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INERTIAL MANIFOLDS FOR DUAL PERTURBATIONS OF THE CAHN-HILLIARD EQUATIONS

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Abstract

In this paper, on the basis of the singular perturbations of the Cahn-Hilliard equtions, by verifying the spectral gap condition, we consider the inertial manifolds for dual perturbations of the Cahn-Hilliard equations.

1. Introduction

In this paper, we consider the inertial manifolds for dual perturbations of the Cahn-Hilliard equations by studying exponential attractor of this equation and verifying spectral gap condition

$$\varepsilon(u_{tt} + u_t) - \alpha\Delta u_t + \Delta^2 u_t - \Delta u^k = f,\ x \in \Omega \subset R^n, \tag{1.1}$$

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$$u(x,\,0) = u_0(x); \quad u_t(x,\,0) = u_1(x), \quad x \in \Omega \subset R^n, \tag{1.2}$$

$$u(x,\,t)|_{\partial\Omega} = 0, \tag{1.3}$$

where $u_0, u_1 \in H_0^1(\Omega) \cap H^2 \cap L^2(\Omega)$, and ε, α are positive constants, especially, $k \geq 2$, $u = u(x,\,t)$ is a real-valued function.

We have known the long-time behavior of the semiflows generated by equation (2.1) when $\varepsilon, \delta = 0$, certainly, we also have discussed the semiflows generated by equation (2.1) when $\delta = 0$. Our motivations for this study reside in part in the fact that equation (2.1) and when $\delta = 0$ are the examples of nonlinear beam equations with viscous dissipation, which are hyperbolic. However, in many situations, it is found that the asymptotic properties of the solutions of the parabolic equations and those of their hyperbolic perturbations are similar, in the next section, we will consider the inertial manifolds for dual perturbations of the Cahn-Hilliard equations for the effects of external *f*.

The rest of this paper is organized as follows: in Section 2, we introduce basic concepts concerning inertial manifolds. In Section 3, we obtain the existence and non-existence of the inertial manifolds.

2. Preliminaries

Let *X* be a Banach space, $L^p(a,\,b;\,X)$ be a function space from $(a,\,b)$ to *X*, and its norm be $\| u \|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}$, $p \geq 1$.

Assume $L^2 = L^2(0,\,\pi)$, $H^m = H^m(0,\,\pi)$, m is a positive integer and $m \geq 1$, $\| \cdot \|_m$ is the norm of H^m, $| \cdot |$ is the norm of L^2. $(\cdot,\,\cdot)$ is an inner product of $L^2(0,\,\pi)$; $\Delta := \dfrac{\partial^2}{\partial x^2}$, $H^\alpha = D\left((-\Delta)^{\frac{\alpha}{2}}\right)$, $\| u \|_\alpha = \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|$, $u \in H^\alpha$. Because of Poincaré inequality, we have $\| u \|_1 = \| \nabla u \|$.

We have discussed the singular perturbations of the Cahn-Hilliard equations:

$$\varepsilon u_{tt} + u_t + \Delta(\Delta u - u^3 + u - \delta u_t) = 0, \tag{2.1}$$

in particular, $\varepsilon \geq 0$, $\delta \geq 0$, $x \in (0, \pi)$, $t > 0$.

On the basis of equation (2.1), we consider equations (1.1), (1.2) and (1.3), first, we have mathematical variable about time, $t \to \sqrt{\varepsilon}t$, hence, equation (1.1) is (2.2),

$$u_{tt} + \sqrt{\varepsilon}u_t - \frac{\alpha}{\sqrt{\varepsilon}}\Delta u_t + \Delta^2 u_t - \Delta u^k = f. \tag{2.2}$$

Assume $U = (u, u_t) = (u, v) \in X$, $U_t = (v, u_{tt})$,

$$A = \begin{pmatrix} 0 & -I \\ \Delta^2 & \sqrt{\varepsilon} - \frac{\alpha}{\sqrt{\varepsilon}}\Delta \end{pmatrix}, \quad F(U) = \begin{pmatrix} 0 \\ g(u) \end{pmatrix}, \quad g(u) = \Delta u^k + f,$$

therefore, we transform equation (1.1) into an equivalent fist-order system of the form

$$U_t + AU = F(u), \quad U \in X. \tag{2.3}$$

Now, we can do priori estimates for equation (2.2).

