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A CODE FOR DISCONNECTED EDGE-COLORED GRAPHS

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Abstract

We extend the definition of code of an edge-colored graph, given in [11] and [9], to the disconnected case and prove that our code keeps the same property of detecting color-isomorphic graphs.

1. Introduction

Edge-colored graphs are a combinatorial tool for representing PL-manifolds. This representation theory, which started in the 70's, makes use of a particular class of edge-colored graphs, called *crystallizations*, and has strict relations, in low dimensions, with other classical or more recent representation theories, such as Heegaard diagrams, special spines ([12]), face-pairing graphs ([1]). On the other hand, crystallization theory can be

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applied to any dimension n and to all compact PL n-manifolds, without restrictions on orientability, boundary, connectedness etc. (see [2], [10] and [3] for a survey about the theory).

Representing manifolds by edge-colored graphs has the further advantage of allowing their encoding and manipulation by computer. The easiest way of representing an edge-colored graph by numerical data is a kind of incident matrix, which is nevertheless too redundant and dependent on a given labelling of the vertex-set and on a given permutation of the set for the coloration of the edges.

As a consequence, this kind of encoding is not suitable for most problems occurring within the theory. For instance, if we want to generate and analyze catalogues of crystallizations, as was done in [4], [6], [8], and [5], then it is necessary to be able to recognize the same combinatorial colored structure independently from different labellings of vertex-sets and permutations of colors.

In order to fulfill this aim, the concept of isomorphism of edge-colored graphs has been introduced in [9]; in the same paper, by extending the procedure introduced in [11] for the bipartite case, an algorithm is presented to compute a numerical code of a connected m-bipartite edge-colored graph, which is invariant under isomorphisms.

In this paper, we extend furthermore both the definition of isomorphism of colored graphs and the algorithm for the code to the case of disconnected graphs.

The algorithm has been implemented in a C++ function, which is part of *Duke* program for manipulating and studying edge-colored graphs[1], as a representation tool for PL-manifolds.

We point out that our concept and results, as in [9], apply to the whole class of edge-colored graphs, without any reference to their possible

[1]Duke program can be downloaded from the WEB page http://cdm.unimo.it/home/matematica/casali.mariarita/DukeIII.htm; details about it are available at the same page.

topological meaning. However, in the last section of the paper we present an alternative definition of isomorphism for colored graphs, together with its application to the computation of the code, and discuss the usefulness of both definitions in view of their applicability to the resolution of topological problems.

2. Preliminaries

An $(n+1)$-*colored graph* is a pair (Γ, γ), where Γ is a multigraph, whose vertices have degree n or $n+1$, and $\gamma : E(\Gamma) \to \Delta_n = \{0, ..., n\}$ a map which is injective on each pair of adjacent edges of Γ. The *boundary* (resp. *internal*) vertices of Γ are the vertices with degree n (resp. $n+1$). The $(n+1)$-colored graph (Γ, γ) is called *without boundary* if all its vertices are internal, i.e., Γ is regular of degree $n+1$. From now on, we assume that, in the case of non-empty boundary, the missing color is always n.

For each $B \subseteq \Delta_n$, we call *B-residues* of (Γ, γ) the connected components of the colored graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$; given an integer $m \in \{1, ..., n\}$, we call *m-residue* of Γ each B-residue of Γ with $\#B = m$.

An $(n+1)$-colored graph (Γ, γ) is said to be *m-bipartite* if all m-residues of Γ are bipartite and there exists at least one $(m+1)$-residue which is not bipartite.

Note that $(n+1)$-bipartite means bipartite in the usual sense and that every $(n+1)$-colored graph is at least 2-bipartite.

In handling m-bipartite graphs we have to be careful with regard to the existence of the boundary; more precisely, for each $(n+1)$-colored graph (Γ, γ), it is useful to define an integer $\overline{m}(\Gamma)$, with $2 \le \overline{m}(\Gamma) \le n+1$, as follows:

$$\overline{m}(\Gamma) = \begin{cases} n & \Gamma \text{ is bipartite and has non-empty boundary,} \\ m & \Gamma \text{ is } m\text{-bipartite or has no boundary.} \end{cases}$$

Definition 1. A graph-isomorphism $\phi : \Gamma \to \Gamma'$ is called an *isomorphism* between the $(n+1)$-colored graphs (Γ, γ) and (Γ', γ') if there exists a permutation σ of Δ_n such that $\sigma \circ \gamma = \gamma' \circ \phi$.

