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CONNECTIONS ON THE FIBRE BUNDLE AND APPLICATION TO THE LAGRANGIAN MECHANICS

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Abstract

In this paper, we use the notion of Grifone's connection over a tangent bundle in order to construct a connection over a sub-bundle. Then we characterize the solutions of non-holonomic Lagrangian mechanics and show that the geodesics of the connection constructed on the sub-bundle are the solutions of the non-holonomic Euler-Lagrange system. Finally, we will prove that the Hamiltonian associated to Lagrangian function is constant along the horizontal curves.

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Introduction

The geometrization of the Euler-Lagrange problem in Lagrangian mechanics holonomic or non-holonomic has been developed specially with Gallissot [4], Klein [9] and Vershik and Faddeev [12]. Grifone and Mehdi [5] constructed a connection on the tangent fibre bundle projected on the sub-manifold with constraints. This projection required specific conditions that this constraint is ideal in the sense of Vershik. In this paper, we take the definition of Grifone's connection [7] and we use it to present an alternative approach over the tangent sub-bundle. The covariant derivative, the torsion and the curvature will be also defined but they required a Lie pre-bracket definition associated to this connection. We also characterize a connection whose geodesics are the solutions of the Euler-Lagrange problem. We finally show that the Hamiltonian associated to a given Lagrangian is preserved along the horizontal paths.

1. Notation and Preliminary Definitions

Let M be a smooth differentiable manifold of dimension n and E be a regular linear tangent sub-bundle of TM over M of dimension p.

For all calculations, we adopt the following conventions:

Summations from 1 to n for Latin indices $i, j, k, \ldots$.

Summations from 1 to p for the Greek indices $\alpha, \beta, \gamma, \ldots$.

Summations from $p+1$ to n for the Latin indices $\bar{i}, \bar{j}, \bar{k}, \ldots$.

1.1. Basis well adapted to E

Let (U, Φ) be chart on M where z is the center and $\pi : E \to M$ be the canonical projection.

Note that we have two structures of fiber bundle over TE:

$$p_E : TE \to E \quad \text{and} \quad \pi^T : TE \to TM;$$

where π^T is the tangent mapping of π and p_E is the canonical projection on E.

Let $(C_1, ..., C_n)$ be a local basis of the vector field on U. Without losing the generality, we can assume that $(C_1, ..., C_p)$ is a local basis of E over U such that

$$C_\alpha(z) = \frac{\partial}{\partial x^\alpha}\bigg|_z, \quad \alpha = 1, ..., p.$$

Therefore, for all $x \in U$, we write:

$$C_i(x) = C_i^j(x)\frac{\partial}{\partial x^j}, \quad \text{for} \quad i, \ j = 1, ..., n$$

so that we have: $C_\alpha(x) = C_\alpha^\beta(x)\dfrac{\partial}{\partial x^\beta} + C_\alpha^{\bar{j}}(x)\dfrac{\partial}{\partial x^{\bar{j}}}.$

In some neighborhood of z, the matrix (C_α^β) is still invertible and we denoted by $(\mathcal{C})$. Thus, we define on U the following vector fields:

$$A_\alpha|_x = ((\mathcal{C}^{-1})_\alpha^\beta C_\beta)\big|_x = \frac{\partial}{\partial x^\alpha}\bigg|_x + B_\alpha^{\bar{i}}\frac{\partial}{\partial x^{\bar{i}}}\bigg|_x,$$

where $B_\alpha^{\bar{i}}|_x = ((\mathcal{C}^{-1})_\alpha^\beta C_\beta^{\bar{i}})|_x$.

The set $(A_1, ..., A_p)$ is also a basis for the vector fields E around the point z in U.

Consider now $A_{\bar{i}}(x) = \dfrac{\partial}{\partial x^{\bar{i}}}\bigg|_x$. Then $(A_1, ..., A_n)$ is basis field on U in TM.

On TM, we already have two coordinates systems:

The classical coordinate system: $(x^1, ..., x^n, y^1, ..., y^n)$.

The adapted coordinate system: $(z^1, ..., z^n, a^1, ..., a^n)$ associated to $(A_1, ..., A_n)$.

Both the systems are characterized by the following relations:

$$z^i = x^i, \ y^\alpha = a^\alpha \quad \text{and} \quad y^{\bar{i}} = a^{\bar{i}} + a^\alpha B_\alpha^{\bar{i}}.$$

Therefore,

$$\frac{\partial}{\partial y^{\alpha}} = \frac{\partial}{\partial a^{\alpha}} - B_{\alpha}^{\bar{i}} \frac{\partial}{\partial a^{\bar{i}}}.$$

According to these bases, we can construct a dual basis (θ^{i}) of (A_i) by taking:

$$\theta^{\bar{i}} = dx^{\bar{i}} - B_{\alpha}^{\bar{i}} dx^{\alpha} \quad \text{and} \quad \theta^{\alpha} = dx^{\alpha},$$

where $(dx^1, \dots, dx^n)$ is the dual basis of $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$.

Here also we have two coordinate systems in T^*M at point (x, ξ):

The classical coordinate system: $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$.

The adapted coordinate system: $(z^1, \dots, z^n, \zeta_1, \dots, \zeta_n)$.

It is now obvious that $\xi_{\bar{i}} = \zeta_{\bar{i}}$ and $\xi_{\alpha} = \zeta_{\alpha} - \zeta_{\bar{i}} B_{\alpha}^{\bar{i}}$.

1.2. Notation

We denote $\mathcal{E}$ the sub-bundle of *TE* defined by $\mathcal{E} = [\pi^T]^{-1}(E)$. Note that the kernel of π^T is equal to $\mathcal{E}^v = \mathcal{E} \cap T^v TM$.

Let J be the almost tangent structure. J is also called the *vertical endomorphism* defined as a tensor field of type $(1, 1)$ on *TM*. By using the classical coordinates, J can be written as $J = dx^i \otimes \frac{\partial}{\partial y^i}$.

Knowing that the vector field A_i^v is the vertical lift of A_i, therefore, the tangent space $T_{(x,a)}E$ is generated by the vector fields $\{A_i, A_{\alpha}^v\}$. We notice that $J(A_i) = \frac{\partial}{\partial a^i}$, so we can deduce that $\mathcal{E}_{(x,a)}$ is generated by $\left\{A_1, \dots, A_p, \frac{\partial}{\partial a^1}, \dots, \frac{\partial}{\partial a^p}\right\}$ and then $\mathcal{E}_{(x,a)}^v$ is generated by $\left\{\frac{\partial}{\partial a^1}, \dots, \frac{\partial}{\partial a^p}\right\}$.

In the classical coordinates, we have:

$$A_\alpha = \frac{\partial}{\partial x^\alpha} + B_\alpha^{\bar{i}} \frac{\partial}{\partial x^{\bar{i}}}$$

and

$$A_\beta^v = \frac{\partial}{\partial y^\beta} + B_\beta^{\bar{i}} \frac{\partial}{\partial y^{\bar{i}}}.$$

We also have

$$[A_\alpha,\ A_\beta^v] = 0.$$

Let us consider now $(z^i,\ a^i)$ the coordinate system defined by

$$z^i = x^i,\ a^\alpha = y^\alpha \ \text{ and } \ a^{\bar{i}} = y^{\bar{i}} - y^\alpha B_\alpha^{\bar{i}}.$$

It allows us to write $[A_\alpha,\ A_\beta^v]$ in this new coordinate system $(z^i,\ a^i)$ of the form

$$[A_\alpha,\ A_\beta^v] = B_\alpha^{\bar{i}} \frac{\partial B_\beta^{\bar{i}}}{\partial x^{\bar{j}}} \frac{\partial}{\partial a^{\bar{j}}}.$$

Since

$$A_\alpha = \frac{\partial}{\partial z^\alpha} + B_\alpha^{\bar{i}} \frac{\partial}{\partial z^{\bar{i}}} + B_\alpha^{\bar{i}} \left(-y^\gamma \frac{\partial B_\gamma^{\bar{j}}}{\partial x^{\bar{i}}} \right) \frac{\partial}{\partial a^{\bar{j}}} \ \text{ and } \ A_\beta^v = \frac{\partial}{\partial a^\beta}.$$

As we know the functions B_α^i are functions of $(x^1, ..., x^n)$ which means $[A_\alpha,\ A_\beta^v] = 0$ at the center z of the chart $(U,\ \Phi)$ and therefore at any point in U. So we get the following proposition:

Proposition 1. *The space* $\mathcal{E}$ *is stable under the action of the almost tangent structure J, i.e.,* $J\mathcal{E} = \mathcal{E}^v$.