Lemma 1. *If* Ω *is a bounded region*, *there exists* $\xi, \eta > 0$, *we have inequation* (2.4),

$$|v|^2 + \|u\|_1^2 + \|u\|_2^2 \leq \xi(|v_0|^2 + \|u_0\|_1^2 + \|u_0\|_2^2) + \eta. \tag{2.4}$$

Among them

$$u \in H_0^1(\Omega) \cap H^2(\Omega) \cap L^2(\Omega), \tag{2.5}$$

$$u_0 \in L^\infty\left(0, T; H_0^1(\Omega) \cap H^2(\Omega) \cap L^2(\Omega)\right), \tag{2.6}$$

$$u_1 \in L^2(0, T; L^2(\Omega)) \cap L^\infty(0, T; L^2(\Omega)). \tag{2.7}$$

Proof. In order to writing, let $v' = \frac{\partial v}{\partial t}$, $v'' = \frac{\partial^2 v}{\partial t^2}$, and so on. For $\varepsilon > 0$, let $v = u_t + \sqrt{\varepsilon} u$, we multiply v for both sides of equation (2.2),

$$(u'', v) + \sqrt{\varepsilon}(u', v) - \frac{\alpha}{\sqrt{\varepsilon}}(\Delta u', v) + (\Delta^2 u, v) - (\Delta u^k, v) = (f, v), \quad (2.8)$$

for equation (2.8), we have

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \frac{\sqrt{\varepsilon}}{2}\frac{d}{dt}\| u \|_1^2 + \frac{1}{2}\frac{d}{dt}\| u \|_2^2 - \frac{\alpha}{\sqrt{\varepsilon}}(\Delta u', u') + \sqrt{\varepsilon}\| u \|^2 - (\Delta u^k, v)$$

$$= (f, v), \quad (2.9)$$

because of Sobolev embedding theorem and Nirenberg-Gagliardo inequation, there are

$$(\Delta u, v) \le |(\Delta u, v)| \le |\Delta u| \cdot |v| \le \frac{1}{2}|\Delta u|^2 + \frac{1}{2}|v|^2, \quad (2.10)$$

$$\| u \|_{L^2}^2 \le C_1 \| u \|_{H^2}^2, \quad (2.11)$$

certainly, there are also

$$-\frac{\alpha}{\sqrt{\varepsilon}}(\Delta u', u') = \frac{\alpha}{\sqrt{\varepsilon}}\| v \|_1^2 + \alpha(\Delta u, v) - \alpha(\nabla v, \nabla u) + \alpha\sqrt{\varepsilon}\| u \|_1^2, \quad (2.12)$$

$$\begin{aligned}(\Delta u^k, v) &\le |\Delta u^k| \cdot |v| \le k(k-1)u^{k-2}|\Delta u| \cdot |v| \\ &\le k(k-1)|u|_{L^{k-2}}^{k-2}\| u \|_2 \cdot |v| \\ &\le C\| u \|_2 \cdot |v| \le \frac{\rho}{2}|v|^2 + \frac{2C}{\rho}\| u \|_2^2, \end{aligned} \quad (2.13)$$

$$(f, v) \le |(f, v)| \le |f| \cdot |v| \le \frac{|f|^2}{2} + \frac{|v|^2}{2}, \quad (2.14)$$

with the help of (2.10), (2.11), (2.12), (2.13) and (2.14), we can see (2.9) that

