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THE EXISTENCE AND UNIQUENESS OF LOCAL AND GLOBAL SOLUTIONS FOR THE IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY TERMS

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Abstract

In this paper, we consider the existence, uniqueness of global solution of impulsive functional differential equations with delay terms by the nonlinear Leray-Schauder alternative theorem.

1. Introduction

The theory and application of impulsive differential equations stimulated by biology, mechanics, ecology and epidemics, etc., have been an important area of investigation in recent years. The theory of delay differential equations has been studied by Benchohra et al. [1], Lakshmikantham et al. [4] Samoilenko and Perestyuk [8] and Rogovchenko [7]. Quinn and

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Carmichael [6] have shown that the controllability problem in Banach space can be converted into a fixed pointed problem for a single-valued mapping. Recently, Hernández [3] invested the existence of mild solutions for a class of partial neutral functional integro-differential equations with unbounded described in the form

$$\begin{cases} \frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F\left(t, x_t, \int_0^t h(t, s, x_s)ds\right), \quad t \in I = [\sigma, T], \\ x_\sigma = \varphi \in B \end{cases}$$

and Ouahab [5] studied the existence and uniqueness of solutions for first order functional differential equations with impulsive effects and multiple delay as follows:

$$\begin{cases} y'(t) = f(t, y_t) + \sum_{i=1}^{n_*} y(t - T_i) \text{ a.e. } t \in J := [0, b] \setminus \{t_1, t_2, ..., t_m\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, ..., m, \\ y(t) = \phi(t), \quad t \in [-r, 0]. \end{cases}$$

In this paper, we study the existence and uniqueness of local and global solutions for first order impulsive abstract functional differential equation as follows by using nonlinear alternative of Leray-Schauder together with the method by Quinn and Carmichael [6]:

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + f(t, x_t), \quad t \in J = [0, b] \setminus \{t_1, t_2, ..., t_m\}, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \\ I_i(x_{t_i}) = x(t_i^+) - x(t_i^-), \quad i = 1, 2, ..., m, \end{cases} \tag{1.1}$$

where $f : J \times B \to X$ is a given function, $A : D(A) \to X$ is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $T(t)$, $t \geq 0$ and a phase space $(\phi \in) B$ will be defined later. We assume that history function $x_t : (-\infty, 0] \to B$ belongs to the abstract phase space B and defined by

$$x_t(\theta) = x(t + \theta), \quad \text{for } \theta \in (-\infty, 0],$$

and $x(t_i^-), x(t_i^+)$ are left limit, right limit, respectively.

2. Preliminaries

In this section, we now introduce the concept of a phase space, the other spaces, alternative of Leray-Schauder and L^1-Carathéodory function.

Throughout this paper, $C([0, b], X)$ is the Banach space of all continuous functions with the norm

$$\| x \|_X = \sup\{| x(t) | : 0 \leq t \leq b\}$$

and let $PC = \{x : (-\infty, b] \to X$, and $x(t_i^-), x(t_i^+)$ exist, $x_i \in C(J_i, X)\}$ with the norm

$$\| x \|_{PC} = \sup\{| x(s) | : 0 \leq s \leq b\},$$

where x_k is the restriction of x to $J_i = (t_i, t_{i+1}]$.

And let $L^1([0, b], X)$ be the Banach space of measurable functions $x : [0, b] \to X$ are Lebesgue integrable with the norm

$$\| x \|_{L^1} = \int_0^b | x(t) | dt \quad \text{for all } x \in L^1([0, b], X).$$

Now we introduce the following definition:

Definition 2.1. The map $f : [0, b] \times B \to X$ is said to be L^1-*Carathéodory function* if

(i) $t \mapsto f(t, x)$ is measurable for each $x \in B$,

(ii) $x \mapsto f(t, x)$ is continuous for $t \in J - \{t_1, t_2, ..., t_m\}$,

(iii) for each $k > 0$, there exists $h_k \in L^1([0, b], R_+)$ such that

$| f(t, x) | \leq h_k(t)$ for all $\| x \|_B \leq k$ and for $t \in J - \{t_1, t_2, ..., t_m\}$.

