,1}} = \#A_4 + \#4D_4,$$

where the points A_4 are the butterflies and D_4 denotes the umbilic points.

5. The 0-stable Multiple Singularities and Fitting Ideals

To compute the 0-stable singularities which are multiple points, we use the Fitting ideals associated to the discriminant of the germ, described by Mond and Pellikaan in [18]. We apply the results given recently in [14], where it is shown how to obtain these singularities using the Fitting ideals. For more details on the definition and main properties of the Fitting ideals, see [18].

5.1. Map germs from $\mathbb{C}^3$ to $\mathbb{C}^2$

For any stable map germ *f* from $\mathbb{C}^3$ to $\mathbb{C}^2$, the discriminant of *f*, $\Delta(f) = f(\Sigma(f))$ is a curve in $\mathbb{C}^2$, possible with isolated singularities (the 0-stable singularities) of two types, the double points set denoted $A_{1,1}$ and the cusps set, denoted A_2. There are two Fitting ideals associated to it, which are denoted by $\mathcal{F}_0(f)$ and $\mathcal{F}_1(f)$. They define in $\Delta(f)$ the following sets:

1. $V(\mathcal{F}_0(f)) := \Delta(f)$ or $\mathcal{F}_0(f)$ is the defining ideal of the discriminant curve.

2. $V(\mathcal{F}_1(f)) = A_{1,1} \cup A_2$, union of the isolated singularities in the target.

First, we show the following formula which relates the ideal $\mathcal{F}_1$ and the 0-stable singularities.

Lemma 5.1. *For any finitely determined map germ of co-rank one* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ *with* $n \geq 2$, *we have*

$$\#A_2 + \#A_{1,1} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2, 0}}{\mathcal{F}_1(f)}.$$

For the special case of map germs from the plane to the plane, this formula was shown by Gaffney and Mond in [4].

Proof. In this case, the critical curve $\Sigma(f)$ is reduced, then from [18, Theorem 5.2], we have that the zero set of the ideal $\mathcal{F}_1(f)$ is determinantal, hence it is Cohen-Macaulay and then we have a flat deformation on the basis of the versal unfolding, from the law of conservation of multiplicity we obtain that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\mathcal{F}_1(f)} = c_2(\#A_2) + c_{1,1}(\#A_{1,1}),$$

where c_i means the contribution of the singularity A_i in the ideal $\mathcal{F}_1(f)$. A straightforward calculation using the normal form of these singularities shows us that the numbers c_i are equal to 1 in these cases. □

Applying the formula to compute the number of points A_2 given before and this we obtain a formula to compute the number of points $A_{1,1}$.

5.2. Map germs from $\mathbb{C}^4$ to $\mathbb{C}^3$

Here the Fitting ideals which appear in the discriminant $\Delta(f) = f(\Sigma(f)) \subset \mathbb{C}^3$ of f are $\mathcal{F}_0(f)$, $\mathcal{F}_1(f)$ and $\mathcal{F}_2(f)$ and they define in $\Delta(f)$ the following sets:

1. $V(\mathcal{F}_0(f)) := \Delta(f)$, here $\mathcal{F}_0(f)$ is the defining ideal of the discriminant.

2. $V(\mathcal{F}_1(f)) = f(D(f)) \cup f(\Sigma^{2,1}(f))$, union of the double points curve and the cuspidal edges curve of f.

3. $V(\mathcal{F}_2(f)) = A_{1,2} \cup A_{1,1,1} \cup A_3$, union of the isolated singularities in the target.

Recently, it is shown in [14] that the formulae relate these ideals and the 0-stable singularities.

Proposition 5.2 [14, Corollary 4.1]**.** *Let* $f : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^m, 0)$ *with* $m \geq 3$, *be a finitely determined map germ of co-rank one with only* *singularities* A_P *in its discriminant*. *Then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_m}{\mathcal{F}_{m-1}(f)} = \sum_{|P|=m} \# A_P.$$

Therefore, when $m = 3$ and $n = 1$, we obtain

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{(\mathbb{C}^3, 0)}}{\mathcal{F}_2(f)} = \# A_{1,2} + \# A_{1,1,1} + \# A_3. \qquad (*)$$

and in [14, Proposition 4.3], the following formula is shown:

$$\# A_{(1,1,1)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I(A_{1,1})^2 : \mathcal{F}_0(f))}. \qquad (**)$$

The notation $(I : J)$ means the quotient ideal of an ideal I by another ideal J in a ring A or

$$(I : J) := \{a \in A;\ aJ \subset I\}.$$

Here $I(A_{1,1})^2$ denotes the defining ideal of the cuspidal curve in the discriminant of f.