$$\frac{d}{dt}\left(\frac{1}{2}|v|^2 + \frac{\sqrt{\varepsilon}}{2}\|u\|_1^2 + \frac{1}{2}\|u\|_2^2\right) \tag{2.15}$$

$$\leq \frac{\rho\sqrt{\varepsilon} - C_1\sqrt{\varepsilon} - 2\alpha + 2}{2\sqrt{\varepsilon}}|v|^2 + \frac{1 - 2\alpha\sqrt{\varepsilon}}{2}\|u\|_1^2$$

$$+ \frac{4C + \rho\alpha - 2\rho\sqrt{\varepsilon}}{2\rho}\|u\|_2^2 + \frac{|f|^2}{2}. \tag{2.16}$$

Assume

$$\frac{\rho\sqrt{\varepsilon} - C_1\sqrt{\varepsilon} - 2\alpha + 2}{2\sqrt{\varepsilon}} > 0, \quad \frac{1 - 2\alpha\sqrt{\varepsilon}}{2} > 0, \quad \frac{4C + \rho\alpha - 2\rho\sqrt{\varepsilon}}{2\rho} > 0, \tag{2.17}$$

let

$$\beta = \min\left\{\frac{1}{2}, \frac{\sqrt{\varepsilon}}{2}\right\}, \tag{2.18}$$

$$\gamma = \max\left\{\frac{\rho\sqrt{\varepsilon} - C_1\sqrt{\varepsilon} - 2\alpha + 2}{2\beta\sqrt{\varepsilon}}, \frac{1 - 2\alpha\sqrt{\varepsilon}}{2\beta}, \frac{4C + \rho\alpha - 2\rho\sqrt{\varepsilon}}{2\beta\rho}\right\}. \tag{2.19}$$

We combine (2.17), (2.18) and (2.19), we have (2.16) that

$$\frac{d}{dt}(|v|^2 + \|u\|_1^2 + \|u\|_2^2) \leq \gamma(|v|^2 + \|u\|_1^2 + \|u\|_2^2) + \frac{|f|^2}{2\beta}, \tag{2.20}$$

because of Gronwall inequation,

$$|v|^2 + \|u\|_1^2 + \|u\|_2^2 \leq \xi(|v_0|^2 + \|u_0\|_1^2 + \|u_0\|_2^2) + \eta, \tag{2.21}$$

therefore, Lemma 1 is proved. □

Definition 1. Assume $S = (S(t))_{t\geq 0}$ is a semiflow in Banach space X, if μ is a finite dimensional Lipschitz manifold in X, and it satisfies the following conditions:

(1) μ is positively invariant about semiflow $S(t)$, that is to say, $S(t)\mu \subseteq \mu,\ t \geq 0$;

(2) μ is exponential attractor trajectory, that is to say, $\forall x \in X$, there are $\gamma' > 0$ and $C' > 0$ such that $\forall t \geq 0$,

$$dist(S(t)x, \mu) \leq C'e^{-\gamma' t}, \quad t \geq 0, \tag{2.22}$$

therefore, μ is an inertial manifold about *s*.

In order to describing the spectral gap condition, first, we consider that the nonlinear term $F : X \to X$ is said to be *bounded* and *whole Lipschitz continuous*, and there is a Lipschitz constant l_F; its operator *A* has a number of positive real characteristic values, the characteristic function is expanded into the corresponding orthogonal space *X*, and assume that $F \in C_b(X, X)$ satisfies the Lipschitz condition:

$$\| F(u) - F(u^*) \|_X \leq l_F \| u - u^* \|_X, \quad u, u^* \in X. \tag{2.23}$$