The isomorphism problem for connected edge-colored graphs has already been solved in [9] by means of a numerical *code*, which distinguishes colored graphs up to isomorphisms. In the following section we will extend this definition to disconnected graphs and we will prove that its property of detecting color-isomorphic graphs is preserved.

3. Encoding Disconnected Colored Graphs

In the following, when the coloration of the edges of a graph is clearly understood, we shall often write Γ instead of (Γ, γ).

Let (Γ, γ) be an $(n+1)$-colored graph with r connected components $\Gamma_1, \ldots, \Gamma_r$ (possibly $r = 1$) and suppose that for each $i = 1, \ldots, r$, $V(\Gamma_i)$ is of order $2q_i$; moreover set $m_i = \overline{m}(\Gamma_i)$. By a vertex-labelling of Γ we mean a bijective map $l : V(\Gamma) \to I_{2q}$, where $q = \sum_{i=1}^{r} q_i$, I_{2q} being a subset of $\mathbb{Z}$, not containing 0, of cardinality $2q$. For sake of simplicity, for each $j \in I_{2q}$, we will denote by v_j the vertex $v \in V(\Gamma)$ such that $l(v) = j$.

For each $(n+1)$-colored graph (Γ, γ) and for each vertex-labelling l of Γ, we define a matrix $\mathcal{A}(\Gamma, \gamma, l) = (a_c^j)$ $(j \in I_{2q}, c \in \Delta_n)$ by

$$a_c^j = \begin{cases} 0 & \text{if } c = n \text{ and } v_j \text{ is a boundary-vertex,} \\ k & \text{if } v_j \text{ and } v_k \text{ are } c\text{-adjacent.} \end{cases}$$

Remark 1. Given two $(n+1)$-colored graphs (Γ, γ) and (Γ', γ'), let us suppose that they are color-isomorphic by the isomorphism ϕ and the color-permutation σ, i.e., $\sigma \circ \gamma = \gamma' \circ \phi$; then there are vertex-labellings l of Γ and l' of Γ' respectively, by means of the same set I_{2q}, such that $l = l' \circ \phi$

and, consequently, $\mathcal{A}(\Gamma, \sigma \circ \gamma, l) = \mathcal{A}(\Gamma', \gamma', l')$. Moreover, it is easy to see that, if Γ and Γ' have non-empty boundary, σ fixes color n.

For each Γ, we define the set of *admissible* permutations $\overline{H}(\Gamma)$, which coincides with the set of all permutations of Δ_n in case $\partial\Gamma = \emptyset$ and with the subset of permutations of Δ_n fixing n, in case $\partial\Gamma \neq \emptyset$.

Let us now consider a permutation $\pi \in \overline{H}(\Gamma)$. For each $i = 1, \ldots, r$ and for each $r_i \in V(\Gamma_i)$, let $N_{r_i, \pi}$ be the vertex-labelling of Γ_i described in [9].

$N_{r_i, \pi}$ has the following properties:

(i) for each $v \in V(\Gamma_i)$, $-q_i \leq N_{r_i, \pi}(v) \leq q_i$, i.e. $I_{2q_i} = \{-q_i, \ldots, -1, 1, \ldots, q_i\}$;

(ii) each pair of $\pi(0)$-adjacent vertices of Γ_i are labelled by opposite integers;

(iii) the vertices of each $\{\pi(0), \ldots, \pi(m_i - 1)\}$-residue of Γ_i belonging to the same bipartition class are labelled by integers having the same sign.

The matrix $\mathcal{A}(\Gamma_i, \pi \circ \gamma_i, N_{r_i, \pi})$ is completely determined by the elements:

a_c^j, for each $j \in \{-q_i, \ldots, -1\}$, for each $c \in \{1, \ldots, m_i - 1\}$;

a_c^j, for each $j \in I_{2q_i}$, for each $c \in \{m_i, \ldots, n\}$.

Therefore, we can define the *code* $c_{r_i, \pi}(\Gamma_i)$ *of* Γ_i, *with respect to the permutation* π *and the root* r_i, as the word of length $(2n - m_i + 1)q_i$ in the alphabet $I_{2q_i} \cup \{0\}$, in the following way.