Further, we can denote J the restriction of J on $\mathcal{E}$.

The Liouville vector field on TM is $C = y^i \frac{\partial}{\partial y^i}$, we can write it in adapted coordinate system by: $C = a^i \frac{\partial}{\partial a^i}$.

Moreover, at any point in E, the Liouville vector field is given by $C = a^\alpha \frac{\partial}{\partial a^\alpha}$. And at any point of T^*M, the Liouville form is given by $\omega = \xi_i dx^i$.

Remark 1. The vertical isomorphism $\xi : T^v TM \to TM$ is expressed in classical coordinate system by: $\xi|_{(x,y)}\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial x^i}\Big|_x$. It induces an isomorphism between $\mathcal{E}^v$ and E, defined by $\xi|_{(x,a)}\left(\frac{\partial}{\partial a^\alpha}\right) = A_\alpha|_x$.

Indeed, $\frac{\partial}{\partial a^\alpha} = \frac{\partial}{\partial y^\alpha} + B^{\bar{i}}_\alpha \frac{\partial}{\partial y^{\bar{i}}}$. Therefore, $\xi|_{(x,a)} \frac{\partial}{\partial a^\alpha} = A_\alpha|_x$.

Definition 1. A spray on E is a vector field S on E such that $JS = C$.

In adapted coordinate system, S can be represented by:

$$S(z, a) = a^\alpha A_\alpha|_{(z,a)} + S^\alpha(z, a)\frac{\partial}{\partial a^\alpha}\Big|_{(z,a)}.$$

The semi-basic tensor Ω is a section of $(\otimes^0_k(TE)) \otimes (\otimes^l_0(T^*E))$ which admits the following locally form:

$$\Omega(x, a) = \Omega^{j_1,\ldots,j_l}_{i_1,\ldots,i_k}(x, a) dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial a^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial a^{j_l}}.$$

Remark 2. A section Z of E can be seen as an application from E to E which verifies $\pi \circ Z = \pi$. In adapted coordinate system, Z can be written by $Z(z, a) = Z^\alpha(z, a) A_\alpha(z)$. We denote $\mathcal{X}(E)$ the set of sections of E.

2. Grifone Connection on the Sub-bundle $\mathcal{E}$

Definition 2. A connection on $\mathcal{E}$ is an homomorphism Γ of $\mathcal{E}$ such that

$$J\Gamma = J \quad \text{and} \quad \Gamma J = -J.$$

In adapted coordinates, we have:

$$\begin{cases} \Gamma(A_\alpha) = A_\alpha - 2\Gamma_\alpha^\beta \dfrac{\partial}{\partial a^\beta} & \text{for } 1 \le \alpha \le p, \\ \Gamma\left(\dfrac{\partial}{\partial a^\alpha}\right) = -\dfrac{\partial}{\partial a^\alpha} & \text{for } 1 \le \alpha \le p. \end{cases} \tag{2.1}$$

A connection Γ is represented by the matrix:

$$\Gamma = \begin{pmatrix} \delta_\alpha^\beta & 0 \\ -2\Gamma_\alpha^\beta & -\delta_\alpha^\beta \end{pmatrix}.$$

The functions Γ_α^β are called the *coefficients of the connection* Γ.

Proposition 2. *A connection* Γ *on* $\mathcal{E}$ *satisfies the following properties*:

(1) $\Gamma^2 = Id\,|_{\mathcal{E}}$.

(2) $\mathcal{E}^v$ *is the eigenspace of* Γ *associated to the eigenvalue* –1.

(3) *Suppose* $\mathcal{H}_\Gamma$ *be the eigenspace of* Γ *associated to the eigenvalue* 1. *Then* $\mathcal{E}$ *splits into the direct sum*:

$$\mathcal{E} = \mathcal{E}^v \oplus \mathcal{H}_\Gamma.$$

The eigenspace $\mathcal{H}_\Gamma$ of Γ associated to the eigenvalue 1 is called the *horizontal space*. We denoted by h_Γ and v_Γ the horizontal and vertical projectors. They are given by:

$$h_\Gamma = \frac{1}{2}(Id_{\mathcal{E}} + \Gamma) \quad \text{and} \quad v_\Gamma = \frac{1}{2}(Id_{\mathcal{E}} - \Gamma).$$

And locally: $h_\Gamma = \begin{pmatrix} \delta_\alpha^\beta & 0 \\ -\Gamma_\alpha^\beta & 0 \end{pmatrix}$ and $v_\Gamma = \begin{pmatrix} 0 & 0 \\ \Gamma_\alpha^\beta & \delta_\alpha^\beta \end{pmatrix}.$

2.1. Properties

As the traditional framework for a connection, we have the following characterization:

Proposition 3. *Let* Γ *be a connection on* $\mathcal{E}$, *there exists only one spray S on E tangent to* $\mathcal{H}_\Gamma$, *it is called the canonical spray S of* Γ.

Indeed, the horizontal projection of any spray gives us a unique spray defined by:

$$h_\Gamma(S)|_{(x,a)} = a^\alpha h_\Gamma(A_\alpha)|_{(x,a)} + S^\alpha(x, a) h_\Gamma\left(\frac{\partial}{\partial a^\alpha}\right)\Bigg|_{(x,a)}$$

$$= a^\alpha (A_\alpha - \Gamma_\alpha^\beta|_{(x,a)}) \frac{\partial}{\partial a^\beta}.$$

So it is obvious that $Jh_\Gamma(S) = C$.

Definition 3. The geodesics of Γ are, by definition, the integrals curves of the canonical spray of Γ.

Proposition 4. *Let* Γ *be a connection on* $\mathcal{E}$. *Then we have the following properties*:

(1) *If* Υ *is semi-basic* (1-1) *tensor on* $\mathcal{E}$, *then* $\Gamma + \Upsilon$ *is a connection on* $\mathcal{E}$.

(2) *Conversely, if* Γ' *is a connection on* $\mathcal{E}$, *then there exists only one semi-basic* (1-1) *tensor field* Υ *such that* $\Gamma' = \Gamma + \Upsilon$ *so that*

$$h_{\Gamma'} = h_\Gamma + \frac{1}{2}\Upsilon \quad \textit{and} \quad v_{\Gamma'} = v_\Gamma - \frac{1}{2}\Upsilon.$$

(3) *If S is the canonical spray of* Γ, *then S will be the canonical spray of* Γ' *if and only if* $\Upsilon(S) = 0$.

Proposition 5. *A connection* Γ *defines only one vectorial form* F_Γ *on* $\mathcal{E}$ *which satisfies*:

$$F_\Gamma J = h_\Gamma \quad and \quad F_\Gamma h_\Gamma = -J,$$

and verifies

$$F_\Gamma^2 = -Id_{\mathcal{E}}.$$

F_Γ *is called almost complex structure associated to* Γ.

2.2. Linear connection

Inspired by the work of Grifone on "stucture presque tangente et connexion" (see [7]), we have recently embarked on the study of a general notion of connection, these connections are defined over fibre bundle.

As in the classical definition, a linear connection on $\mathcal{E}$ is defined as the sub-bundle of $\mathcal{E}$ transverse with $\mathcal{E}^v$ and verifies:

For all $t \in \mathbb{R}$, $t \neq 0$, we have $\delta_{t_*}(\mathcal{H}_{(x,u)}) = \mathcal{H}_{\delta_t(x,u)}$, where $\delta_t(x, u) = (x, tu)$.

In this case, we deduce the following proposition:

Proposition 6. *Let* Γ *be a connection on E. Then the following statements are equivalent*:

(i) *The connection* Γ *is linear*.