Next, we will assume that a phase space B satisfies the following axioms:

(A) If $x : (-\infty, b] \to X$, $b > 0$, $x_0 \in B$ and $x(t_i^-)$, $x(t_i^+)$ exist with $x(t_i) = x(t_i^-)$, $i = 1, 2, ..., m$, then for every t in $[0, b]\backslash\{t_1, ..., t_m\}$ the following conditions hold:

(i) $x_t \in B$ and x_t is continuous on $[0, b]\backslash\{t_1, ..., t_n\}$,

(ii) $| x(t) | \leq H\| x_t \|_B$,

(iii) $\| x_t \|_B \leq K(t)\sup\{\| x(s) \|_X : 0 \leq s \leq t\} + M(t)\| x_0 \|_B$,

where $H \geq 0$ is a constant, $K : [0, +\infty) \to [0, +\infty)$ is continuous function and $M : [0, +\infty) \to [0, +\infty)$ is locally bounded, and H, K, M are independent of $x(t)$.

(A1) For the function $x(\cdot)$ in (A), x_t is a B-valued continuous function on $[0, b]\backslash\{t_1, ..., t_m\}$.

(A2) B is complete.

In this paper, we assume that

$$\| \phi \|_B = \sup\{\| \phi(t) \|\} < \infty$$

and let

$$\Omega = \{x : (-\infty, b] \to X \mid x \in B \cap PC\},$$

let $\| \cdot \|_\Omega$ be the seminorm in Ω defined by

$$\| x_t \|_\Omega = \| x_0 \|_B + \sup\{| x(t) | : 0 \leq t \leq b\}, \quad x \in \Omega.$$

Next, we introduce the Leray-Schauder's alternative theorem.

Theorem 2.2 [8]. *Let D be a convex subset of a Banach space Z and $F : D \to D$ be a completely continuous map. Then, either the set $\{x \in D : x = \lambda F(x),$ for some $0 < \lambda < 1\}$ is unbounded or F has a fixed point in D.*

3. The Existence of Mild Solutions for (1.1)

Assume that $x, x' \in \Omega$, and $x : J \to X$ is a solution of (1.1). Then from semigroup theory, we have

$$x(t) = T(t)\phi(0) + \int_0^t T(t-s) f(s, x_s) ds, \quad t \in [0, t_1),$$

and so

$$x(t_1^-) = T(t_1)\phi(0) + \int_0^{t_1} T(t_1 - s) f(s, x_s) ds, \quad t \in [0, t_1).$$

Since $I_i(x(t_i)) = x(t_1^+) - x(t_1^-)$, for $t \in (t_1, t_2)$, we get

$$\begin{aligned} x(t) &= T(t-t_1)x(t_1^+) + \int_{t_1}^t T(t-s) f(s, x_s) ds \\ &= T(t-t_1)\{x(t_1^-) + I_1(x(t_1))\} + \int_{t_1}^t T(t-s) f(s, x_s) ds \\ &= T(t-t_1)\left[T(t_1)\phi(0) + \int_0^{t_1} T(t_1-s) f(s, x_s) ds + I_1(x(t_1))\right] \\ &\quad + \int_{t_1}^t T(t-s) f(s, x_s) ds \\ &= T(t)\phi(0) + \int_0^t T(t-s) f(s, x_s) ds + T(t-t_1) I_1(x(t_1)). \end{aligned}$$

Iterating in similar fashion, we can show

$$x(t) = T(t)\phi(0) + \int_0^t T(t-s) f(s, x_s) ds + \sum_{0<t_i<t} T(t-t_i) I_i(x(t_i)), \quad t \in J.$$

To show the existence of mild solutions for system (1.1), we need to define the mild solution of system (1.1).

Definition 3.1. A function $x \in \Omega$ is said to be a *mild solution* of system (1.1) if (1.1) is verified and

$$x(t) = T(t)\phi(0) + \int_0^t T(t-s) f(s, x_s) ds + \sum_{0<t_i<t} T(t-t_i) I_i(x(t_i)), \quad t \in J.$$

To get our results, we assume the following:

(H1) There exists constant $M \geq 1$ such that $\| T(t) \| \leq M$, $t \in J$.