Definition 2. Assume the point spectrum of operator *A* can be divided into the following two parts σ_1 and σ_2, of which σ_1 is finite and such that if

$$\wedge_1 = \sup\{Re\lambda \,|\, \lambda \in \sigma_1\}, \quad \wedge_2 = \inf\{Re\lambda \,|\, \lambda \in \sigma_2\}, \tag{2.24}$$

$$X_i = \text{span}\{w_j \,|\, \lambda_j \in \sigma_j\}, \quad j = 1, 2, \tag{2.25}$$

span says expansion into space symbols, $Re\lambda$ denotes the real part of a complex number λ, and

$$\wedge_1 - \wedge_2 > 4l_F, \tag{2.26}$$

then the orthogonal decomposition

$$X = X_1 \oplus X_2, \tag{2.27}$$

holds, with continuous orthogonal projections $P_1 : X \to X_1$ and $P_2 : X \to X_2$. Hence, the operator *A* is said to satisfy the spectral gap condition.

Lemma 2. *Let* $g(u) = \Delta u^k$ $(k \geq 3)$, $g : H^2 \cap H_0^1 \to H$ *is said to be bounded and whole Lipschitz continuous function.*

Proof. $\forall u,\ u^* \in H^2 \cap H_0^1$,

$$
\begin{aligned}
|\, g(u) - g(u^*) \,| &= |\, \Delta u^k - \Delta u^{*k} \,| \\
&= |\, k(k-1)u^{k-2}(\nabla u)^2 - k(k-1)u^{k-2}(\nabla u^*)^2 \\
&\quad + ku^{k-1}\Delta u - ku^{k-1}\Delta u^* \,| \\
&\le |\, k(k-1)u^{k-2}(\nabla u)^2 - k(k-1)u^{k-2}(\nabla v)^2 \,| \\
&\quad + |\, ku^{k-1}\Delta u - ku^{k-1}\Delta u^* \,| \\
&\le k(k-1)(|\, u^{k-2} \,|_{L^\infty} + |\, u^{*k-2} \,|_{L^\infty})(|\, \nabla u + \nabla u^* \,|)(|\, \nabla u - \nabla u^* \,|) \\
&\quad + k(|\, u^{k-1} \,|_{L^\infty} + |\, u^{*k-1} \,|_{L^\infty})(|\, \Delta u - \Delta u^* \,|) \\
&\le l_0^{k-2} |\, \nabla u - \nabla u^* \,| + l_0^{k-1} |\, \Delta u - \Delta u^* \,| \\
&\le l_0(|\, \nabla u - \nabla u^* \,| + |\, \Delta u - \Delta u^* \,|).
\end{aligned}
$$

Let $l = l_0$. Then l is a Lipschitz coefficient of function $g(u)$. Therefore, with the help of Lemma 1, Lemma 2 is proved. □

For first-order system of the form (2.3), the solution can be expressed in the characteristic function of A,

$$
U_t = \sum_{j=1}^{\infty} (u(t),\ w_j) w_j = \sum_{j=1}^{\infty} C_j(t) w_j,
$$

$P_n : H \to \operatorname{span}\{w_1, \ldots, w_j\}$, $Q_n = I - P_n$. Let $p = P_n u = Pu$, $q = Q_n u = Qu$.

Definition 3. Assume arbitrary solutions $u(t)$ and $\overline{u(t)}$ of equation (2.3) satisfy:

(1) because of cone invariance $|\, q(0) - \bar{q}(0) \,| \le |\, p(0) - \bar{p}(0) \,|$, we have $|\, q(t) - \bar{q}(t) \,| \le |\, p(t) - \bar{p}(t) \,|$, $t > 0$ and

(2) attenuation properties: because of $|q(t) - \bar{q}(t)| \geq |q(0) - \bar{q}(0)|$, $(\exists t > 0)$, we have

$$|q(t) - \bar{q}(t)| \leq |q(0) - \bar{q}(0)| e^{-kt}, \tag{2.28}$$

therefore, question (2.3) has strong squeezing properties.