Definition 2. Let $w_{i,c}^-$ (resp. $w_{i,c}^+$) be the word of length q_i obtained by juxtaposition of the elements a_c^j, $j \in \{-q_i, \ldots, -1\}$ (resp. a_c^j, $j \in \{1, \ldots, q_i\}$)

of $\mathcal{A}(\Gamma_i, \pi \circ \gamma_i, N_{r_i, \pi})$ in the order induced by the columns. Define

$$c_{r_i, \pi}(\Gamma_i) = w^-_{i,1} \cdots w^-_{i,n} w^+_{i,m_i} \cdots w^+_{i,n}.$$

The *code* $c(\Gamma_i, \pi)$ of Γ_i with respect to π (or simply the π-code of Γ_i) is defined as the lexicographic maximum among the codes $c_{r_i, \pi}(\Gamma_i)$, for $r_i \in V(\Gamma_i)$.

Let us state now a property of the roots corresponding to the π-code of a connected graph, which is useful in its effective computation and will justify our subsequent definitions.

Proposition 1 [9]. *Let* $(\overline{\Gamma}, \overline{\gamma})$ *be a connected* $(n+1)$*-colored graph,* π *be an admissible permutation of* Δ_n *and suppose* $c(\overline{\Gamma}, \pi) = c_{r, \pi}(\overline{\Gamma})$. *Then* r *belongs to a* $\{\pi_0, \pi_1\}$*-residue of maximum length among all* $\{\pi_0, \pi_1\}$*-residues of* $\overline{\Gamma}$.

Let $C = \{\Gamma_1, ..., \Gamma_r\}$ be the set of the connected components of the graph Γ; for each $i = 1, ..., r$, we denote by λ_i the maximum length of the $\{\pi_0, \pi_1\}$-residues of Γ_i.

Let us consider, now, the following ordering "$\prec$" of the set C:

for each $i, j = 1, ..., r$, $\Gamma_j \prec \Gamma_i$ iff one of the following conditions holds:

$\lambda_i < \lambda_j$

$\lambda_i = \lambda_j$ and $q_i < q_j$

$\lambda_i = \lambda_j$, $q_i = q_j$ and $c(\Gamma_i, \pi)$ is lexicographically less than $c(\Gamma_j, \pi)$.

If $c(\Gamma_i, \pi) = c(\Gamma_j, \pi)$, then we choose indifferently the ordering $\Gamma_i \prec \Gamma_j$ or $\Gamma_j \prec \Gamma_i$, since both choices have the same consequence on the following definitions.

The ordering of C defines a labelling N_π of Γ induced by the labellings of the Γ_i's corresponding to their π-codes; more precisely, for each $i = 1, \ldots, r$ and for each $v \in V(\Gamma_i)$, if v is labelled $k \in \{-q_i, \ldots, -1, 1, \ldots, q_i\}$ by $N_{r_i,\pi}$ (r_i being the root such that $c(\Gamma_i, \pi) = c_{r_i,\pi}(\Gamma_i)$), then $N_\pi(v) = k + s$, where

$$s = \begin{cases} \sum_{\Gamma_j \prec \Gamma_i} q_j & \text{if } k > 0, \\ -\sum_{\Gamma_j \prec \Gamma_i} q_j & \text{if } k < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We point out that the information which allow to reconstruct $(\Gamma, \pi \circ \gamma)$ from $\mathcal{A}(\Gamma, \pi \circ \gamma, N_\pi)$, are encoded in the following $(2n - \overline{m}(\Gamma) + 1)q$ elements of the matrix:

$$a_c^j, \text{ for each } j \in \{-q, \ldots, -1\}, \text{ for each } c \in \{1, \ldots, \overline{m}(\Gamma) - 1\}$$

$$a_c^j, \text{ for each } j \in I_{2q}, \text{ for each } c \in \{\overline{m}(\Gamma), \ldots, n\}.$$

From now on we call them the *essential elements* of $\mathcal{A}(\Gamma, \pi \circ \gamma, N_\pi)$.

We define the code $c(\Gamma, \pi)$ of Γ with respect to π as the word of length $(2n - \overline{m}(\Gamma) + 1)q$ in the alphabet $I_{2q} \cup \{0\}$, obtained by juxtaposition of the essential elements of $\mathcal{A}(\Gamma, \pi \circ \gamma, N_\pi)$ in the order as described in Definition 2.