(H2) $f : J \times B \to X$ is Carathéodory function.

(H3) There exist a function $p \in L^1(J, R_+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$\| f(t, x_t) \|_X \leq p(t)\psi(\| x_t \|_B) \text{ for a.e. } t \in J \text{ and each } x_t \in B$$

with

$$\int_0^{t_1} K_b M p(s) ds < \int_{\beta(0)}^{\infty} \frac{1}{\psi(s)} ds,$$

where $K_b = \sup\{| K(t) | : t \in [0, b]\}$.

Theorem 3.2. *Suppose* (H1)-(H3) *hold, then the system* (1.1) *has at least one solution on* $(-\infty, b]$.

Proof. Transform equation (1.1) into a fixed point theorem.

Let the operator $\Phi : \Omega \to \Omega$ be defined by

$$\Phi(x)(t) = \begin{cases} T(t)\phi(0) + \int_0^t T(t-s) f(s, x_s) ds \\ \qquad\qquad + \sum_{0<t_i<t} T(t-t_i) I_i(x(t_i)), & t \in J, \\ \phi(t), & t \in (-\infty, 0], \end{cases}$$

and let $y(\cdot) : (-\infty, b] \to B$ be the function defined by

$$y(t) = \begin{cases} T(t)\phi(0), & t \in J, \\ \phi(t), & t \in (-\infty, 0], \end{cases}$$

for each $z \in PC$ with $z(0) = 0,$ we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & t \in J, \\ 0, & t \in (-\infty, 0]. \end{cases}$$

Then we can have $x(t) = \bar{z}(t) + y(t)(t \in J),$ and in similar fashion, $x_t = \bar{z}_t + y_t(t \in J),$ and so the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t T(t-s) f(s, \bar{z}_s + y_s) ds + \sum_{0<t_i<t} T(t-t_i) I_i(\bar{z}(t_i) + y(t_i)), \quad t \in J.$$

Let $\Omega_0 = \{z \in \Omega : z_0 = 0\}.$ Then for any $z \in \Omega_0,$ we have

$$\| z \|_{\Omega_0} = \| z_0 \|_{\Omega_0} + \sup\{| z(t) | : t \in J\} = \sup\{| z(t) | : t \in J\}$$

and so $(\Omega_0, \| \cdot \|_{\Omega_0})$ is a Banach space.

Define the operator $P : \Omega_0 \to \Omega_0$ as follows:

$$(Pz)(t) = \begin{cases} \displaystyle\int_0^t T(t-s) f(s, \bar{z}_s + y_s) ds \\ \qquad\qquad + \displaystyle\sum_{0<t_i<t} T(t-t_i) I_i(\bar{z}(t_i) + y(t_i)), & t \in J, \\ 0, & (-\infty, 0]. \end{cases}$$

Then that the operator Φ has a fixed point is equivalent to that P has a fixed point, and so we will verify that P has a fixed point by using the Leray-Schauder alternative.

Step 1. P is continuous.

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $\Omega_0.$ Then

$$|(Pz_n)(t) - (Pz)(t)| \leq \int_0^b T(t-s)| \, f(s, \bar{z}_{n_s} + y_s) - f(s, \bar{z}_s + y_s)|ds$$

$$+ M \sum_{0<t_i<t} | \, I_i(\bar{z}_n(t_i) + y(t_i)) - I_i(\bar{z}(t_i) + y(t_i))|,$$

and since f is L^1-Carathéodory, we have

$$\|(Pz_n) - (Pz)\|_{\Omega_0} \leq M \| \, f(s, \bar{z}_{n_s} + y_s) - f(s, \bar{z}_s + y_s)\|_{L^1}$$

$$+ M \sum_{0<t_i<t} | \, I_i(\bar{z}_n(t_i) + y(t_i)) - I_i(\bar{z}(t_i) + y(t_i))|.$$

Obviously, the right side of the above equation converges zero as $n \to \infty$.

Step 2. P maps bounded sets into bounded sets in Ω_0.