Lemma 3. *If question* (2.3) *satisfies strong squeezing properties*, *then there exists a Lipschitz function* $\Phi : P_n H \to Q_n H$,

$$|\Phi(p_1) - \Phi(p_2)| \leq |p_1 - p_2|, \quad \forall p_1, p_2 \in P_n H \tag{2.29}$$

and attractor $A \subset graph(\Phi)$.

Proof. For $u, v \in A$, we have $|Qu - Qv| \leq |Pu - Pv|$, otherwise, because of equation (2.28) and invariance of operator A, there exist $u_t, v_t \in A$, $u = S(t)u_t$, $v = S(t)v_t$ for any t, therefore,

$$|u - v| \leq e^{-kt} |u_t - v_t|,$$

A is finite $(|u| \leq R, \forall u \in A)$, $|u - v| \leq 2Re - kt$, $\forall t > 0$, therefore, $u = v$. We define a Lipschitz function $\Phi : PA \to QA$, $\Phi(Pu) = Qu$. Because of squeezing properties, (2.29) is proved. □

Definition 4. If there is a bounded absorbing ball $B(0, \rho)$ in Hilbert space H, $B(0, \rho) \cap PH$ is a positive constant and $\forall t \geq 0$, $PS(t)[PH] = PH$, that is to say, $\forall p \in PH$, there exists $p_0 \in PH$, we have $p = P(S(t) p_0)$, therefore, equation (2.3) has similar preparation conditions.

Lemma 4. *Assume equation* (2.3) *has strong squeezing properties and similar preparation conditions*, *therefore*, *equation* (2.3) *has an inertial manifold* $\mu = graph(\Phi)$, *Lipschitz function* $\Phi : P_n H \to Q_n H$,

$$|\Phi(p) - \Phi(\bar{p})| \leq |p - \bar{p}|, \quad \forall p, \bar{p} \in P_n H.$$

Proof. Let $\Phi_0 = 0$ and $\mu_0 = P_n H = graph(\Phi_0)$, $S(t)$ be semigroups, therefore,

$$\mu_t = S(t)\mu_0 = \{S(t)\mu_0 \,|\, u_0 \in \mu_0\},$$

we will prove $\mu_t = graph(\Phi_t)$, Lipschitz constant of Φ_t is less than or equal to 1. For $\forall u, \bar{u} \in \mu_0$, there is $q = \bar{q} = 0$, therefore, $|q - \bar{q}| \leq |p - \bar{p}|$. According to strong squeezing properties, we know that

$$|q(t) - \bar{q}(t)| \leq |p(t) - \bar{p}(t)|.$$

Therefore, $|q_1 - q_2| \leq |p_1 - p_2|$, $\forall u_1, u_2 \in \mu_t$. For $p \in P\mu_t$, there are unique $\Phi_t(p)$ and $p + \Phi_t(p) \in \mu_t$. Because of similar preparation conditions, $\mu_t = graph(\Phi_t)$, $P\mu_t = PH$.

Further, we consider the state of Φ_t $(t \to \infty)$. There are $u = p + \Phi_t(p)$ and $\bar{u} = p + \Phi_\tau(p)$, $\tau > t$. Because $u = S(t)u_0 \in \mu_t$, $u_0 \in PH$. $\bar{u} = S(\tau)\overline{u_0} \in \mu_t$, $\overline{u_0} \in PH$.

Assuming $\Phi_t(p) \neq \Phi_\tau(p)$, we have

$$|QS(t)u_0 - QS(\tau)\overline{u_0}| = |\Phi_t(p) - \Phi_\tau(p)| > 0 = |PS(t)u_0 - PS(t)\overline{u_0}|.$$

And because of the exponential decay,

$$|\Phi_t(p) - \Phi_\tau(p)| \leq |Qu_0 - QS(\tau - t)\overline{u_0}|e^{-kt} \leq |QS(\tau - t)\overline{u_0}|e^{-kt}.$$