(ii) *For any vector field Z on E tangent to* $\mathcal{E}$, *we have*

$$[C, \Gamma]Z = [C, \Gamma Z] - \Gamma[C, Z] = 0.$$

(iii) *In an adapted coordinate system, the coefficients of* Γ *are as form*

$$\Gamma_\alpha^\beta(x, a) = a^\gamma \Gamma_{a\gamma}^\beta(x) \quad for\ all \quad \gamma = 1, \ldots, p. \tag{2.2}$$

3. Lie Pre-bracket

3.1. Pre-bracket of Lie on *E*

When $E = TM$ therefore, $\mathcal{E} = TTM$, we can define a connection through a spray according to [7]: suppose that *S* is a spray on *TM*, then the connection associated to *S*, $\Gamma_S = [J, S]$, is defined by

$$\forall X \in \mathcal{X}(TM) \quad [J, S]X = [JX, S] - J[X, S].$$

The connection Γ_S is called *canonical connection* associated to *S*. Unfortunately, if $E \neq TM$, then $\mathcal{E}$ is no more stable by Lie bracket act. Since the previous expression is not defined on $\mathcal{E}$. Our main goal now is to give an additional structure on *E*, in order to associate a canonical connection Γ_S to a given *S*. This new structure is called the *Lie new-bracket*.

Definition 4. A Lie pre-bracket on *E* is an application:

$$[\ ,\ ]_E : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E),$$

$\mathbb{R}$ -bilinear, antisymmetric and verify the Leibniz formula given by:

$$[X, fY]_E = X(f)Y + f[X, Y]_E, \tag{3.1}$$

for all *X*, *Y* vector fields on *M* that are tangent to *E* and for all *f* functions on *M*.

Remark 3. There is a simple and natural way to construct a Lie pre-bracket on *E*. Indeed, if we split *TM* into the direct sum *E* and *F*, for any vector field *E*. Let $q : TM \to E$ be the associated canonical projection. We can define a Lie pre-bracket on *E* by:

$$[X, Y]_E = q[X, Y],$$

where $[X, Y]$ is the classical Lie bracket on *M* of the vectors fields *X* and *Y* on *M* but its value is in *E*.

We renew the approach of pre-bracket done by [8] and [10]:

Let $X \in \mathcal{X}(E)$ be a vector field on M with values in E. The restriction of the inner product i_X on E^* defines a function on E^*. Then di_X is an element of T^*E^*. Let P be a bivector on E^* which means P is a section of the fibre bundle $\wedge^2 T^*E^*$.

Obviously, we can associate to the bivector P a bracket $\{,\}_P$ on E defined by: $\{f, g\}_P = P(df, dg)$, where f and g are functions on E^*. In general, this bracket does not verify the Jacobi's identity unless if P is Poisson tensor which is equivalent to say that the bracket of Schouten of P vanishes.

The given of a new lie bracket $[,]_E$ on E is equivalent to give a linear bivector P on E^* which verifies the following propositions:

$$i_{[X,Y]_E} = P(di_X, di_Y) = \{i_X, i_Y\}_P,$$

$$X(f) \circ \pi^* = \{i_X, f \circ \pi^*\}_P, \tag{3.2}$$

for all X, Y tangent to E, for all functions f on M, where π^* is a canonical projection of E^* on M.

To define a Lie pre-bracket on E means to define a bracket between the vector fields A_α. In an adapted coordinate system, we have $i_{A_\alpha} = \zeta_\alpha$. According to (3.2), P will have the local form:

$$P = \frac{1}{2} C^\gamma_{\alpha\beta} \zeta_\gamma \frac{\partial}{\partial \zeta_\alpha} \wedge \frac{\partial}{\partial \zeta_\beta} + A_\alpha \wedge \frac{\partial}{\partial \zeta_\alpha}. \tag{3.3}$$

Corollary 1. *There exists on E a canonical pre-bracket* $[,]_0$ *defined by*

$$[A_\alpha, A_\beta]_0 = 0.$$

Recall that on T^*M, we have a canonical Poisson bracket associated to the Poisson tensor π_0. Of all the pre-brackets on E, there is one that is intrinsic defined through:

$$\pi_0 = \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \xi_i} = \Sigma_\beta \left(\frac{\partial}{z^\beta} + \xi_{\bar{i}} \frac{\partial B_\alpha^{\bar{i}}}{\partial x^\beta} \frac{\partial}{\partial \zeta_\alpha} \right) \wedge \frac{\partial}{\partial \zeta_\beta}$$

$$+ \Sigma_{\bar{j}} \left(\frac{\partial}{\partial z^{\bar{j}}} + \xi_{\bar{i}} \frac{\partial B_\alpha^{\bar{i}}}{\partial x^{\bar{j}}} \frac{\partial}{\partial \zeta_\alpha} \right) \wedge B_\alpha^{\bar{j}} \frac{\partial}{\partial \zeta_\alpha}.$$

By restrict it on *E*, and at the center of the chart already chosen, we obtain:

$$\pi_0 \big|_z = \Sigma_\beta \frac{\partial}{z^\beta} \bigg|_z \wedge \frac{\partial}{\partial \zeta_\beta} \bigg|_z = A_\beta \wedge \frac{\partial}{\partial \zeta_\beta} \bigg|_z .$$

This bivector defines eventually the pre-bracket $[\ ,\ ]_0$ by $[A_\alpha, A_\beta]_0 = 0$ (see [3, 2]).

3.2. Lie pre-bracket on $\mathcal{E}$

In the sequel, we are dealing to use the Lie pre-bracket *P* on *E*. We will define now a pre-bracket on $\mathcal{E}$ which only depends on *P* ([1, 3, 11]).

Proposition 7. *There exists a unique linear bivector* Π *on* $\mathcal{E}^*$ *such that the pre-bracket* $[\ ,\ ]_\Pi$ *satisfies the two following properties*:

(1) $[Y, Z]_\Pi = [Y, Z]$ *for any section Z and vertical section Y in* $\mathcal{E}$.

(2) Π *is projected on P*:

(i) $[Z, Z']_\Pi \big|_{(x, u)} = [pi^T(Z), pi^T(Z')]_P \big|_{(x)}$ *for any sections Z and* Z' *of* $\mathcal{E}$ *and* $(x, u) \in E$. *We say that the sections* $pi^T(Z)$ *and* $pi^T(Z')$ *are, respectively, the projections of the sections Z and* Z' *on E.*

(ii) $\tilde{Z}(f \circ \pi) = Z(f) \circ \pi$ *for any function f on M and any section* $\tilde{Z}$ *of* $\mathcal{E}$ *projected on the section Z of E.*

We shall give the following lemma in order to give sense to the first property:

Lemma 1. *For all vertical vector Y and all section Z of* $\mathcal{E}$, $[Y, Z]$ *is tangent to* $\mathcal{E}$.

Proof. The vector field $[Y, Z]$ is defined on *TE*. Moreover, $\mathcal{E}^v$ is stable by the Lie bracket, so we have: $\left[A_\alpha, \frac{\partial}{\partial a^\alpha}\right] = 0,$ hence $[Y, Z]$ is tangent to $\mathcal{E}$. □

3.3. Canonical connection associated to a spray

For the further study, we assume that the bivector Π on $\mathcal{E}^*$ is the bivector associated to a given bivector *P* on E^*.

Proposition 8. *Given a spray S on E, the form*

$$\Gamma_S(X) = [J, S]_\Pi(X) = [JX, S] - J[X, S]_\Pi,$$

is a connection on $\mathcal{E}$.

Proof. It is obvious that $[X, S]_\Pi$ is tangent to $\mathcal{E}$, and Lemma 1 implies that $[JX, S]$ is tangent to $\mathcal{E}$ as well. Therefore, Γ_S is defined as an endomorphism of $\mathcal{E}$.