Finally the *code* $c(\Gamma)$ of Γ is defined as the lexicographic maximum of $c(\Gamma, \pi)$ among all admissible permutations π of Δ_n.

Remark 2. Note that, given Γ_i and Γ_j with $q_i = q_j$ and $\lambda_i = \lambda_j$, if $m_i > m_j$, then $\Gamma_i \prec \Gamma_j$.

From Proposition 1 and the definition of $c(\Gamma)$, the following result is straightforward.

Proposition 2. *Let* (Γ, γ) *be an* $(n+1)$*-colored graph and* π *be an admissible permutation of* Δ_n *such that* $c(\Gamma) = c(\Gamma, \pi)$. *Then there is a* $\{\pi_0, \pi_1\}$*-residue of maximum length among all* $\{\sigma_0, \sigma_1\}$*-residues of* Γ, *with* σ *admissible permutation of* Δ_n.

It is clear from the above proposition that the computation of $c(\Gamma)$ can be restricted to the admissible permutations π containing a $\{\pi_0, \pi_1\}$-residue of maximum length.

Proposition 3. *Let* Γ *and* Γ' *be* $(n+1)$*-colored graphs. Then* Γ *and* Γ' *are color-isomorphic if and only if* $c(\Gamma) = c(\Gamma')$.

Proof. If Γ and Γ' are color-isomorphic through the isomorphism ϕ and the permutation σ, by Remark 1 and our definition of the code, it is easy to see that, for each admissible permutation π and for each $i = \{1, \ldots, r\}$, $c(\Gamma_i, \pi) = c(\phi(\Gamma_i), \pi \circ \sigma)$ (r being the number of connected components of Γ and Γ'), hence $c(\Gamma) = c(\Gamma')$, since the orders of the connected components and the maximum lengths of residues are invariant under colored isomorphisms.

Conversely if $c(\Gamma) = c(\Gamma')$, note that, as an easy consequence of the definition of code, Γ and Γ' have the same number of connected components and $\overline{m}(\Gamma) = \overline{m}(\Gamma')$.

Let π (resp. π') be the permutation of Δ_n such that $c(\Gamma) = c(\Gamma, \pi)$ (resp. $c(\Gamma') = c(\Gamma', \pi')$). Let $(\Gamma_1, \ldots, \Gamma_r)$ (resp. $(\Gamma'_1, \ldots, \Gamma'_r)$) be the r-ple of connected components of Γ (resp. Γ') with the ordering induced by $c(\Gamma, \pi)$ (resp. $c(\Gamma', \pi')$); then for each $i = 1, \ldots, r$, we have $c(\Gamma_i, \pi) = c(\Gamma'_i, \pi')$. As a consequence, by Proposition 2.10 of [9], there exists a colored isomorphism ϕ_i between Γ_i and Γ'_i corresponding to a permutation σ such that $\pi' = \pi \circ \sigma$. The pair (ϕ, σ), where $\phi_{|\Gamma_i} = \phi_i$ for each $i = 1, \ldots, r$, determines a colored isomorphism between Γ and Γ'.

Figure 1 (resp. Figure 2) shows, as an example, a 3-bipartite (resp. 2-bipartite) 4-colored disconnected graph and its code[2].

Remark 3. It is obvious that if Γ and Γ' are color-isomorphic, then there is a bijective correspondence between color-isomorphic connected components of Γ and Γ'. The converse is not generally true. Figure 3 shows an example of two 3-bipartite 3-colored graphs Γ and Γ' with two connected components Γ_1, Γ_2 and Γ'_1, Γ'_2 respectively. It is easy to see that for each $i = 1, 2, \Gamma_i$ is color-isomorphic to Γ'_i (i.e. their codes coincide), but $c(\Gamma) \neq c(\Gamma')$. In fact the isomorphism between Γ_1 and Γ'_1 is relative to the permutation (012) of Δ_2, while that between Γ_2 and Γ'_2 is relative to the distinct permutation (102). The existence of a color-isomorphism between Γ and Γ' implies that all their connected components are color-isomorphic through the same permutation.