For the continuous function $K(t)$ and locally bounded $M(t)$,

$$\|\bar{z}_s + y_s\|_{\Omega_0} \leq \|\bar{z}_s\|_{\Omega_0} + \|y_s\|_{\Omega_0}$$

$$\leq K_b(\sup\{|z(s)| : 0 \leq s \leq b\} + |\phi(0)|) + M_b \|\phi\|_{\Omega_0}, \qquad \text{(E1)}$$

where $M_b = \sup\{|M(t)| : t \in [0, b]\}$. Thus, $\|x\|_B$ is bounded, and so by (H3) and definition of space PC,

$$\|(Pz)\|_{\Omega_0} \leq M\psi(\|x\|_B)\int_0^b p(s)ds + M\sum_{i=1}^m \sup\{| \, I_i(\bar{z}(t_i) + y(t_i))|\}$$

is bounded, too.

Step 3. P maps bounded sets into equicontinuous sets of Ω_0. For each $t_* \in [0, b]$, we have

$$|(Pz)(t_* + \varepsilon) - (Pz)(t_*)|$$

$$\leq \int_{t_*}^{t_*+\varepsilon} |T(t_* + \varepsilon - s) - T(t_* - s)|| \, f(s, \bar{z}_s + y_s)|ds$$

$$+ \sum_{0 \text{ or } t_i < t_*, \, t_* + \varepsilon < t_i} |T(t_* + \varepsilon - s) - T(t_* - s)|| \, I_i(\bar{z}(t_i) + y(t_i))|.$$

Since f is L^1-Carathéodory function, we understand that $|(Pz)(t_* + \varepsilon) - (Pz)(t_*)|$ converges to zero independently of $z \in \Omega_0$ as $\varepsilon \to 0$. And so from Arzelá-Ascoli theorem, we can say that the image of $P : \Omega_0 \to \Omega_0$ is relatively compact.

Step 4. There exist a priori bounds on solutions.

Let z be a solution of the equation $z = \lambda P(z)$ for some $\lambda \in (0, 1)$. Then for all $t \in [0, t_1]$,

$$z(t) = \lambda \int_0^t T(t-s) f(s, \bar{z}_s + y_s) ds.$$

And so

$$|z(t)| \le \int_0^t |T(t-s)||f(s, \bar{z}_s + y_s)| ds \le M \int_0^t p(s)\psi(\|\bar{z}_s + y_s\|_B) ds. \quad \text{(E2)}$$

Denote by $\omega(t)$ the right hand side of (E1). Then we have $\|\bar{z}_s + y_s\|_{\Omega_0} \le \omega(t)$ and so

$$|z(t)| \le M \int_0^t p(s)\psi(\omega(s)) ds, \quad t \in [0, t_1].$$

Substituting (E2) for (E1), we get

$$\omega(t) \le K_b M \left[\int_0^t p(s)\psi(\omega(s)) ds + |\phi(0)| \right] + M_b \|\phi\|_{\Omega_0}, \quad t \in [0, t_1]. \quad \text{(E3)}$$

Denote by $\beta(t)$ the right hand side of equation (E3). Then we have $\omega(t) \le \beta(t)$ and $\beta(0) = M_b \|\phi\|_{\Omega_0}$, $\beta'(t) = K_b M p(t)\psi(\omega(t)) \le K_b M p(t)\psi(\beta(t))$. Thus for each $t \in [0, t_1]$,

$$\int_{\beta(0)}^{\beta(t)} \frac{1}{\psi(s)} ds \le \int_0^{t_1} K_b M p(s) ds < \int_{\beta(0)}^{\infty} \frac{1}{\psi(s)} ds.$$

Thus, from (H3) there exists a constant G_* such that $\beta(t) \le G_*$, $t \in [0, t_1]$ and so $\| \bar{z}_t + y_t \|_{\Omega_0} \le \omega(t) \le G_*$, $t \in [0, t_1]$. Hence, by equation (E1), there exists a constant G_1 such that

$$\| z \|_{\Omega_0} \le \frac{G_*}{K_b} = G_1.$$

Now for $t \in (t_1, t_2]$,

$$z(t) = \lambda \int_{t_1}^{t} T(t-s) f(s, \bar{z}_s + y_s) ds$$

and since $z(t_1^+) = z(t_1) + I_1(\bar{z}(t_1) + x(t_1))$,

$$\begin{aligned} | z(t_1^+) | &\le | z(t_1) | + | I_1(\bar{z}(t_1) + x(t_1)) | \\ &\le G_1 + \sup\{| I_i(u) | : | u | \le G_1 + M | \phi(0) |\}. \end{aligned}$$