However $J^2 = 0$, we get:

$$J(\Gamma_S(X)) = J[JX, S] = JX,$$

$$\Gamma_S(JX) = -J[JX, S] = -JX. \qquad \square$$

Proposition 9. *In an adapted coordinate system, using the previous notations of* (2.1), *the coefficients of the connection* Γ_S *associated to* $S = a^\alpha A_\alpha + S^\beta \frac{\partial}{\partial a^\beta}$ *are*:

$$\Gamma_\alpha^\beta = \frac{1}{2}\left(a^\gamma C_{\alpha\gamma}^\beta - \frac{\partial S^\beta}{\partial a^\alpha}\right) \;\; for \;\; \gamma = 1, \dots, p. \qquad (3.4)$$

Proof. According to (2.1), we have $\Gamma_S(A_\alpha) = A_\alpha - 2\Gamma_\alpha^\beta \frac{\partial}{\partial a^\beta}$. To find Γ_α^β, it is sufficient to calculate $\Gamma_S(A_\alpha)$:

$$[JA_\alpha, S] = A_\alpha + \frac{\partial S^\beta}{\partial a^\alpha} \frac{\partial}{\partial a^\beta},$$

$$J[A_\alpha, S]_\Pi = a^\gamma C_{a\gamma}^\beta \frac{\partial}{\partial a^\beta}. \qquad \square$$

Remark 4. It is important to note that the canonical spray of the connection Γ_S, for a given S on E, is $h_{\Gamma_S}(S) = \frac{1}{2}(S + [C, S])$ which is not, in general, S.

Definition 5. A spray S is quadratic if $[C, S] = S$.

Proposition 10. *The following statements are equivalent*:

(i) *S is quadratic*,

(ii) *S has the form of*:

$$S = a^\alpha A_\alpha + \frac{1}{2} a^\gamma a^\lambda S_{\gamma\lambda}^\beta(x) \frac{\partial}{\partial a^\beta},$$

(iii) Γ_S *is linear*,

(iv) *S is the canonical spray of* Γ_S.

4. Covariant Derivative, Torsion and Curvature

4.1. Covariant derivative

Definition 6. A covariant derivative on E is a map $D : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E)$ such that

$$D_{fX+gY}Z = fD_X Z + gD_Y Z, \text{ where } X, Y, Z \in E \text{ and } f, g \in C^{+\infty}(M).$$

Proposition 11. *There exists an isomorphism between the set of connections on E and the set of covariant derivative on E.*

Proof. For every connection Γ on *E*, there exists an associated covariant derivative defined by:

$$D_X Y\,|_x = \xi\,|_{(x,a)}(v_\Gamma(Y^T(X))) \quad \forall X = X^\alpha A_\alpha,\ Y = Y^\beta A_\beta \in E.$$

It is easy to see that

$$Y^T(X)\Big|_{Y(x)} = \left(X^\alpha A_\alpha + X^\alpha (A_\alpha . Y^\beta) \frac{\partial}{\partial a^\beta} \right)\Bigg|_{Y(x)}.$$

So, it is in $\mathcal{E}$. Although $v_\Gamma(Y^T(X)) = \left(X^\alpha \Gamma_\alpha^\beta + X^\alpha (A_\alpha . Y^\beta) \frac{\partial}{\partial a^\beta} \right)\Bigg|_{Y(x)}$ is in $\mathcal{E}^v$. Using the act of $\xi\,|_{(x,a)}$, the local expression of $D_X Y$ holds:

$$D_X Y\,|_x = (X^\alpha \Gamma_\alpha^\beta(x, Y(x)) + X^\alpha A_\alpha . Y^\beta) A_\beta\,\Big|_x.$$

This expression of the covariant derivative allows us to verify immediately the properties of the covariant derivative.

Conversely, if *D* is a covariant derivative on *E*, then the functions Γ_α^β of the connection are given by:

$$\Gamma_\alpha^\beta(x, X(x)) = (D_{A_\alpha} X)^\beta - A_\alpha . X^\beta.$$

One can easily check that the covariant derivative associated to Γ is equal to *D*. □

4.2. Parallel transport

Let $c : [0, T] \to M$ be a curve of class C^2. We denote $\chi_c(E)$ the set of vector fields in *E* along *c*.

Proposition 12. *For any connection D on E, we can eventually associate* *a derivation* $D_{\dot{c}}: \chi(E) \to \chi(E)$ *such that: for all* $X \in \chi_c(E)$, *we have* $D_{\dot{c}} X = D_{\dot{c}} \widetilde{X}$ *as* $\widetilde{X}$ *is an extension of X on M.*

Proof. It is easy to see that this derivation does not depend on the choice of the extension. Indeed, if $\widetilde{X}'$ is another extension of X. We have $\widetilde{X}' = \widetilde{X} + fZ$, where $f(c(t)) = 0$ and Z is a vector field on M,

$$\begin{aligned} D_{\dot{c}}(fZ) &= \xi(v_\Gamma(fZ)^T(\dot{c}(T))) \\ &= \xi(v_\Gamma(df|_{\dot{c}(t)}\ Z(c(t)) + f|_{c(t)}\ Z^T(\dot{c}(t))) \\ &= \xi\left(v_\Gamma\left(\frac{d(foc(t))}{dt}\right)Z(c(t)) + (foc(t))Z^T(\dot{c}(t))\right) = 0. \end{aligned}$$ □

Proposition 13. *A curve c is a geodesic for the connection* Γ *if and only if* $D_{\dot{c}}\dot{c} = 0$.

Proof. Indeed, the associated spray to Γ is $S = a^\alpha A_\alpha - a^\alpha \Gamma_\alpha^\beta(x, a)\frac{\partial}{\partial a^\beta}$. As in the natural case, we can easily show that the integral curves of S are solution of the system $D_{\dot{c}}\dot{c} = 0$. □

4.3. Torsion and curvature of Γ

Definition 7. A weak torsion associated to Γ and to a bracket form $[\ ,\ ]_\Pi$, is the 2 vectorial form t on $\mathcal{E}$ defined by $t = \frac{1}{2}[J, \Gamma]_\Pi$.

For any $X \in \mathcal{E}$ and $Y \in \mathcal{E}$, we have

$$\begin{aligned} [J, \Gamma]_\Pi(X, Y) = {} & [JX, \Gamma Y]_\Pi + [\Gamma X, JY]_\Pi - J[\Gamma X, Y]_\Pi - J[X, \Gamma Y]_\Pi \\ & - \Gamma[JX, Y]_\Pi - \Gamma[X, JY]_\Pi + J\Gamma[X, Y]_\Pi + \Gamma J[X, Y]_\Pi \\ = {} & [JX, \Gamma Y]_\Pi + [\Gamma X, JY]_\Pi - J[\Gamma X, Y]_\Pi \\ & - J[X, \Gamma Y]_\Pi - \Gamma[JX, Y]_\Pi - \Gamma[X, JY]_\Pi. \end{aligned}$$

Proposition 14. *The weak torsion is skew-symmetric and semi-basic tensor*.

Proof. From the definition, it is clear that t is skew-symmetric.

The torsion is also semi-basic. Indeed, let $V \in \mathcal{E}^{v}$ and $X \in \mathcal{E}$, so:

$$\begin{aligned}[J, \Gamma]_{\Pi}(X, V) = {} & [JX, \Gamma V]_{\Pi} + [\Gamma X, JV]_{\Pi} - J[\Gamma X, V]_{\Pi} \\ & - J[X, \Gamma V]_{\Pi} - \Gamma[JX, V]_{\Pi} - \Gamma[X, JV]_{\Pi} \\ = {} & -[JX, V]_{\Pi} - J[\Gamma X, V]_{\Pi} + J[X, V]_{\Pi} + [JX, V]_{\Pi} \\ = {} & 2J[v_{\Gamma} X, V] = 0.\end{aligned}$$