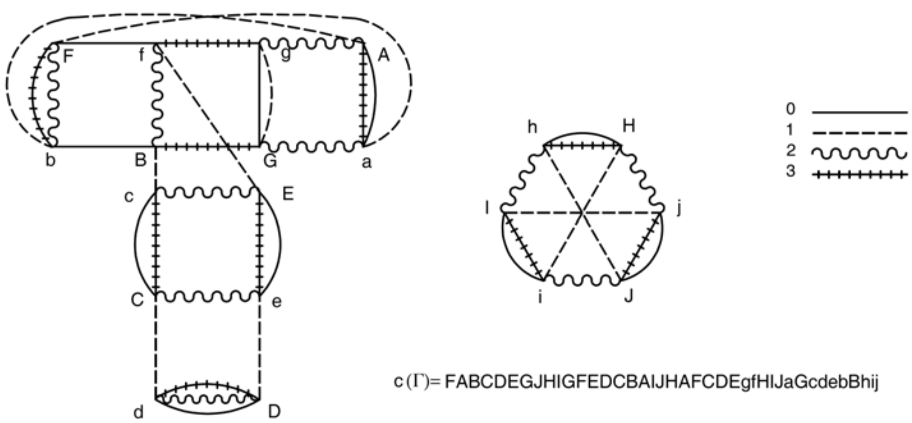

Figure 1

[2]In the figures, because of the low number of vertices of the examples, we use letters instead of integers, in order to display the code in a more compact way: capital letters correspond to positive numbers, small letters to negative ones.

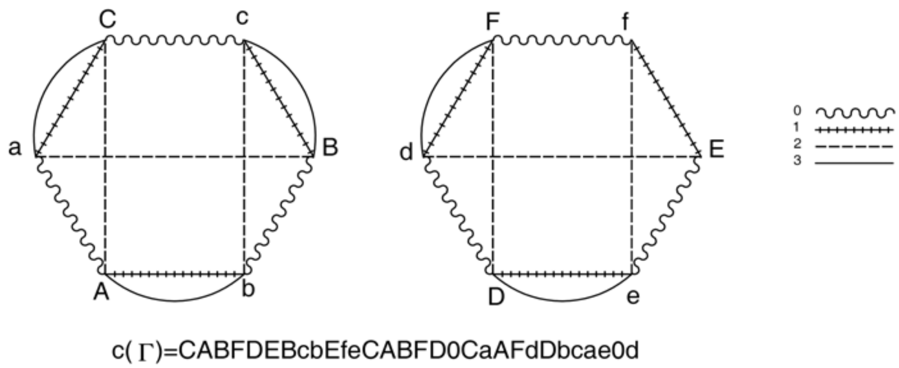

Figure 2

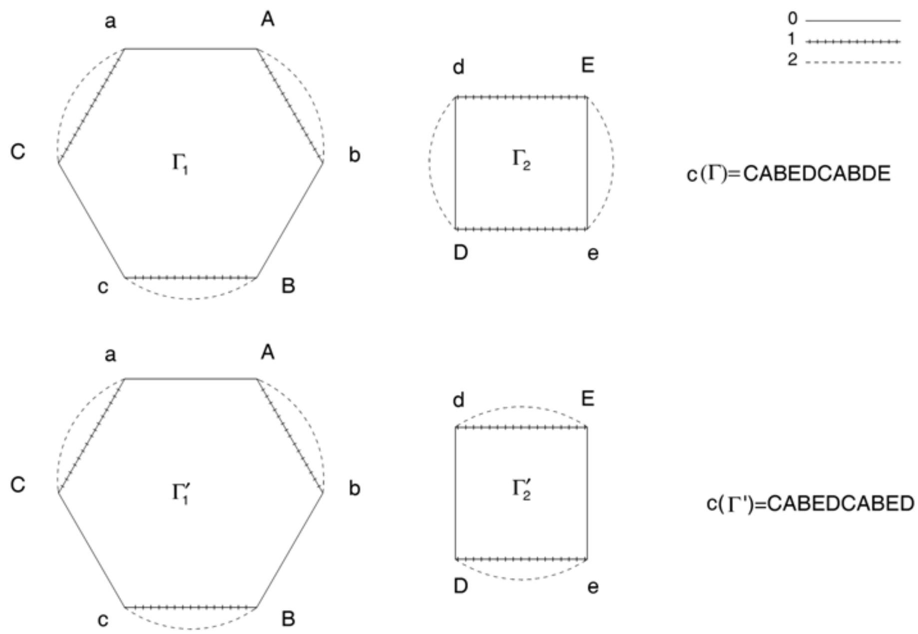

Figure 3

4. Colored Graphs and Manifolds Triangulations

From Remark 3, it comes natural to define a different concept of color-isomorphism as follows:

Definition 3. Two $(n+1)$-colored graphs Γ and Γ' are *weakly color-isomorphic* if there exists a bijective correspondence between the set of connected components of Γ and the set of connected components of Γ', such that corresponding components are color-isomorphic.