Assume $u_0 \in PH$, because of similar preparation conditions $S(t)u_0 \in B(0, \rho) \cap PH$, $t \geq 0$,

$$|QS(\tau - t)\overline{u_0}| \leq \rho \quad \text{and} \quad |\Phi_t - \Phi_\tau|_\infty \leq \rho e^{-kt}, \quad \tau > t.$$

This indicates that the sequences of $\{\Phi_n\}$ are the sequences of Cauchy, therefore, it converges to the Lipschitz function Φ. We have limit

$$|\Phi_t - \Phi|_\infty \leq \rho e^{-kt}, \quad \tau > t,$$

therefore, the graph of Φ is called μ, that is to say, $\mu = graph(\Phi)$.

Assume $u_0 \in \mu$, $u_0 = p + \Phi(p)$. We consider approximation of u_0, $u_0^t \in \mu_t$, $u_0^t = p + \Phi(p)$, $S(\tau)u_0^t \in \mu_{t+\tau}$. Let $t \to \infty$, by the continuous dependence of the corresponding initial value of the solution,

$$S(\tau)u_0^t \to S(\tau)u_0.$$

Φ_t converges Φ uniformly, for $S(\tau)u_0^t = P[S(\tau)u_0^t] + \Phi_{t+\tau}(P[S(\tau)u_0^t])$, we have limit $P[S(\tau)u_0] + \Phi(P[S(\tau)u_0])$. Therefore, $S(\tau)u \in \mu$, μ is invariant manifold.

Finally, we prove manifold μ is exponential attraction. We consider initial conditions $u_0 \in B(0, \rho)$, $u = S(t)u_0 = p + q$, $\forall \bar{u} \in \mu$, $\bar{u} = p + \Phi(p)$, we have

$$|Qu - Q\bar{u}| > 0 = |Pu - P\bar{u}|,$$

therefore,

$$|u - \bar{u}| = |q - \bar{q}| \leq |Qu_0 - \Phi(Z)|e^{-kt},$$

$S(t)[Z + \Phi(Z)] = \bar{u}$, and

$$dist(S(t)u_0, \mu) \leq |u - \bar{u}| \leq (\rho + \|\Phi\|_\infty)e^{-kt}.$$

Initial conditions u_0 do not necessarily within $B(0, \rho)$ more generally, there exists $t_0(Y)$ $(Y \subset H)$ which is a bounded set, $u_0 \in Y$, we can get $S(t)u_0 \in B(0, \rho)$. Therefore,

$$dist(S(t)u_0, \mu) = dist(S(t - t_0)[S(t_0)u_0], \mu)$$

$$\leq (\rho + \|\Phi\|_\infty)e^{-k(t-t_0)} \leq C(Y)e^{-kt}.$$

Constant C depends on bounded set Y.

Assume nonlinear $F(u)$ is whole Lipschitz in equation (2.3), with the help of Lemma 2, we have

$$| F(u) - F(v) | \leq C_1 | u - v |, \quad u, v \in H. \tag{2.30}$$

□

Lemma 5. *Assume there is an n that makes characters* λ_n *and* λ_{n+1} *satisfy the following condition*:

$$\lambda_{n+1} - \lambda_n > 4C_1, \tag{2.31}$$

therefore, *squeezing properties are established*, $k \geq \lambda_n + 2C_1$, k *is in equation* (2.28).

Proof. Assume u, $\bar{u}$ are the solutions of equation (2.3), $w = u - \bar{u}$, and w is in bounded cone, that is to say,

$$\{(u, \bar{u}) | | Q(u - \bar{u}) | \leq P | u - \bar{u} | \}.$$

First, the trajectory cannot leave the cone, we need to prove that $\frac{du}{dt}(| Qw | - | Pw |)$ (when $| Qw | = | Pw |$) is negative. w satisfies