Locally, let us consider $[A_{\alpha}, A_{\beta}] = C^{\gamma}_{\alpha\beta} \dfrac{\partial}{\partial a^{\gamma}}$,

$$\begin{aligned}[J, \Gamma]_{\Pi}(A_{\alpha}, A_{\beta}) = {} & [JA_{\alpha}, \Gamma A_{\beta}]_{\Pi} + [\Gamma A_{\alpha}, JA_{\beta}]_{\Pi} - J[\Gamma A_{\alpha}, A_{\beta}]_{\Pi} \\ & - J[A_{\alpha}, \Gamma A_{\beta}]_{\Pi} - \Gamma[JA_{\alpha}, A_{\beta}]_{\Pi} - \Gamma[A_{\alpha}, JA_{\beta}]_{\Pi} \\ = {} & \left[\frac{\partial}{\partial a^{\alpha}}, A_{\beta} - 2\Gamma^{\gamma}_{\beta}\frac{\partial}{\partial a^{\gamma}}\right] + \left[A_{\alpha} - 2\Gamma^{\gamma}_{\alpha}\frac{\partial}{\partial a^{\gamma}}, \frac{\partial}{\partial a^{\beta}}\right] \\ & - J\left[A_{\alpha} - 2\Gamma^{\gamma}_{\alpha}\frac{\partial}{\partial a^{\gamma}}, A_{\beta}\right] - J\left[A_{\alpha}, A_{\beta} - 2\Gamma^{\gamma}_{\beta}\frac{\partial}{\partial a^{\gamma}}\right] \\ & - \Gamma\left[\frac{\partial}{\partial a^{\alpha}}, A_{\beta}\right] - \Gamma\left[A_{\alpha}, \frac{\partial}{\partial a^{\beta}}\right] \\ = {} & \left(2\frac{\partial \Gamma^{\gamma}_{\alpha}}{\partial a^{\beta}} - 2\frac{\partial \Gamma^{\gamma}_{\beta}}{\partial a^{\alpha}} - 2C^{\gamma}_{\alpha\beta}\right)\frac{\partial}{\partial a^{\gamma}}.\end{aligned}$$

Therefore,

$$t(A_{\alpha}, A_{\beta}) = \left(\frac{\partial \Gamma^{\gamma}_{\alpha}}{\partial a^{\beta}} - \frac{\partial \Gamma^{\gamma}_{\beta}}{\partial a^{\alpha}} - C^{\gamma}_{\alpha\beta}\right)\frac{\partial}{\partial a^{\gamma}}. \qquad \square$$

Corollary 2. *As in the standard connection theory, with the linear connection* Γ, *one can associate the torsion determined by*:

$$T_{\Gamma}(A_{\alpha}, A_{\beta})|_{x} = \xi|_{(x, a)} t(A_{\alpha}, A_{\beta}) = D_{A_{\alpha}} A_{\beta} - D_{A_{\beta}} A_{\alpha} - [A_{\alpha}, A_{\beta}]_{\Pi}.$$

Proof. Given a linear connection Γ means $\Gamma_{\alpha}^{\beta}(x, a) = a^{\gamma}\Gamma_{\alpha\gamma}^{\beta}(x)$. Then

$$t(A_{\alpha}, A_{\beta})|_{(x, a)} = (\Gamma_{\alpha\beta}^{\gamma}|_{x} - \Gamma_{\beta\alpha}^{\gamma}|_{x} - C_{\alpha\beta}^{\gamma})(x)\frac{\partial}{\partial a^{\gamma}}\bigg|_{(x, a)}.$$

Therefore, $\xi|_{(x, a)} t(A_{\alpha}, A_{\beta}) = D_{A_{\alpha}} A_{\beta} - D_{A_{\beta}} A_{\alpha} - [A_{\alpha}, A_{\beta}]_{\Pi}$. □

Definition 8. The tension of Γ is the 1-vectorial form $H = \frac{1}{2}[C, \Gamma]_{\Pi}$.

Locally, H can be represented in an adapted basis by

$$H = \left(\Gamma_{\alpha}^{\beta} - a^{\gamma}\frac{\partial\Gamma_{\alpha}^{\beta}}{\partial a^{\gamma}}\right)\theta^{\alpha} \otimes \frac{\partial}{\partial a^{\beta}}.$$

The local expression of H shows that the tension is independent of the Lie pre-bracket Π.

In the linear case, the tension vanishes.

Definition 9. The strong-torsion T of Γ is given by $T = i_{S}t - H$, for an arbitrary spray S.

Locally, T can be expressed in an adapted basis by:

$$T_{(x, a)}X = X^{\alpha}\left(a^{\gamma}\frac{\partial\Gamma_{\gamma}^{\beta}}{\partial a^{\alpha}} - \Gamma_{\alpha}^{\beta} + a^{\gamma}C_{\alpha\gamma}^{\beta}\right)\frac{\partial}{\partial a^{\beta}}.$$

For $(x, a) \in E$ and $X \in \mathcal{E}|_{(x, a)}$.

In the linear case, we get

$$T_{(x, a)}X = X^{\alpha}a^{\gamma}(\Gamma_{\gamma\alpha}^{\beta} - \Gamma_{\alpha\gamma}^{\beta} + C_{\alpha\gamma}^{\beta})\frac{\partial}{\partial a^{\beta}}.$$

The torsion coincides as well with the image of the vertical isomorphism ξ of the strong torsion. Indeed,

$$T(X, Y) = \xi|_X T(Y) = D_X Y - D_Y X - [X, Y]_\Pi.$$

Definition 10. The curvature of a connection Γ is given by $R = -\frac{1}{2}[h, h]_\Pi$.

Proposition 15. *The curvature* $R = -\frac{1}{2}[h, h]_\Pi$ *is semi-basic and skew-symmetric form.*

Proof. From the definition of R we can obviously see that R is a skew-symmetric form.

The curvature is semi-basic form as well. Indeed, for $V \in \mathcal{E}^v$ and $X \in \mathcal{E}$, we have:

$$\begin{aligned}[h, h]_\Pi(X, V) &= [hX, hV]_\Pi + [hX, hV]_\Pi - h[hX, V]_\Pi - h[X, hV]_\Pi \\ &\quad - h[hX, V]_\Pi - h[X, hV]_\Pi + h[X, V]_\Pi + h[X, V]_\Pi \\ &= -2h[hX, V] + 2h[X, V] = 0.\end{aligned}$$

The local expression of the curvature in an adapted coordinate system is:

$$R(A_\alpha, A_\beta) = \left(\Gamma_\beta^\delta \frac{\partial \Gamma_\alpha^\gamma}{\partial a^\delta} - \Gamma_\alpha^\delta \frac{\partial \Gamma_\beta^\gamma}{\partial a^\delta} + A_\alpha.\Gamma_\beta^\gamma - A_\beta.\Gamma_\alpha^\gamma - C_{\alpha\beta}^\gamma\right)\frac{\partial}{\partial a^\gamma}. \qquad \square$$

5. Spray Associated to a Regular Lagrangian

5.1. Legendre transformation

Definition 11. A Lagrangian L on E is a map $L : E \to \mathbb{R}$ which is C^∞ on E, C^0 on the zero-section, and verifies $L(0) = 0$,

Based on [6], we will construct a Legendre transformation.

Lemma 2. *Let* $\widetilde{L} : TM \to \mathbb{R}$ *be a differentiable function on TM. Then there exists a unique differentiable map* $\widetilde{\Lambda} : TM \to T^*M$ *such that*

- $q_M \circ \widetilde{\Lambda} = p_M$,
- $\widetilde{\Lambda}$ *of rank* $2n$,
- $\widetilde{\Lambda}^*\omega = d_J \widetilde{L}$,

where ω *is the Liouville form on* T^*M.

In addition, the following diagram is commutative

$$\begin{tikzcd} TM \arrow[r, "\tilde{\Lambda}"] \arrow[d, "p_M"'] & T^*M \arrow[dl, "q_M"] \\ M & \end{tikzcd}$$

From [3], in order, we get the following result:

Lemma 3. *Let* $L : E \to \mathbb{R}$ *be a Lagrangian. There exists a unique map* $\Lambda : E \to T^*M$ *such that the following diagram is commutative*

$$\begin{tikzcd} E \arrow[r, "\Lambda"] \arrow[d, "\pi"'] & T^*M \arrow[dl, "q_M"] \\ M & \end{tikzcd}$$

Although $i_J(dL|_{\mathcal{E}}) = \Lambda^*\omega|_{\mathcal{E}}$, *where* $dL|_{\mathcal{E}}$ *and* $\Lambda^*\omega|_{\mathcal{E}}$ *are the restrictions of* dL *and* $\Lambda^*\omega$ *on* $\mathcal{E} \subset TE$, *respectively.*

Proof. Take $\widetilde{L}$ as a differential extension of L on TM. From the previous theorem, there exists a unique map $\widetilde{\Lambda} : TM \to T^*M$ verifying $\widetilde{\Lambda}^*\omega = i_J d\widetilde{L}$.