Thus, in similar fashion as the above proof we can show that there exists $G_2 > 0$ such that

$$\| z \|_{\Omega_0} \le G_2 \quad \text{for} \quad t \in [t_1, t_2].$$

Iterating this process, we can show that there exists constant G such that $\| z \|_{\Omega_0} \le \max\{G_1, G_2, \ldots, G_m\} = G$.

Let $U = \{x \in PC : \| z \|_{\Omega_0} < G + 1\}$. Then $\overline{U} \to \Omega_0$ is continuous and completely continuous. For the choice of U, there is no $z \in \partial U$ such that $z = \lambda P(z)$ for some $\lambda \in (0, 1)$. By the nonlinear alternative of Leray-Schauder, we can see that P has a fixed point z in U. Hence, Φ has a fixed point x which is a solution of system (1.1).

Now assume following hypotheses to give the uniqueness of the system (1.1):

(H4) There exists an $a \in L^1([0, b], R_+)$ such that

$$| f(t, x) - f(t, y) | \leq e^{-a|\sin t|} \| x - y \|_B \text{ for all } x, y \in B \text{ and } t \in J,$$

where $a \in R_+$ is sufficiently large number.

(H5) There exist constants $d_i \geq 0$, $i = 1, 2, ..., m$ such that

$$| I_i(x) - I_i(y) | \leq d_i | x - y | \text{ for all } x, y \in X.$$

Theorem 3.3. *Suppose that hypotheses* (H4), (H5) *hold. If* $\sum_{i=1}^{m} Md_i < 1$, *then equation* (1.1) *has unique solution.*

Proof. Let $P : \Omega_0 \to \Omega_0$ be defined as in Theorem 3.2. We shall show that Φ is a contraction. For $z, z^* \in \Omega_0$ and for each $t \in [0, b]$, let

$$(Pz)(t) = \int_0^t T(t-s) f(s, \bar{z}_s + y_s) ds$$

$$+ \sum_{0<t_i<t} T(t - t_i) I_i(\bar{z}(t_i) + y(t_i)), \quad t \in J.$$

Then

$$| (Pz)(t) - (Pz^*)(t) |$$

$$\leq \int_0^t | T(t-s) \| f(s, \bar{z}_s + y_s) - f(s, \bar{z}_s^* + y_s) | ds$$

$$+ \sum_{0<t_i<t} | T(t - t_i) \| I_i(\bar{z}(t_i) + y(t_i)) - I_i(\bar{z}^*(t_i) + y(t_i)) |$$

$$\leq \int_0^t Me^{-a|\sin s|} \| \bar{z}_s - \bar{z}_s^* \|_B ds + \sum_{0<t_i<t} Md_i | z(t_i) - z^*(t_i) |$$

$$\leq \int_0^t Me^{-a|\sin s|} \sup | z_s - z_s^* | ds + \sum_{0<t_i<t} Md_i | z(t_i) - z^*(t_i) |$$

and so, by the Jordan's inequality,

$$|(Pz)(t) - (Pz^*)(t)| \le M\left(\frac{b\pi}{a} + \sum_{i=1}^{m} d_i\right)\| z - z^* \|.$$

Thus, P is contraction and so has a unique fixed point z, that is, system (1.1) has a unique solution.