$$\frac{dw}{dt} + Aw = F(u) - F(\bar{u}). \tag{2.32}$$

Let $p = Pw$, $q = Qw$, because of equation (2.32),

$$\frac{dp}{dt} + Ap = PF(u) - PF(\bar{u}), \tag{2.33}$$

$$\frac{dq}{dt} + Aq = QF(u) - QF(\bar{u}), \tag{2.34}$$

and equation (2.33) takes the inner product by $p = Pw$,

$$\frac{1}{2}\frac{d}{dt}| p |^2 + \| p \|^2 = (PF(u) - PF(\bar{u}), p) = -(PF(\bar{u}) - PF(u), p),$$

$$\frac{1}{2}\frac{d}{dt}| p |^2 = -\| p \|^2 - (PF(\bar{u}) - PF(u), p) \geq -\lambda_n | p |^2 - C_1 | w || p |.$$

When $|q(0)| = |p(0)|$,

$$\left(\frac{d}{dt}|p|\right)_{t=0} \geq (-\lambda_n + 2C_1)|q|,$$

$$\frac{1}{2}\frac{d}{dt}|q|^2 + \|q\|^2 = (QF(u) - QF(\bar{u}), q),$$

$$\frac{1}{2}\frac{d}{dt}|q|^2 \leq -\lambda_{n+1}|q|^2 + C_1|q||w|. \tag{2.35}$$

When $|q(0)| = |p(0)|$, $\left(\frac{1}{2}\frac{d}{dt}|q|^2\right)_{t=0} \leq -\lambda_{n+1}|q|^2 + 2C_1|q|$. Therefore,

$$\left(\frac{d}{dt_+}|q|\right)_{t=0} \leq -(\lambda_{n+1} - 2C_1)|q|.$$

When $t = 0$, $\frac{d}{dt_+}(|q| - |p|)_{t=0} \leq -(\lambda_{n+1} - \lambda_n - 4C_1)|q(0)|$. We prove invariance of cone by the spectrum gap condition (2.31).

We have $|q| \geq |p|$ without cone, by equation (2.35), we get

$$\frac{1}{2}\frac{d}{dt}|q|^2 \leq -\lambda_{n+1}|q|^2 + C_1|q||w|.$$

Let $k = \lambda_{n+1} - 2C_1$, and with the help of inequation of Gronwall, we can get

$$|q(t)| \leq |q(0)|e^{-kt}.$$

□

3. Inertial Manifolds

In this section, we will discuss many cases of the parameters, then we obtain the existence and non-existence of inertial manifolds.

3.1. Existence

To determine the characteristic value of the matrix operator A, we have the inner product on X first,

$$(U, V)_X = (\Delta u, \Delta \overline{y}) + (\overline{z}, v), \tag{3.1}$$

$U = (u, v)$, $V = (y, z) \in X$, $\overline{y}$, $\overline{z}$ are conjugated for y, z,

$$(AU, U)_X = (-\Delta v, \Delta \overline{u}) + \left(\overline{v}, \Delta^2 u + \left(\sqrt{\varepsilon} - \frac{\alpha}{\sqrt{\varepsilon}}\Delta\right)v\right)$$

$$= \sqrt{\varepsilon}|v|^2 + \frac{\alpha}{\sqrt{\varepsilon}}\|v\|_1^2. \tag{3.2}$$

Therefore, the operator A is monotonically increasing, and $(AU, U)_X$ is a nonnegative real number.

To determine the eigenvalues of A, we observe the eigenvalue equation

$$AU = \lambda U, \quad U = (u, v) \in X$$

is equivalent to the system

$$\begin{cases} -v = \lambda u, \\ \Delta^2 u + \left(\sqrt{\varepsilon} - \dfrac{\alpha}{\sqrt{\varepsilon}}\Delta\right)v = \lambda u. \end{cases} \tag{3.3}$$