If f is of co-rank one, we also see in [14]:

$$\# A_{1,2} := \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)} - \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I(A_{1,1})^2 : \mathcal{F}_0(f))} - \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{J_{2,1,1}(f)}. \qquad (***)$$

Example 5.3. Let $f(x, y, u, v) := (x, y, yu + xv + uv^2 + u^3)$. First, we obtain $\# A_3 = 3$ using Theorem 4.3. To compute the numbers $\# A_{1,1,1}$ and $\# A_{2,1}$ of f, we compute the Fitting ideals of the discriminant:

$$\mathcal{F}_0(f) = \langle 27Z^4 + 36X^2YZ^2 + 4Y^3Z^2 + 4X^6 + 8X^4Y^2 + 4X^2Y^4 \rangle,$$

$$\begin{aligned}\mathcal{F}_1(f) = \langle & 3X^2Z - Y^2Z,\ 9XZ^2 + 2X^3Y + 2XY^3,\ 3YZ^2 + 2X^4 \\ & + 2X^2Y^2,\ 9Z^3 + 2X^2YZ + 2Y^3Z,\ 3X^5 + 2X^3Y^2 - XY^4 \rangle\end{aligned}$$

and

$$\mathcal{F}_2(f) = \langle X^2,\ XY,\ Y^2,\ XZ,\ YZ,\ Z^2 \rangle.$$

To get the number of triple points, we compute the defining ideal of the cuspidal edge, which is generated by $\{3X^2 - Y^2,\ 27Z^2 + 8Y^3\}$ and apply the formula (∗∗) above $\#A_{1,1,1} := \dim_{\mathbb{C}} \dfrac{\mathcal{O}_3}{\left(\mathcal{F}_2(f) : I\left(f\left(\sum^{n+1,1}(f)\right)\right)\right)} = 0.$

From equation (∗∗∗), we get $\#A_{1,2} := 1,$ since $\dim_{\mathbb{C}} \dfrac{\mathcal{O}_3}{\mathcal{F}_2(f)} = 4.$

6. Invariants of the 1-stable Singularities

We see in [20] that there are some invariants associated to the one-dimensional singularities which appear in the discriminant of f that can be computed in terms of the number of 0-stable singularities. These invariants are the polar multiplicities and the Euler obstruction. We apply the results of [20] to show here how to compute some of these invariants in terms of the weights and degrees.

6.1. The cuspidal curve $f(\Sigma^{2,1}(f))$

We recover the result of [20] which shows the relationship among the first polar multiplicity of the cuspidal curve $f(\Sigma^{(n-2,1)}(f))$ and the number of A_n points.

Proposition 6.1 [20, Corollary 4.2]**.** *Let* $f(x_1, x_2, x_3, x_4) = (x_1, x_2, g(x_1, x_2, x_3, x_4))$ *be a finitely determined, quasi-homogeneous, co-rank one map germ with weights* $w_1, w_2, w_3, w_4,$ *where* $w_1 < w_2,$ $w_3 < w_4$ *and let d be the weighted degree of g with* $d > w_1$ *and* $d > w_2.$

The first polar multiplicity at the origin of the cuspidal curve $f(\Sigma^{(n-2,1)}(f))$ *denoted* $m_1(f(\Sigma^{2,1}(f)))$ *can be computed as*

$$m_1(f(\Sigma^{2,1}(f))) = \sum_{\rho=2}^{4}\prod_{\nu=1}^{4}\left(\frac{D_\rho}{w_\nu}-1\right)\prod_{\substack{\kappa=2\\ \kappa\neq\rho}}^{4}\left(\frac{1}{\frac{D_\rho}{D_\kappa}-1}\right)\left[\prod_{\mu=2}^{4}\left(\frac{1}{\frac{D_\mu}{D_1}-1}\right)+1\right] - \#A_3$$

with $D_1 = w_1$, $D_2 = 2(d - w_3 - w_4)$, $D_3 = d - w_3$ *and* $D_4 = d - w_4$.