4. The Existence and Uniqueness of Global Solutions

In this section, we show that the existence and uniqueness of global solution of impulsive functional differential equations with infinite delay by the nonlinear alternative in Fréchet spaces. We consider the following system:

$$\begin{cases} \frac{d}{dt} x(t) = Ax(t) + f(t, x_t), \quad t \in J = [0, \infty] \backslash \{t_1, t_2, \ldots\}, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \\ I_i(x_{t_i}) = x(t_i^+) - x(t_i^-), \quad i = 1, 2, \ldots, \end{cases} \tag{4.1}$$

where $f : J \times B \to X$ is a given function, $A : D(A) \to X$ is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $T(t),\ t \ge 0$ and a phase space $(\phi \in) B$ will be defined later. We assume that history function $x_t : (-\infty, 0] \to B$ belongs to the abstract phase space B and defined by

$$x_t(\theta) = x(t + \theta), \quad \text{for } \theta \in (-\infty, 0],$$

and $x(t_i^-),\ x(t_i^+)$ are left limit, right limit, respectively.

◎ Let X be a Fréchet space with a family of seminorms $\{\| \cdot \|_n,\ n \in N\}$. $Y \subset X$ is said to be *bounded* if for each $n \in N$, there exists $M_n > 0$ such that $\| y \|_n \le$ for all $y \in Y$.

◎ Let $\{(X^n, \|\cdot\|_n)\}$ be a sequence of Banach space. For each $n \in N$, equivalence relation R_n by $x(R_n)y \Leftrightarrow \|x - y\|_n = 0$ for all $x, y \in X$.

◎ Let $X^n = (X/R_n, \|\cdot\|_n)$ be a quotient space of X and let $[x]_n$ be the equivalence class of $x \in X^n$ for each $x \in X$.

We now introduce contraction, nonlinear alternative of Leray-Schauder type in Fréchet spaces, L^1-Carathéodory function and axioms for phase spaces.

Definition 4.1. A function $f : X \to X$ is *contraction* if for each $n \in N$, there exists $c_n \in (0, 1)$ such that

$$\| f(x) - f(y) \|_n \leq c_n \| x - y \|_n \quad \text{for all } x, y \in X.$$

Theorem 4.2. *Let X be a Fréchet space and* $Y \subset X$ *be a closed subset in X and let* $S : Y \to X$ *be a contraction such that* $S(Y)$ *is bounded. Then one of the following holds*:

(i) *S has a unique fixed point,*

(ii) *there exist* $\lambda \in (0, 1)$, $n \in N$ *and* $x \in \partial_n Y^n$ *such that* $\| x - \lambda S(x) \|_n = 0$.

Definition 4.3. The map $f : [0, b] \times B \to X$ is said to be L^1-*Carathéodory function* if

(i) $t \mapsto f(t, x)$ is measurable for each $x \in B$,

(ii) $x \mapsto f(t, x)$ is continuous for $t \in J - \{t_1, t_2, \ldots\}$,

(iii) for each $k > 0$, there exists $h_k \in L^1([0, b], R_+)$ such that

$$| f(t, x) | \leq h_k(t) \text{ for all } \| x \|_B \leq k \text{ and for } t \in J - \{t_1, t_2, \ldots\}.$$

Axioms 4.4. Assume that phase space B satisfies the following axioms:

(A) If $x : (-\infty, \infty] \to X$, $x_0 \in B$ and $x(t_i^-)$, $x(t_i^+)$ exist with $x(t_i) = x(t_i^-)$, $i = 1, 2, \ldots,$ then for every t in $[0, \infty]\backslash\{t_1, t_2, \ldots\}$, the following conditions hold:

(i) $x_t \in B$ and x_t is continuous on $[0, \infty)\backslash\{t_1, t_2, \ldots\}$,

(ii) $|x(t)| \leq H\| x_t \|_B$,

(iii) $\| x_t \|_B \leq K(t)\sup\{\| x(s)\|_X : 0 \leq s \leq t\} + M(t)\| x_0 \|_B$,

where $H \geq 0$ is a constant, $K : [0, +\infty) \to [0, +\infty)$ is continuous function and $M : [0, +\infty) \to [0, +\infty)$ is locally bounded, and H, K, M are independent of $x(t)$.

(A1) For the function $x(\cdot)$ in (A), x_t is a B-valued continuous function on $[0, \infty)\backslash\{t_1, t_2, \ldots\}$.

(A2) B is complete.