In other words, two colored graphs are weakly color-isomorphic if they are isomorphic component by component (obviously this is the same as Definition 1 in the case of connected graphs). This definition is equivalent to require the existence of a bijective correspondence between the two sets formed by the codes of the connected components of Γ and Γ' respectively.

We think interesting to discuss the usefulness of each definition in view of the connection between colored graphs and triangulations of manifolds.

Let K be a pseudocomplex triangulating a closed PL n-manifold M [3]. K is called a *colored triangulation* of M if there is a labelling of its vertices by means of Δ_n, which is injective on each simplex of K. The dual 1-skeleton of K is a multigraph $\Gamma(K)$ which inherits an edge-coloration from that of K. $\Gamma(K)$ is said to *represent* M. Moreover, for each $c \in \Delta_n$, each $(\Delta_n \setminus \{c\})$-residue of $\Gamma(K)$ represents the boundary of a particular regular neighbourhood (called *disjoint link*) of a c-labelled vertex of K, which is an $(n-1)$-sphere; as a consequence $\Gamma(K)$ can only be m-bipartite with $m = n+1$ (iff M is orientable) or $m = n$. If, for each $c \in \Delta_n$, K has only one c-colored vertex, then it is called *contracted*. In this case, for each $c \in \Delta_n$, the subgraph $\Gamma_{\hat{c}}(K)$ of $\Gamma(K)$ obtained by deleting all c-colored edges, is connected and $\Gamma(K)$ is called *contracted*, too, or a *crystallization* of M (see [2], [10] or [3] for a survey on this representation theory).

The following statement is easily proved.

Proposition 4. *Color-isomorphic* (*resp. weakly color-isomorphic*) *graphs correspond to the same triangulation and represent the same manifold*.

---

[3] $(n+1)$-colored graphs are a representation tool for compact PL n-manifolds; in this section, for simplicity, we will fix on the closed case, although definitions and results can be extended to the case of non-empty boundary with only slight modifications.

Therefore, the topological meaning of $(n+1)$-colored graphs representing *n*-manifolds seems to support the preference for Definition 3, since, in the case of non-connected manifolds it is highly important to recognize the possible largest number of colored graphs representing the same manifold, disregarding the fact that different components correspond through different colorations. However the choice of the more restrictive Definition 1 is not only justified by its extending directly the connected one, but has also its topological backgrounds.

In fact, an interesting problem in the theory of colored graphs is that of generating all possible $(n+1)$-colored graphs with a fixed number of vertices representing manifolds, in order to obtain catalogues of triangulations.

More precisely, since each *n*-manifold admits a crystallization, given $p > 1$, we are interested in generating the set $\mathcal{C}_{2p}^{n+1}$, of all contracted $(n+1)$-colored graphs without boundary Γ, with $2p$ vertices, representing manifolds, i.e. such that $\Gamma_{\hat{c}}$ represents a $(n-1)$-sphere for each $c \in \Delta_n$.

Let us denote by Σ_{2p}^{r}(resp. $\overline{\Sigma}_{2p}^{r}$) the set of all connected (resp. possibly disconnected) *r*-colored graphs, with $2p$ vertices, representing (resp. whose connected components represent) $(r-1)$-spheres.

One way to solve our problem is to generate the set Σ_{2p}^{n} and *complete* each of its elements by adding the $(n+1)$-colored edges in all possible way, so as to obtain a manifold. Again, in order to construct Σ_{2p}^{n}, we start by adding *n*-colored edges to each element of the set $\overline{\Sigma}_{2p}^{n-1}$ so as to obtain spheres[4].

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[4]The problem of recognizing $(n-1)$-colored graphs representing spheres is dealt with differently depending on *n*. It is obviously easy for $n = 4$; for $n > 4$, it can be solved by manipulating the graph in order to obtain a contracted one (which is always possible) and by comparison, if there exists a catalogue of all crystallizations of $(n-2)$-spheres with $2p$ vertices. For instance, this happens for $n = 5$ and $p < 16$ (see [8]).