Now, we use this formula and the formula given in Subsection 4.3 for the number of points A_3 to compute directly the first polar multiplicity $m_1(f(\Sigma^{2,1}(f)))$ in terms of the weights and degrees.

Corollary 6.2. *Let* $f(x_1, x_2, x_3, x_4) = (x_1, x_2, g(x_1, x_2, x_3, x_4))$ *be a finitely determined, quasi-homogeneous, co-rank one map germ with weights* w_1, w_2, w_3, w_4, *where* $w_1 < w_2$, $w_3 < w_4$ *and let d be the weighted degree of g with* $d > w_1$ *and* $d > w_2$. *Then*

$$m_1(f(\Sigma^{2,1}(f))) = \sum_{\rho=2}^{4}\prod_{\nu=1}^{4}\left(\frac{D_\rho}{w_\nu}-1\right)\prod_{\substack{\kappa=2\\ \kappa\neq\rho}}^{4}\left(\frac{1}{\frac{D_\rho}{D_\kappa}-1}\right)$$

$$\cdot\left[\prod_{\mu=2}^{4}\left(\frac{1}{\frac{D_\mu}{D_1}-1}\right)+1\right] - \frac{P_4 d^4 + P_3 d^3 + P_2 d^2 + P_1 d + P_0}{\omega_1\omega_2\omega_3\omega_4}$$

with $P_4 = 16$, $P_3 = -3 - 55\omega_4 - 55\omega_3$, $P_2 = \frac{15}{2}\omega_3 + 63\omega_3^2 + 63\omega_4^2 + 139\omega_3\omega_4 + \frac{15}{2}\omega_4$, $P_1 = -24\omega_3^3 - 108\omega_3^2\omega_4 - 24\omega_4^3 - 12\omega_3\omega_4 - 108\omega_3\omega_4^2 - \frac{9}{2}\omega_3^2 - \frac{9}{2}\omega_4^2$ *and* $P_0 = 24\omega_3^3\omega_4 + 24\omega_3\omega_4^3 + 45\omega_3^2\omega_4^2 + \frac{9}{2}\omega_3^2\omega_4 + \frac{9}{2}\omega_3\omega_4^2$.

We also see in [20] that the local Euler obstruction of the cuspidal curve of any weighted homogeneous co-rank one map germ is calculated in terms of the weights and degrees of *f* and also in terms of the number of points A_3.

We remember that, in fact, the local Euler obstruction for varieties, introduced in [12] by MacPherson in a purely obstructional way, is an invariant that is also associated to the polar multiplicities and Tráng and Teissier in [11] showed that the local Euler obstruction is an alternate sum of the multiplicity of the local polar varieties.

Theorem 6.3 (Tráng and Teissier [11]). *Let X be a reduced analytic space at $0 \in \mathbb{C}^{n+1}$ of dimension d. Then*

$$Eu_0(X) = \sum_{i=0}^{d-1} (-1)^{d-i-1} m_i(X),$$

where $m_i(X)$ denotes the absolute polar multiplicity of the polar variety $P_i(X)$.

We see in [10] the formula for the Euler obstruction at the origin of $f(\Sigma^{n-2,1}(f))$.

Proposition 6.4 [10]. *Let $f \in \mathcal{O}(n, 3)$, $n > 3$ be a finitely determined map germ. Then*

$$Eu_0(f(\Sigma^{n-2,1}(f))) = \#A_3 - \mu(\Sigma^{n-2,1}(f)) + 1 + m_1(f(\Sigma^{n-2,1}(f))).$$

And from the second equation of the Proposition 6.1 we have the following formula to compute the Euler obstruction in terms of the weights and degrees.