In this section, we assume that

$$\| \phi \|_B = \sup\{\| \phi(t) \|\} < \infty.$$

Let

$$\Omega = \{x : (-\infty, \infty) \to X \mid x \in B \cap PC\},$$

$$\Omega_i = \{x \in \Omega : \sup| x(t) | < \infty,\ t \in J_i\},$$

where $J_i = (-\infty, t_i]$ and let $\| \cdot \|_\Omega$ be the seminorm in Ω defined by

$$\| x_t \|_\Omega = \| x_0 \|_B + \sup\{| x(t) | : 0 \leq t < \infty,\ x \in \Omega\}.$$

To show the existence of solution for system (4.1), we need to define the solution of system (4.1).

Definition 4.5. A function $x \in \Omega$ is said to be the *solution* of system (4.1) if (4.1) is verified and

$$x(t) = T(t)\phi(0) + \int_0^t T(t-s) f(s, x_s) ds + \sum_{0<t_i<t} T(t-t_i) I_i(x(t_i)), \quad t \in J.$$

To get our results, we assume the following:

(H6) There exists constant $M \geq 1$ such that $\| T(t) \| \leq M$, $t \in J$.

(H7) $f : J \times B \to X$ is Carathéodory function.

(H8) There exist a function $p \in L^1(J, R_+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$\| f(t, x_t) \|_X \leq p(t)\psi(\| x_t \|_B) \text{ for a.e. } t \in J \text{ and each } x_t \in B$$

with

$$\int_0^{t_1} K_i M p(s) ds < \int_{\beta(0)}^{\infty} \frac{1}{\psi(s)} ds,$$

where

$$K_i = \sup\{| K(t) | : t \in [0, t_i], i = 1, 2, \cdot\}.$$

(H9) There exists an $a \in L^1([0, b], R_+)$ such that

$$| f(t, x_t) - f(t, y_t) | \leq e^{-a|\sin t|} \| x_t - y_t \|_B \text{ for all } x_t, y_t \in B \text{ and } t \in J,$$

where $a \in R_+$ is sufficiently large number.

(H10) There exist constants $d_i \geq 0$, $i = 1, 2, ..., m$ such that

$$| I_i(x) - I_i(y) | \leq d_i | x - y | \text{ for all } x, y \in X.$$

Theorem 4.6. *Suppose* (H6)-(H10) *hold and* $\sum_{i=1}^{\infty} Md_i < 1$, *then the system* (4.1) *has a unique solution*.

Proof. Transform equation (1.1) into a fixed point theorem.

Let the operator $\Phi : \Omega \to \Omega$ be defined by

$$\Phi(x)(t) = \begin{cases} T(t)\phi(0) + \int_0^t T(t-s) f(s, x_s) ds \\ \qquad\qquad + \sum_{0<t_i<t} T(t-t_i) I_i(x(t_i)), & t \in [0, \infty), \\ \phi(t), & t \in (-\infty, 0], \end{cases}$$

and let $y(\cdot) : (-\infty, \infty) \to B$ be the function defined by

$$y(t) = \begin{cases} T(t)\phi(0), & t \in J, \\ \phi(t), & t \in (-\infty, 0], \end{cases}$$

for each $z \in PC$ with $z(0) = 0,$ we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & t \in J, \\ 0, & t \in (-\infty, 0]. \end{cases}$$

Then we can have $x(t) = \bar{z} + y(t)(t \in J),$ and in similar fashion, $x_t = \bar{z}_t + y_t(t \in J),$ and so the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t T(t-s) f(s, \bar{z}_s + y_s) ds + \sum_{0<t_i<t} T(t-t_i) I_i(\bar{z}(t_i) + y(t_i)), \quad t \in J.$$

Let $\Omega_0 = \{z \in \Omega_i : z_0 = 0\}$. Then for any $z \in \Omega_0,$ we have

$$\| z \|_{\Omega_0} = \| z_0 \|_B + \sup\{| z(t) | : t \in J\} = \sup\{| z(t) | : t \in J\}$$

and so $(\Omega_0, \| \cdot \|_{\Omega_0})$ is a Banach space. Let $\overline{\Omega}_0 = \{z \in \Omega : z_0 = 0\}$. Then $\overline{\Omega}_0$ is Fréchet space with a family of seminorms $\| \cdot \|_{\Omega_0}$. Define the operator $P : \overline{\Omega}_0 \to \overline{\Omega}_0$ as follows:

$$(Pz)(t) = \begin{cases} \int_0^t T(t-s) f(s, \bar{z}_s + y_s) ds \\ \qquad\qquad + \sum_{0<t_i<t} T(t-t_i) I_i(\bar{z}(t_i) + y(t_i)), & t \in J, \\ 0, & (-\infty, 0]. \end{cases}$$