Let $\iota : E \hookrightarrow TM$ be the canonical injection. By choosing $\Lambda = \tilde{\Lambda} \circ \iota$, Λ verifies the results of the lemma. Due to the local expression, Λ is unique. Indeed,

$$i_J(dL|_{\mathcal{E}}) = \left(\frac{\partial L}{\partial x^i} dx^i + \frac{\partial L}{\partial a^\beta} da^\beta\right) \circ \left(dx^\alpha \otimes \frac{\partial}{\partial a^\alpha}\right)$$

$$= \frac{\partial L}{\partial a^\alpha} dx^\alpha.$$

Assume that $\Lambda(x^i, a^\alpha) = (x^i, \zeta_i) = (x^i, \Lambda_1(x, a), ..., \Lambda_n(x, a))$. Let U be a vector field tangent to $\mathcal{E}$ on E:

$$U|_{(x,a)} = X^\alpha A_\alpha + Y^\beta \frac{\partial}{\partial a^\beta}.$$

Then

$$(i_J dL)(U) = \frac{\partial L}{\partial a^\alpha} X^\alpha \quad \text{and} \quad (\Lambda^* \omega)(U) = \omega(\Lambda_* U) = \zeta_\alpha X^\alpha.$$

Hence,

$$\zeta_\alpha = \frac{\partial L}{\partial a^\alpha},$$

$$\zeta_{\bar{i}} = 0.$$

Consider now Λ and $\overline{\Lambda}$ the two maps verify the results of the lemma. Therefore:

$$\Lambda(x, a) = (x, \Lambda_1(x, a), ..., \Lambda_n(x, a))$$

$$= \left(x, \left.\frac{\partial L}{\partial a^1}\right|_{(x,a)}, ..., \left.\frac{\partial L}{\partial a^p}\right|_{(x,a)}, 0, ..., 0\right).$$

Similarly,

$$\overline{\Lambda}(x, a) = (x, \overline{\Lambda}_1(x, a), \ldots, \overline{\Lambda}_n(x, a))$$

$$= \left(x, \left.\frac{\partial L}{\partial a^1}\right|_{(x, a)}, \ldots, \left.\frac{\partial L}{\partial a^p}\right|_{(x, a)}, 0, \ldots, 0 \right).$$

Thus, $\Lambda_i = \overline{\Lambda}_i$ and we deduce that Λ is unique. □

Definition 12. The map defined in Lemma 3 is called *Legendre transformation* of L. The Lagrangian L on E is regular if the Legendre transformation Λ has a maximal rank.

Remark 5. The Lagrangian L is regular if and only if $\det\left(\frac{\partial^2 L}{\partial a^\alpha \partial a^\beta}\right) \neq 0$, since the Jacobian matrix of Λ is given by

$$\begin{pmatrix} Id & 0 \\ \left(\frac{\partial^2 L}{\partial x^i \partial a^\beta}\right) & \left(\frac{\partial^2 L}{\partial a^\alpha \partial a^\beta}\right) \end{pmatrix}.$$

5.2. Spray associated to L

Lemma 4. *Let L be a Lagrangian on E. Let us consider the* 2-*form*:

$$\Omega_L = \Lambda^* d\omega.$$

The form Ω_L has the following properties:

(1) $\Omega_L(JX, Y) + \Omega_L(X, JY) = 0$ *for any vector fields X and Y on E tangent to $\mathcal{E}$. In particular, the restriction of Ω_L on the sub-bundle $\mathcal{E}^v$ vanishes*.

(2) *The rank of Ω_L is at most* $2p$. *The restriction of Ω_L on the sub-bundle $\mathcal{E}$ has a maximal rank $2p$ if and only if L is regular*.

Proof. (1) To prove the first property, it will be enough to represent locally the restriction of Ω_L on $\mathcal{E}$. The proof will be elementary. We notice first that $\Omega_L = \Lambda^* d\omega = d(\Lambda^* \omega) = d(i_J dL)$. Therefore, in an adapted coordinate system (x^i, a^α), we have:

$$\Omega_L = \frac{\partial^2 L}{\partial x^i \partial a^\alpha} dx^i \wedge dx^\alpha + \frac{\partial^2 L}{\partial a^\beta \partial a^\alpha} da^\beta \wedge dx^\alpha$$

$$= \sum_{\gamma<\alpha} \left(\frac{\partial^2 L}{\partial x^\gamma \partial a^\alpha} - \frac{\partial^2 L}{\partial x^\alpha \partial a^\gamma} \right) dx^\gamma \wedge dx^\alpha + \frac{\partial^2 L}{\partial x^{\bar{i}} \partial a^\alpha} dx^{\bar{i}} \wedge dx^\alpha$$

$$+ \frac{\partial^2 L}{\partial a^\beta \partial a^\alpha} da^\beta \wedge dx^\alpha$$

$$= \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^\gamma \partial a^\alpha} - \frac{\partial^2 L}{\partial x^\alpha \partial a^\gamma} \right) dx^\gamma \wedge dx^\alpha + \frac{\partial^2 L}{\partial x^{\bar{i}} \partial a^\alpha} dx^{\bar{i}} \wedge dx^\alpha$$

$$+ \frac{\partial^2 L}{\partial a^\beta \partial a^\alpha} da^\beta \wedge dx^\alpha.$$

By restriction into $\mathcal{E}$, we obtain:

$$\Omega_L \big|_{\mathcal{E}} = \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^\alpha \partial a^\beta} - \frac{\partial^2 L}{\partial x^\beta \partial a^\alpha} \right) dx^\alpha \wedge dx^\beta + \frac{\partial^2 L}{\partial a^\alpha \partial a^\beta} da^\alpha \wedge dx^\beta.$$

(2) Using the local representation of Ω_L found in (1), its rank is at most $2p$. Furthermore, the rank will attempt the maximal value if and only if the exterior power Ω_L^p of order p is no-zero. In an adapted coordinate system:

$$\Omega_L^p = \pm \det \left(\frac{\partial^2 L}{\partial a^\alpha \partial a^\beta} \right) dx^1 \wedge \cdots \wedge dx^p \wedge da^1 \wedge \cdots \wedge da^p.$$

Then Ω_L has a rank $2p$ if and only if $\det \left(\frac{\partial^2 L}{\partial a^\alpha \partial a^\beta} \right)$ is no-zero which means L is regular. □

Theorem 1. *There exists a unique spray S on E tangent to $\mathcal{E}$ and solution of the equation*

$$i_S \Omega_L = -d(\Theta_C L - L)|_{\mathcal{E}};$$

where Θ_C is the lie derivative with respect to Liouville vector field C on TE.

Proof. In an adapted coordinate system on E, we have:

$$\Theta_C L - L = a^\alpha \frac{\partial L}{\partial a^\alpha} - L.$$

Therefore,

$$d(\Theta_C L - L)|_{(x,a)} = \left(a^\alpha \frac{\partial^2 L}{\partial a^\beta \partial a^\alpha} \right) da^\beta + \left(a^\alpha \frac{\partial^2 L}{\partial x^i \partial a^\alpha} - \frac{\partial L}{\partial x^i} \right) dx^i.$$

On the other hand, a vector field S on E tangent to $\mathcal{E}$ can be written by:

$$S = S^\alpha A_\alpha + \bar{S}^\alpha \frac{\partial}{\partial a^\alpha}.$$

Similarly to the proof of Lemma 1, the restriction of Ω_L to $\mathcal{E}$ can be written by

$$\Omega_L = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^\alpha \partial a^\beta} - \frac{\partial^2 L}{\partial x^\beta \partial a^\alpha} \right) dx^\alpha \wedge dx^\beta + \frac{\partial^2 L}{\partial a^\alpha \partial a^\beta} da^\alpha \wedge dx^\beta.$$

The equation $i_S \Omega_L = -d(\Theta_C L - L)|_{\mathcal{E}}$ gives:

$$\begin{cases} S^\alpha \dfrac{\partial^2 L}{\partial a^\alpha \partial a^\beta} = a^\alpha \dfrac{\partial^2 L}{\partial a^\alpha \partial a^\beta}, & (1) \\[2ex] S^\alpha \left(\dfrac{\partial^2 L}{\partial x^\alpha \partial a^\beta} - \dfrac{\partial^2 L}{\partial x^\beta \partial a^\alpha} \right) + \bar{S}^\alpha \dfrac{\partial^2 L}{\partial a^\beta \partial a^\alpha} = \dfrac{\partial L}{\partial x^\beta} - a^\alpha \dfrac{\partial^2 L}{\partial x^\beta \partial a^\alpha}. & (2) \end{cases}$$

Since the matrix $\left(\dfrac{\partial^2 L}{\partial a^\alpha \partial a^\beta}\right)$ is invertible, (1) implies

$$S^\alpha = a^\alpha.$$

According to (2), we get

$$\overline{S}^\alpha \frac{\partial^2 L}{\partial a^\alpha \partial a^\beta} = \frac{\partial L}{\partial x^\beta} - a^\alpha \frac{\partial^2 L}{\partial x^\alpha \partial a^\beta}.$$

Therefore, S is a spray on E. Further, this spray will be denoted by S_L. Because Ω_L is symplectic on $\mathcal{E}$ then S_L is unique. □

6. Lagrangian Connections

Let L be a regular Lagrangian on E and Ω_L the associated symplectic form.