Corollary 6.5 [20]. *Let $f(x_1, x_2, x_3, x_4) = (x_1, x_2, g(x_1, x_2, x_3, x_4))$ be a finitely determined, quasi-homogeneous, co-rank one map germ with weights w_1, w_2, w_3, w_4, where $w_1 < w_2$, $w_3 < w_4$ and let d be the weighted degree of g with $d > w_1$ and $d > w_2$. Then*

$$Eu_0(f(\Sigma^{2,1}(f))) = \sum_{\rho=1}^{4} \prod_{\nu=1}^{4} \left(\frac{D_\rho}{w_\nu} - 1 \right) \prod_{\substack{\kappa=1 \\ \kappa \neq \rho}}^{4} \left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1} \right) + 1$$

with $D_1 = w_1$, $D_2 = 2d - 2(\omega_3 + \omega_4)$, $D_3 = d - w_3$, $D_4 = d - w_4$.

6.2. The double points curve $f(D(f))$

We consider now the invariants associated to the double points curve $f(D(f))$.

We shall use the equation $V(\mathcal{F}_1) = f(D(f)) \cup f(\Sigma^{2,1}(f))$ and the results of [10], where it is shown how the polar multiplicities are related with the 0-stable singularities of the discriminant.

Now, we show how the polar multiplicities of $f(D_1^2(f \mid \Sigma(f)))$ are related. For this set, there are two polar multiplicities and the relation between them is given in terms of the Milnor number of the set $f(D_1^2(f \mid \Sigma(f)))$ and also of the number of isolated singularities of *f*.

Theorem 6.6 [10, Theorem 4.6]**.** *Let* $f(\mathbb{C}^4, 0) \to (\mathbb{C}^3, 0)$ *be a finitely determined map germ of co-rank one*. *Then*

$$2(m_0(f(D^2(f \mid \Sigma(f)))) - m_1(f(D^2(f \mid \Sigma(f)))))$$

$$= -\mu(D_1^2(f \mid \Sigma(f))) + 3\#A_{(1,2)} + 3\#A_3 + 6\#A_{(1,1,1)} + 1.$$

Therefore, from this formula and the formula given by Tráng and Teissier, we get a formula to compute the Euler obstruction in terms of all 0-stable singularities, points A_3, $A_{(1,2)}$ and $A_{(1,1,1)}$ and also of the Milnor number of the double points curve.

Corollary 6.7.

$$Eu_0(f(D^2(f))) = \frac{-\mu(D_1^2(f \mid \Sigma(f))) + 3\#A_{(1,2)} + 3\#A_3 + 6\#A_{(1,1,1)} + 1}{2}.$$

7. Appendix

The main purpose of this Appendix is to give an example which shows clearly how can we construct the free resolution of an ideal in terms of the modules of syzygies. This method is very powerful to compute such resolutions and is done in terms of the standard bases of the modules. We fix here the notation of Section 4 which is based in the notation of [6].

We shall recover next the definition of the syzygies.

Definition 7.1. Let R be an arbitrary ring. A *syzygy* between k elements $h_1, \ldots, h_k$ of an R-module M is a k-tuple $(r_1, \ldots, r_k) \in R^k$ satisfying $\sum_{i=1}^{k} r_i h_i = 0$.

The set of all syzygies between $h_1, \ldots, h_k$, denoted by $syz(h_1, \ldots, h_k)$, is a submodule of R^k, it is the kernel of the ring homomorphism $\phi : F_1 := \bigoplus_{i=1}^{k} R\varepsilon_i \to M$ with $\varepsilon_i \mapsto r_i$, where $\{\varepsilon_1, \ldots, \varepsilon_k\}$ denotes the canonical basis of R^k. If $I = \langle h_1, \ldots, h_k \rangle_R$, then we provide the following:

Definition 7.2. $syz(I) := syz(h_1, \ldots, h_k) := \ker(\phi)$, the module of syzygies of I with respect to the generators $\{h_1, \ldots, h_k\}$. Besides we define inductively the kth syzygy module: $syz_k(I) := syz(syz_{k-1}(I))$, setting $syz_0(I) := I$.

The existence of such syzygies is guaranteed by the Buchberger's criterion for the existence of a standard basis of $syz(I)$.

The first step is the construction of a standard basis for the first $syz(I)$.