This process can be repeated recursively, taking as initial step $\overline{\Sigma}_{2p}^{2}$ i.e. the set of all graphs with $2p$ vertices, whose connected components are bicolored cycles: for each $r = 3, \ldots, n-1,$ the construction of $\overline{\Sigma}_{2p}^{r}$ is performed by adding to each element of $\overline{\Sigma}_{2p}^{r-1}$ the r-colored edges in order to obtain disjoint unions of spheres.

Note that at the last step only connected graphs are required, since we want the final objects to be crystallizations.

Of course, due to the large amount of possible addings of edges, all constructions are done by computer and, to save computational time, it is important to start, at each step, from the smallest essential set $\overline{\Sigma}_{2p}^{r}$ $(\text{for } r < n)$ or Σ_{2p}^{n}. Here *essential* means "assuring that no n-manifold admitting a contracted triangulation with $2p$ vertices can be left out from the final catalogue".

For this purpose, only Definition 1 assures that two disconnected color-isomorphic r-colored graphs give rise to the same set of $(r+1)$-colored graphs.

Figure 4 shows that this does not happen for Definition 3; in fact, it presents two connected 4-colored graphs $\overline{\Gamma}$ and $\overline{\Gamma}'$ arising from the 3-colored graphs Γ and Γ' respectively of Figure 3, by adding corresponding sets of 4-colored edges.

Although Γ and Γ' are color-isomorphic by Definition 3 (but they are not by Definition 1), $\overline{\Gamma}$ and $\overline{\Gamma}'$ are not color-isomorphic, as it is clear from their codes. As a consequence the set $\overline{\Sigma}_{10}^{3}$ must contain both Γ and Γ'.

Catalogues $\mathcal{C}_{2p}^{4}$ for $1 \leq p \leq 15$ have already been generated, analyzed and all represented 3-manifolds have been recognized (see [4], [5], [6], [8]). Catalogues $\mathcal{C}_{2p}^{5}$ for $1 \leq p \leq 10$ have been generated and partial results, with regard to classification of the involved 4-manifolds, have already been obtained ([7]).

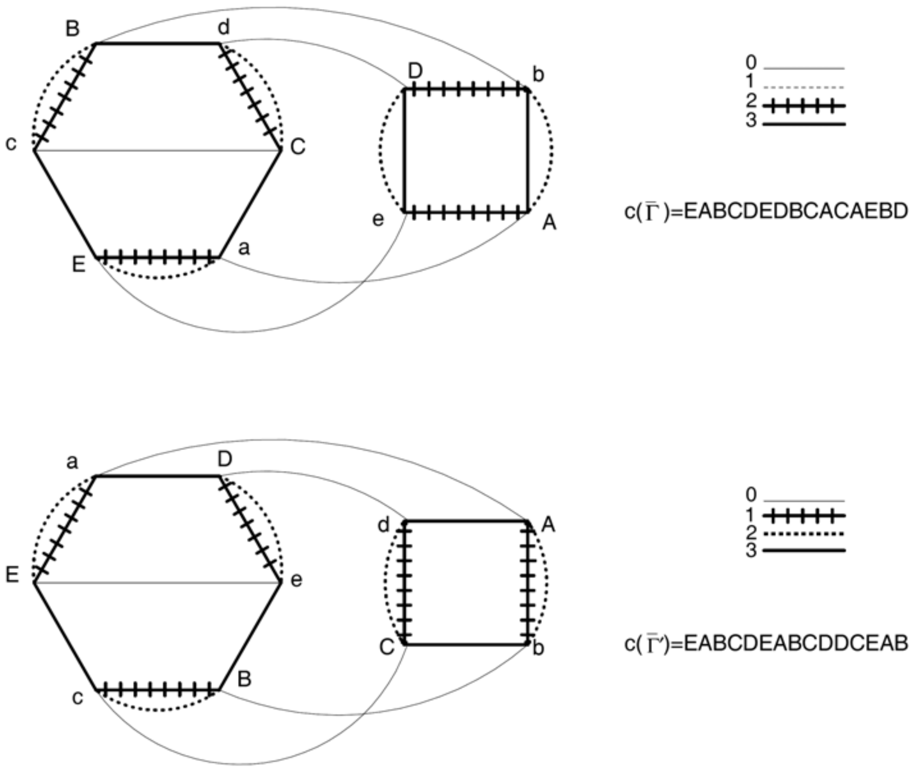

Figure 4

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