Lemma 5. *We have the following properties*:

(1) $\mathcal{E}^v$ *is a Lagrangian sub-bundle with respect to* Ω_L.

(2) *There exists a unique metric* g_L *on* $\mathcal{E}^v$ *defined by* $g_L(Y, Y') = \Omega_L(Y, Z')$, *where* Z' *verifies* $JZ' = Y'$.

(3) *The kernel of* Ω_L *is a supplement of* $\mathcal{E}$ *in TE*.

Proof. (1) Let Y and Y' be two vertical vector fields and let X be a vector field such that $JX = Y$, we have

$$\Omega(Y, Y') = \Omega(JX, Y') = -\Omega(X, JY') = 0.$$

Then $\mathcal{E}^v$ is a Lagrangian sub-bundle of $\mathcal{E}$.

(2) g_L is a Riemannian pseudo-metric on $\mathcal{E}^v$:

- g_L is well defined. Indeed, if Y' is a vector field on E such that

$JY = JY'$, then $\Omega_L(X, J(Y - Y')) = 0$. We obtain $\Omega_L(JX, Y - Y') = 0$ therefore $g_L(JX, JY) = g_L(JX, JY')$.

- g_L is symmetric:

$$g_L(JX, JY) = \Omega_L(JX, Y) = -\Omega_L(X, JY)$$

$$= \Omega_L(JY, X) = g_L(JY, JX).$$

- g_L is not degenerated because Ω_L has a maximal rank.

(3) For $\Omega_L : TE \to T^*E$, we have: $\dim(T^*E) = \dim(Ker\,\Omega_L) + \dim(\mathrm{Im}\,\Omega_L)$. We deduce from the property (2) of Ω_L that $\dim(\mathrm{Im}\,\Omega_L) = rk(\Omega_L) = 2p$, then $\dim(Ker\,\Omega_L) = n - p$ and $Ker(\Omega_L) \cap \mathcal{E} = \{0\}$. Thus $Ker\,(\Omega_L) \oplus \mathcal{E} = TE$. □

Recall that the geodesics of a connection where S_L is its canonical spray, are the integral curves of S_L. Our main goal now is to find a connection which has a canonical spray S_L. For that purpose, we first recall the notion of the Lagrangian connections.

Definition 13. A connection Γ is called *Lagrangian* if the associated horizontal space is Lagrangian sub-bundle with respect to Ω_L.

We can easily prove that Γ is Lagrangian if and only if $i_\Gamma(\Omega_L|_\mathcal{E}) = 0$ which is equivalent to $i_{h_\Gamma}(\Omega_L|_\mathcal{E}) = \Omega_L|_\mathcal{E}$ and to $i_{v_\Gamma}(\Omega_L|_\mathcal{E}) = \Omega_L|_\mathcal{E}$.

Theorem 2. *Let L be a regular Lagrangian and S be a spray. There exists a Lagrangian connection with respect to* Ω_L *such that its canonical spray is S.*

Proof. Suppose that $\Gamma = [J, S]_\Pi + \Upsilon$ is a connection and S is its canonical spray. The connection Γ is Lagrangian if and only if

$$i_\Gamma(\Omega_L|_\mathcal{E}) = 0$$

which means

$$i_{[J,S]_\Pi}(\Omega_L|_{\mathcal{E}}) + i_\Upsilon(\Omega_L|_{\mathcal{E}}) = 0. \qquad (6.1)$$

Take now the vertical vector field $U = S_L - S, U$. Knowing that the connection associated S_L, $\Gamma_L = [J, S_L]$, is Lagrangian therefore (6.1) is equivalent to

$$i_\Upsilon(\Omega_L|_{\mathcal{E}}) - i_{[J,U]_\Pi}(\Omega_L|_{\mathcal{E}}) = 0.$$

Moreover, the Lie pre-bracket $[\,,\,]_\Pi$ has the same value of the classical lie bracket if one of the two vector fields is vertical. Thus $[J, U]_\Pi = [J, U]$. By using the fact $i_J(\Omega_L|_{\mathcal{E}}) = 0$, and according to the Frolicher-Nijenhuis theory, we obtain

$$i_{[J,U]}(\Omega_L|_{\mathcal{E}}) = i_J\Theta_U(\Omega_L|_{\mathcal{E}}).$$

Consequently, for all X and Y tangent to $\mathcal{E}$, the previous condition will be:

$$g_L(\Upsilon X, JY) + (\Theta_U\Omega_L)(JX, Y) = g_L(\Upsilon Y, JX) + (\Theta_U\Omega_L)(JX, X).$$

The problem now is to find the semi-basic symmetric 2-form, ϑ:

$$\vartheta(X, Y) = g_L(\Upsilon X, JY) + (\Theta_U\Omega_L)(JX, Y) \qquad (6.2)$$

which verify

$$\vartheta(S, Y) = -g_L(S^*, JY) + (\Theta_U\Omega_L)(C, Y), \qquad (6.3)$$

with $\Upsilon S_L = \Upsilon S = -S^* = S - [C, S]_\Pi = S - [C, S]$.

Let us consider the semi-basic symmetric 2-form $\vartheta = i_C\Omega_L \odot \overline{\varpi}$, where $\overline{\varpi}$ is a scalar semi-basic 1-form and $\odot$ is the symmetric product. We will show that $\overline{\varpi}$ exists and verifies the condition (6.3). Since $\vartheta(S, Y) = \vartheta(S_L, Y)$, (6.3) is equivalent to:

$$g_L(C, C)\overline{\varpi} + i_{S_L}\overline{\varpi} i_C\Omega_L = -i_{S^*}\Omega_L + i_C\Theta_U\Omega_L. \qquad (6.4)$$

By applying S_L on (6.4), we obtain

$$2i_{S_L}\overline{\omega}g_L(C, C) = -g_L(S^*, C) - (\Theta_U\Omega_L)(S_L, C). \tag{6.5}$$

By replacing the value of $i_{S_L}\overline{\omega}$ found from (6.5) in (6.4), we get

$$\overline{\omega} = \frac{1}{g_L(C, C)}\left[-i_{S^*}\Omega_L + i_C\Theta_U\Omega_L - \frac{g_L(S^*, C) + (\Theta_U\Omega_L)(S_L, C)}{2g_L(C, C)} i_C\Omega_L\right].$$

$\square$

Theorem 3. *Let L be a regular Lagrangian on E, Γ be a Lagrangian connection on $\mathcal{E}$. The spray associated to Γ is S_L if and only if $d_{h_\Gamma}\mathcal{H}|_{\mathcal{E}} = 0$, where $\mathcal{H} = \Theta_C L - L$ is the Hamiltonian.*

Proof. Since Γ is a Lagrangian connection therefore $i_{h_\Gamma}\Omega_L|_{\mathcal{E}} = \Omega_L|_{\mathcal{E}}$.