If $R = K[x]$, $f \in R^r \backslash \{0\}$ and > denotes a module ordering, then f can be written uniquely as $f = cx^{\alpha}e_i + f^*$, with $c \in K\backslash\{0\}$ and $x^{\alpha}e_i > x^{\alpha^*}e_j$ for any non-zero term $c^* x^{\alpha^*} e_j$ of f^*.

So we define: the *leading monomial* by $LM(f) = x^{\alpha}e_i$, the *leading coefficient* by $LC(f) = c$, the leading term by $LT(f) = cx^{\alpha}e_i$ and the *tail* of f by $tail(f) := f - LT(f)$. Also, if $G \subset R^r$, then we define the leading submodule of $\langle G \rangle$ by $L(G) = \langle LM(g) : g \in G\backslash\{0\}\rangle_R$.

Now, let $G = \{f_1, ..., f_k\}$ be a standard base of $I = \langle f_1, ..., f_k \rangle$, with $f_i \in R^r \setminus \{0\}$, $\forall i \in \{1, ..., k\}$ (i.e., $G \subset I$ and the leading ideals coincide: $L(I) = L(G)$; we note that in the global case, it is also called *Groebner base*). For each $i \neq j$ such that f_i and f_j have their leading terms in the same component, i.e., $LM(f_i) = x^{\alpha_i} e_\nu$, $LM(f_j) = x^{\alpha_j} e_\nu$, we define the monomial $m_{ji} = x^{\lambda - \alpha_i} \in R$, where

$$\lambda := lcm(\alpha_i, \alpha_j) := (\max(\alpha_i^1, \alpha_j^1), ..., \max(\alpha_i^r, \alpha_j^r))$$

is the least common multiple of α_i and α_j, $c_i = LC(f_i)$, $c_j = LC(f_j)$. Then the s-polynomial of f_i and f_j is given by $spoly(f_i, f_j) = m_{ji} f_i - \frac{c_i}{c_j} m_{ij} f_j$. We can assume that $spoly(f_i, f_j)$ has a standard representation: $m_{ij} f_i - \frac{c_i}{c_j} m_{ij} f_j = \sum_{\nu-1}^{k} a_\nu^{(ij)} f_\nu$, $a_\nu^{(ij)} \in R$.

Now, for $i < j$ such that $LM(f_i)$ and $LM(f_j)$ involve the same component, define

$$s_{ij} = m_{ji} \varepsilon_i - \frac{c_i}{c_j} m_{ij} \varepsilon_j - \sum_{\nu-1}^{k} a_\nu^{ij} f_\nu.$$

It is possible to show that $s_{ij} \in syz(I)$.

With these notation, the construction of a base for $syz(I)$ is described below.

Theorem 7.3 ([6, Theorem 2.5.9], see also [1, Theorem 15.10])**.** *Let* $G = \{h_1, ..., h_k\}$ *be a set of generators of* $I \subset R^r$.

Let $P := \{(i, j), 1 \leq i < j \leq k$ *such that the leading terms of the* r_i *and* r_j *involve in the same component*$\}$ *and let* $J \subset P$.

Assume that $NF(spoly(h_i, h_j) \mid G_{ij}) = 0$ *for some* $G_{ij} \subset G$, $(i, j) \in J$ *and for* $i = 1, \ldots, r$ *we have the equality*

$$\langle \{m_{ij}\varepsilon_i \mid (i, j) \in J\} \rangle = \langle \{m_{ij}\varepsilon_i \mid (i, j) \in P\} \rangle.$$

Then the following statements hold: G *is a standard basis of* I (*Buchberger's criterion*) *and* $S := \{s_{ij} \mid (i, j) \in J\}$ *is a standard basis of* $syz(I)$.

Example 7.4. Let F be the finitely determined co-rank one map germ from $(\mathbb{C}^4, 0)$ to $(\mathbb{C}^3, 0) : F(x, y, u, v) := (x, y, yu + xv + uv^2 + u^3)$. Note that this map is quasi-homogeneous of type $(2, 2, 1, 1; 3)$ and $J_{2,1,1}(F)$ has the standard base $I = \langle y, x, v^2, uv, u^2 \rangle$.

The first syzygy module M_1 is formed by the generators of the ideal $I = \langle y, x, v^2, uv, u^2 \rangle$, so $M_1 = (y \;\; x \;\; v^2 \;\; uv \;\; u^2) \in M_{1\times 5}(\mathcal{O}_4)$.