Then that the operator Φ has a fixed point is equivalent to that P has a fixed point, and so we will verify that P has a fixed point by using the Leray-Schauder alternative. Now we will use following four steps in similar fashion as Theorem 3.2.

Step 1. Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $\overline{\Omega}_0$. By the hypothesis that f is L^1-Carathéodory function, we can see that P is continuous by calculating $|(Pz_n)(t) - (Pz)(t)|$.

Step 2. Under the condition of the assumption (H8) and the definition of space PC, we will show that $\| x \|_B$ is bounded, and so $\| (Pz) \|_{\overline{\Omega}_0}$ is bounded, too. Hence, P maps bounded sets into bounded sets in $\overline{\Omega}_0$.

Step 3. Since f is L^1-Carathéodory function, we understand that $|(Pz)(t_* + \varepsilon) - (Pz)(t_*)|$ converges to zero independently of $z \in \Omega_0$ as $\varepsilon \to 0$. Thus, P maps bounded sets into equicontinuous sets of Ω_0 and so from Arzelá-Ascoli theorem, we can say that the image of $P : \Omega_0 \to \Omega_0$ is relatively compact.

Step 4. Let z be a solution of the equation $z = \lambda P(z)$ for some $\lambda \in (0, 1)$. Then for all $t \in [t_{i-1}, t_i]$, from

$$z(t) = \lambda \int_0^t T(t-s) f(s, \bar{z}_s + y_s) ds,$$

we will calculate $\| z \|$. Under the condition of (H8), there exists G_1 such that $\| z \|_{\overline{\Omega}_0} \le G_1$ for $t \in [0, t_1]$ and generally there exists G_i such that $\| z \|_{\overline{\Omega}_0}$

$\leq G_i$ for $t \in [t_{i-1}, t_i]$. Thus, we can show that there exists constant G such that $\| z \|_{\overline{\Omega}_0} \leq \max\{G_1 \cdot G_2, \ldots, G_m\} = G$. Let $U = \{x \in PC : \| z \|_{\overline{\Omega}_0} < G + 1\}$. Then $\overline{U} \to \Omega_0$ is continuous and completely continuous. For the choice of U, there is no $z \in \partial U$ such that $z = \lambda P(z)$ for some $\lambda \in (0, 1)$. That is, there exist a priori bounds on solutions. By the nonlinear alternative of Leray-Schauder, we can see that P has a fixed point z in U. Hence, Φ has a fixed point x which is a solution of system (4.1).

Let z be such a solution. Then there exists some $\overline{\lambda} \in (0, 1)$ such that $z = \overline{\lambda}\overline{P}(z)$. In succession, we can put constant $\overline{G_i} > 0$ such that $\| z \|_{\Omega_i} \leq \overline{G_i}$, $i = 1, 2, \ldots$. Now let

$$F = \{z \in \overline{\Omega}_0 : \{\sup | z(t) | : t \in J_i\} \leq \overline{G_i} + 1, \ i = 1, 2, \ldots\}.$$

Then F is a closed subset of $\overline{\Omega}_0$. And for $z, z^* \in \overline{\Omega}_0$ and for each $t \in [0, t_i]$, by verifying

$$| (Pz)(t) - (Pz^*)(t) | \leq M\left(\frac{b\pi}{a} + \sum_{i=1}^{m} d_i\right)\| z - z^* \|,$$

we can say that $P : F \to \Omega_i$ is *contraction* and so from the choice of F, there is no $z \in \partial F^n$ such that $z = \overline{\lambda}\overline{P}(z)$ for some $\overline{\lambda} \in (0, 1)$. Thus by Theorem 4.2, P has a unique fixed point z, that is, system (4.1) has a unique solution.

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