On the other hand, $d_{h_\Gamma}\mathcal{H} = i_{h_\Gamma}d\mathcal{H}$, gives us:

$$\begin{aligned} i_{h_\Gamma}(i_{S_L}\Omega_L + d\mathcal{H})|_{\mathcal{E}} &= (i_{S_L}i_{h_\Gamma}\Omega_L - i_{h_\Gamma S_L}\Omega_L + i_{h_\Gamma}d\mathcal{H})|_{\mathcal{E}} \\ &= (i_{S_L}\Omega_L - i_{h_\Gamma S_L}\Omega_L + d_{h_\Gamma}\mathcal{H})|_{\mathcal{E}}\ . \end{aligned} \tag{6.6}$$

We notice that $i_{S_L}\Omega_L + d\mathcal{H} = 0$, the writing in (6.6) the proof is done by using:

- Given a spray S_L of Γ, then $h_\Gamma S_L = S_L$ therefore $d_{h_\Gamma}\mathcal{H}|_{\mathcal{E}} = 0$.

- If $d_{h_\Gamma}\mathcal{H}|_{\mathcal{E}} = 0$, then $i_{S_L}\Omega_L - i_{h_\Gamma S_L}\Omega_L = 0$. This implies that $i_{v_\Gamma S_L}\Omega_L = 0$ which means that $i_{v_\Gamma S_L}\Omega_L(Y) = 0$ for all Y tangent to $\mathcal{E}$, but since Ω_L has a maximal rank on $\mathcal{E}$ therefore $v_\Gamma S_L = 0$ and $S_L = h_\Gamma S_L$. $\square$

7. Application

We shall give an example to illustrate the previous results. Consider the case where $M = \mathbb{R}^3$ and let E be the sub-bundle of $TM = \mathbb{R}^6$ generated, at any point $x = (x^1, x^2, x^3) \in M$, by

$$\begin{cases} A_1|_x = \left.\dfrac{\partial}{\partial x^1}\right|_x, \\ A_2|_x = \left.\dfrac{\partial}{\partial x^2}\right|_x + x^1 \left.\dfrac{\partial}{\partial x^3}\right|_x, \\ A_3|_x = \left.\dfrac{\partial}{\partial x^3}\right|_x, \end{cases}$$

where $A_i|_x = A_i^j \left.\dfrac{\partial}{\partial x^j}\right|_x$ for $1 \le i, j \le 3$.

Recall the notation introduced in first section in this paper, we can easily verify that $\{A_1, A_2, A_3\}$ is an adapted basis on E.

Suppose that $\mathcal{A}$ is the transition matrix from the adapted basis $\{A_i\}_{1\le i\le 3}$ to the canonical basis $\left\{\dfrac{\partial}{\partial x^i}\right\}_{1\le i\le 3}$. So $\mathcal{A}$ is given by:

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x^1 & 1 \end{pmatrix}.$$

To find the dual basis $\{\theta^i\}_{1\le i\le 3}$ of $\{A_i\}_{1\le i\le 3}$, we need to find the transition matrix ${}^t\mathcal{A}^{-1}$ matrix from the adapted dual basis $\{\theta^i\}_{1\le i\le 3}$ to the canonical dual basis $\{dx^i\}_{1\le i\le 3}$ which is given by:

$${}^t\mathcal{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -x^1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{cases} \theta^1|_x = dx^1|_x, \\ \theta^2|_x = dx^2|_x, \\ \theta^3|_x = -x^1 dx^2|_x + dx^3|_x. \end{cases}$$

Set up the Lagrangian:

$$L : E \to \mathbb{R}$$

$$(x, a) \mapsto L(x, a) = (a^1)^2 f(x^1, x^2, x^3) + (a^2)^2 g(x^1, x^2, x^3)$$
$$+ U(x^1, x^2, x^3),$$

where *f*, *g* and *U* are two no-zero functions on *M*.

Since the $\det\left(\dfrac{\partial^2 L}{\partial a^\alpha \partial a^\beta}\right) = 4fg \neq 0,$ so *L* is regular.

Let Λ be the Legendre transformation associated to *L*:

$$\Lambda : E \to T^*M$$

$$(x, a) \mapsto \Lambda(x, a) = 2a^1 f dx^1 + 2a^2 g dx^2.$$

The symplectic form $\Omega_L|_{\mathcal{E}}$ is given by:

$$\Omega_L|_{\mathcal{E}} = \Lambda^* d\omega|_{\mathcal{E}} = \left(2a^2 \frac{\partial g}{\partial x^1} - 2a^1 \frac{\partial f}{\partial x^2}\right) dx^1 \wedge dx^2$$
$$+ 2 f da^1 \wedge dx^1 + 2g da^2 \wedge dx^2.$$

The canonical spray S_L of *L*, is written by:

$$S_L(x, a) = a^1 A_1 + a^2 A_2 + S^1(x, a)\frac{\partial}{\partial a^1} + S^2(x, a)\frac{\partial}{\partial a^2},$$

where S^α is given by the expression:

$$S^{\alpha} \frac{\partial^2 L}{\partial a^{\alpha} \partial a^{\beta}} = \frac{\partial L}{\partial x^{\beta}} - a^{\alpha} \frac{\partial^2 L}{\partial x^{\alpha} \partial a^{\beta}}.$$

For $\beta = 1, 2.$ Then

$$S^1 = \frac{1}{2f}\left[-(a^1)^2 \frac{\partial f}{\partial x^1} + (a^2)^2 \frac{\partial g}{\partial x^1} + \frac{\partial U}{\partial x^1} - 2a^1 a^2 \frac{\partial f}{\partial x^2}\right],$$

$$S^2 = \frac{1}{2g}\left[+(a^1)^2 \frac{\partial f}{\partial x^2} - (a^2)^2 \frac{\partial g}{\partial x^2} + \frac{\partial U}{\partial x^2} - 2a^1 a^2 \frac{\partial g}{\partial x^1}\right].$$

The associated metric g_L of L is defined by $g_L(X, Y) = \Omega_L(X, Z),$ where $JZ = Y$. The matrix of g_L is given by:

$$\begin{pmatrix} 2f & 0 \\ 0 & 2g \end{pmatrix}.$$

Since $[A_1, A_2] = \frac{\partial}{\partial x^3} = A_3,$ E is not stable with respect to the Lie bracket.

We can associate the canonical Lie pre-bracket to E verifying $[A_1, A_2]_0 = 0.$

The coefficients of the connection $\Gamma_L = [J, S_L]$ are:

$$\Gamma_1^1(x, a) = -\frac{1}{2}\frac{\partial S^1}{\partial a^1} = \frac{1}{2f}\left\{a^1 \frac{\partial f}{\partial x^1} + a^2 \frac{\partial f}{\partial x^2}\right\},$$

$$\Gamma_1^2(x, a) = -\frac{1}{2}\frac{\partial S^2}{\partial a^1} = \frac{1}{2g}\left\{a^2 \frac{\partial g}{\partial x^1} - a^1 \frac{\partial f}{\partial x^2}\right\},$$

$$\Gamma_2^2(x, a) = -\frac{1}{2}\frac{\partial S^2}{\partial a^2} = \frac{1}{2g}\left\{a^2 \frac{\partial g}{\partial x^2} + a^1 \frac{\partial g}{\partial x^1}\right\},$$

$$\Gamma_2^1(x, a) = -\frac{1}{2}\frac{\partial S^1}{\partial a^2} = \frac{1}{2f}\left\{a^1 \frac{\partial f}{\partial x^2} - a^2 \frac{\partial g}{\partial x^1}\right\}.$$

On the other hand, for any vector fields $X(x) = X^1(x)A_1 + X^2(x)A_2$ and $Y(x) = Y^1(x)A_1 + Y^2(x)A_2$ on M tangents to E, the covariant derivative D associated to the connection Γ_L is characterized by:

$$D_X Y = [X^\alpha \Gamma_\alpha^\beta(x, Y(x)) + X^\alpha A_\alpha(Y^\beta)]A_\beta.$$

In particular,

$$D_{A_1}A_2 = \frac{1}{2f}\left\{\frac{\partial f}{\partial x^1} - \frac{\partial g}{\partial x^1} + 2\frac{\partial f}{\partial x^2}\right\}A_1 + \frac{1}{2g}\left\{\frac{\partial g}{\partial x^2} - \frac{\partial f}{\partial x^2} + 2\frac{\partial g}{\partial x^1}\right\}A_2.$$

Finally, by using the local representation of the weak torsion $t = \frac{1}{2}[J, \Gamma]_0$ and the tension $H = \frac{1}{2}[C, \Gamma]_0$, we get $t = 0$ and $H = 0$, and the strong torsion is also zero, since $T = i_S t - H$.

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