Numbering the elements of $G : g_1 = y$, $g_2 = x$, $g_3 = v^2$, $g_4 = uv$ and $g_5 = u^2$. We are admitting a monomial ordering $>$ such that $LM(g_1) > LM(g_2) > LM(g_3) > LM(g_4 > LM(g_5))$.

The respective monomials $m_{ij}\varepsilon_i$, $1 \le i < j \le 5$ are given in the following table:

$i \diagdown j$	2	3	4	5
1	$x\epsilon_1$	$v^2\epsilon_1$	$uv\epsilon_1$	$u^2\epsilon_1$
2	–	$v^2\epsilon_2$	$uv\epsilon_2$	$u^2\epsilon_2$
3	–	–	$u\epsilon_3$	$u^2\epsilon_3$
4	–	–	–	$u\epsilon_4$

Hence, we may choose

$$J := \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (4, 5)\}$$

and compute

$$s_{1,2} = m_{2,1}\varepsilon_1 - m_{1,2}\varepsilon_2 - spoly(g_1, g_2) = x\varepsilon_1 - y\varepsilon_2 - 0 = (x, -y, 0, 0, 0),$$

$$s_{1,3} = m_{3,1}\varepsilon_1 - m_{1,3}\varepsilon_3 - spoly(g_1, g_3) = v^2\varepsilon_1 - y\varepsilon_3 - 0 = (v^2, 0, -y, 0, 0),$$

$$s_{1,4} = m_{4,1}\varepsilon_1 - m_{1,4}\varepsilon_4 - spoly(g_1, g_4) = uv\varepsilon_1 - y\varepsilon_4 - 0 = (uv, 0, 0, -y, 0),$$

$$s_{1,5} = m_{5,1}\varepsilon_1 - m_{1,5}\varepsilon_5 - spoly(g_1, g_5) = u^2\varepsilon_1 - y\varepsilon_5 - 0 = (u^2, 0, 0, 0, -y),$$

$$s_{2,3} = m_{3,2}\varepsilon_2 - m_{2,3}\varepsilon_3 - spoly(g_2, g_3) = v^2\varepsilon_2 - x\varepsilon_3 - 0 = (0, v^2, -x, 0, 0),$$

$$s_{2,4} = m_{4,2}\varepsilon_2 - m_{2,4}\varepsilon_4 - spoly(g_2, g_4) = uv\varepsilon_2 - x\varepsilon_4 - 0 = (0, uv, 0, -x, 0),$$

$$s_{2,5} = m_{5,2}\varepsilon_2 - m_{2,5}\varepsilon_5 - spoly(g_2, g_5) = u^2\varepsilon_2 - x\varepsilon_5 - 0 = (0, u^2, 0, 0, -x),$$

$$s_{3,4} = m_{4,3}\varepsilon_3 - m_{3,4}\varepsilon_4 - spoly(g_3, g_4) = u\varepsilon_3 - v\varepsilon_4 - 0 = (0, 0, u, -v, 0),$$

$$s_{4,5} = m_{5,4}\varepsilon_4 - m_{4,5}\varepsilon_5 - spoly(g_4, g_5) = u\varepsilon_4 - v\varepsilon_5 - 0 = (0, 0, 0, u, -v).$$

The set $S := \{s_{1,2}, s_{1,3}, s_{1,4}, s_{1,5}, s_{2,3}, s_{2,4}, s_{2,5}, s_{3,4}, s_{4,5}\}$ is an interreduced standard basis for $syz(I) := M_2$. Therefore, by Theorem 7.3, the second syzygy module is generated by the columns of the matrix

$$M_2 = syz(I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x & v^2 & uv & u^2 \\ 0 & 0 & v^2 & uv & u^2 & -y & 0 & 0 & 0 \\ 0 & u & -x & 0 & 0 & 0 & -y & 0 & 0 \\ u & -v & 0 & -x & 0 & 0 & 0 & -y & 0 \\ -v & 0 & 0 & 0 & -x & 0 & 0 & 0 & -y \end{pmatrix} \in M_{5\times 9}(\mathcal{O}_4).$$

Now, numbering the generators of $syz(I)$ we call:

$$h_1 = (0, 0, 0, u, -v),\ h_2 = (0, 0, u, -v, 0),\ h_3 = (0, v^2, -x, 0, 0),$$

$$h_4 = (0,\ uv,\ 0,\ -x,\ 0),\ h_5 = (0,\ u^2,\ 0,\ 0,\ -x),\ h_6 = (x,\ -y,\ 0,\ 0,\ 0),$$

$$h_7 = (v^2,\ 0,\ -y,\ 0,\ 0),\ h_8 = (uv,\ 0,\ 0,\ -y,\ 0),\ h_9 = (u^2,\ 0,\ 0,\ 0,\ -y).$$

So, we see that the set M of pairs $(i,\ j),\ 1 \leq i < j \leq 9$ such that the leading monomials of the ith and jth elements of S involve the same components consisting of 7 elements:

$$M = \{(3,\ 4),\ (4,\ 5),\ (6,\ 7),\ (6,\ 8),\ (6,\ 9),\ (7,\ 8),\ (8,\ 9)\}$$

and the respective monomials $m_{ij}\varepsilon_i$, for $1 \leq i < j \leq 9$, are given in the following tables:

$i \backslash j$	4	5
3	$u\epsilon_3$	$\cancel{u^2\epsilon_3}$
4	–	$u\epsilon_4$
5	–	–

,

$i \backslash j$	7	8	9
6	$v^2\epsilon_6$	$uv\epsilon_6$	$u^2\epsilon_6$
7	–	$u\epsilon_7$	$\cancel{u^2\epsilon_7}$
8	–	–	$u\epsilon_8$
9	–	–	–

We compute

$$s_{3,4}^{(1)} = m_{4,3}\varepsilon_3 - m_{3,4}\varepsilon_4 - spoly(h_3,\ h_4) = u\varepsilon_3 - v\varepsilon_4 - (-x)\varepsilon_2$$

$$= (0,\ x,\ u,\ -v,\ 0,\ 0,\ 0,\ 0,\ 0),$$

$$s_{4,5}^{(1)} = m_{5,4}\varepsilon_4 - m_{4,5}\varepsilon_5 - spoly(h_4,\ h_5) = u\varepsilon_4 - v\varepsilon_5 - (-x)\varepsilon_1$$

$$= (x,\ 0,\ 0,\ u,\ -v,\ 0,\ 0,\ 0,\ 0),$$

$$s_{6,7}^{(1)} = m_{7,6}\varepsilon_6 - m_{6,7}\varepsilon_7 - spoly(h_6,\ h_7) = v^2\varepsilon_6 - x\varepsilon_7 - (-y)\varepsilon_3$$

$$= (0,\ 0,\ y,\ 0,\ 0,\ v^2,\ -x,\ 0,\ 0),$$

$$s_{6,8}^{(1)} = m_{8,6}\varepsilon_6 - m_{6,8}\varepsilon_8 - spoly(h_6,\ h_8) = uv\varepsilon_6 - x\varepsilon_8 - (-y)\varepsilon_4$$

$$= (0,\ 0,\ 0,\ y,\ 0,\ uv,\ 0,\ -x,\ 0),$$

$$s^{(1)}_{6,9} = m_{9,6}\varepsilon_6 - m_{6,9}\varepsilon_9 - spoly(h_6, h_8) = u^2\varepsilon_6 - x\varepsilon_9 - (-y)\varepsilon_5$$

$$= (0, 0, 0, 0, y, u^2, 0, 0, -x),$$

$$s^{(1)}_{7,8} = m_{8,7}\varepsilon_7 - m_{7,8}\varepsilon_8 - spoly(h_7, h_8) = u\varepsilon_7 - v\varepsilon_8 - (-y)\varepsilon_2$$

$$= (0, y, 0, 0, 0, 0, u, -v, 0),$$

$$s^{(1)}_{8,9} = m_{9,8}\varepsilon_8 - m_{8,9}\varepsilon_9 - spoly(h_8, h_9) = u\varepsilon_8 - v\varepsilon_9 - (-y)\varepsilon_1$$

$$= (y, 0, 0, 0, 0, 0, 0, u, -v).$$

The set $S^{(1)} := \{s^{(1)}_{3,4}, s^{(1)}_{4,5}, s^{(1)}_{6,7}, s^{(1)}_{6,8}, s^{(1)}_{6,9}, s^{(1)}_{7,8}, s^{(1)}_{8,9}\}$ is an interreduced standard basis for $syz(syz(I)) = syz(M_2) := M_3$. Therefore, by Theorem 7.3, the third syzygy module is generated by the columns of the matrix

$$M_3 = syz(M_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & x & y & 0 & 0 \\ 0 & 0 & y & u & 0 & 0 & 0 \\ 0 & y & 0 & -v & 0 & u & 0 \\ y & 0 & 0 & 0 & 0 & -v & 0 \\ u^2 & uv & v^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & u & 0 & 0 \\ 0 & -x & 0 & 0 & -v & 0 & u \\ -x & 0 & 0 & 0 & 0 & 0 & -v \end{pmatrix} \in M_{9\times 7}(\mathcal{O}_4).$$

Now, numbering the generators of $syz(syz(I)) : l_1 = (0, 0, 0, 0, y, u^2, 0, 0, -x)$, $l_2 = (0, 0, 0, y, 0, uv, 0, -x, 0)$, $l_3 = (0, 0, y, 0, 0, v^2, -x, 0, 0)$, $l_4 = (0, x, u, -v, 0, 0, 0, 0, 0)$, $l_5 = (0, y, 0, 0, 0, 0, u, -v, 0)$, $l_6 = (x, 0, 0, u, -v, 0, 0, 0, 0)$, $l_7 = (y, 0, 0, 0, 0, 0, 0, u, -v)$, we see that the set N of pairs (i, j), $1 \le i < j \le 7$ such that the leading monomials of the ith and jth elements of $S^{(1)}$ involve the same components, consists of 2 elements:

$N = \{(4, 5), (6, 7)\}$ and the respective monomials $m_{ij}\varepsilon_i, 1 \leq i < j \leq 7,$ are given in the following tables:

$i \diagdown j$	5
4	$y\epsilon_4$
5	–

,

$i \diagdown j$	7
6	$y\epsilon_6$
7	–

We compute

$$s_{4,5}^{(2)} = m_{5,4}\varepsilon_4 - m_{4,5}\varepsilon_5 - spoly(l_4, l_5) = y\varepsilon_4 - x\varepsilon_5 + u\varepsilon_3 - v\varepsilon_2$$

$$= (0, -v, u, y, -x, 0, 0),$$

$$s_{6,7}^{(2)} = m_{7,6}\varepsilon_6 - m_{6,7}\varepsilon_7 - spoly(l_6, l_7) = y\varepsilon_6 - x\varepsilon_7 + u\varepsilon_2 - v\varepsilon_1$$

$$= (-v, u, 0, 0, 0, y, -x).$$

The set $S^{(2)} := \{s_{4,5}^{(2)}, s_{6,7}^{(2)}\}$ is an interreduced standard basis for $syz(syz(syz(I))) = syz(M_3) := M_4$. Therefore, by Theorem 7.3, the fourth syzygy module is generated by the columns of the matrix

$$M_4 = syz(M_3) = \begin{pmatrix} 0 & -v \\ -v & u \\ u & 0 \\ y & 0 \\ -x & 0 \\ 0 & y \\ 0 & -x \end{pmatrix} \in M_{7\times 2}(\mathcal{O}_4).$$

Thus, we obtain the free resolution for $\dfrac{\mathcal{O}_4}{J_{2,1,1}(f)}$,

$$0 \to \bigoplus_{i=1}^{2} \mathcal{O}_4 \xrightarrow{M_4} \bigoplus_{i=1}^{7} \mathcal{O}_4 \xrightarrow{M_3} \bigoplus_{i=1}^{9} \mathcal{O}_4 \xrightarrow{M_2} \bigoplus_{i=1}^{5} \mathcal{O}_4 \xrightarrow{M_4} \mathcal{O}_4 \xrightarrow{\pi} \frac{\mathcal{O}_4}{J_{2,1,1}(f)} \to 0.$$

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