Thus, u must solve the eigenvalue problem

$$\begin{cases} \lambda^2 u + \left(\dfrac{\alpha}{\sqrt{\varepsilon}} - \sqrt{\varepsilon}\right)u\lambda + \Delta^2 u = 0, \\ u(0) = u(\pi) = 0, \quad \Delta u(0) = \Delta u(\pi) = 0. \end{cases} \tag{3.4}$$

We easily see that (3.3) has, for each positive integer j, the pair of eigenvalues

$$U_j^{\pm} = (u_j, v_j) = (u_j, -\lambda_j^{\pm} u_j), \quad u_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx), \tag{3.5}$$

so that A does have countable set of eigenvalues, with $\Re u_j^{\pm} > 0$ for all j. Because of the first of (3.3), the corresponding eigenfunctions have the form $U_j^{\pm} = (u_j, -\lambda_j^{\pm} u_j)$, with $u_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$. For future reference, we note that for all $j > 1$,

$$\| u_j \|_1 = j, \quad \| u_j \|_{-1} = \frac{1}{j}, \tag{3.6}$$

therefore, we substitute u of (3.4) by $u_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$ and do inner product with $(-\Delta)^{-1} u_j(x)$, with the help of (3.6),

$$\lambda^2 - \left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right)\lambda + j^4 = 0, \tag{3.7}$$

$$\therefore \lambda_j^{\pm} = \frac{\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right) \pm \sqrt{\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right)^2 - 4j^4}}{2}$$

$$= \frac{\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right) \pm \sqrt{\left[\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right) + 2j^2\right] \cdot \left[\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right) - 2j^2\right]}}{2} \in C, \tag{3.8}$$

C is the complex domain.

For (3.8), when $\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right) - 2j^2 \geq 0$, eigenvalues $\lambda_j^{\pm}$ are real numbers; when $\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right) - 2j^2 < 0$, eigenvalues $\lambda_j^{\pm}$ of A are complex. And they have the same real part $\frac{\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2}{2}$, in this case, there is no decomposition of eigenvalues of A, the spectral gap condition (2.26) is not valid, therefore, existence of inertial manifolds of equation (2.3) cannot be assured.

Furthermore, we consider $\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right) - 2j^2 \geq 0$, assuming ε is sufficiently small, we only consider $\frac{\alpha}{\sqrt{\varepsilon}} j^2 - 2j^2 \geq 0$, that is to say, when

$\frac{\alpha}{\sqrt{\varepsilon}} \geq 2$, some eigenvalues $\lambda_j^{\pm}$ of A are different positive real numbers, therefore, the spectral gap condition can be valid in this case.

We first prove that the spectral gap condition can be valid when ε is sufficiently small, finally, we prove the existence of inertial manifolds of equation (2.3).

Theorem 3.1. *Assume that* ε *and* α *satisfy* $0 < \frac{\alpha}{\sqrt{\varepsilon}} < 2$, l *Lipschitz constant of* $g(u)$ *in Lemma* 2, *let* $N_1 \in N$, *when* $N \geq N_1$, *we have*

$$\frac{\alpha}{\sqrt{\varepsilon}}[(N+1)^2 - N^2] > 8l, \tag{3.9}$$

therefore, *operator A satisfies the spectral gap condition* (2.26).

Proof. For (2.3) and (3.1), $U = (u, \bar{u})$, $V = (v, \bar{v}) \in X$, therefore,

$$\| F(u) - F(v) \|_X = \| g(u) - g(v) \|_H \leq l \| u - v \|_{H^2}, \tag{3.10}$$

that is to say, $l_F \leq l_0$. According to (3.8), $\lambda_j^{\pm}$ is a real number, necessary and sufficient condition of which is $\sqrt{\varepsilon} \geq \left(2 - \frac{\alpha}{\sqrt{\varepsilon}}\right) j^2$. If $2 - \frac{\alpha}{\sqrt{\varepsilon}} > 0$, then A has finite $2N_0$ characteristic roots at most, when $N_0 = 0$, $\sqrt{\varepsilon} < \left(2 - \frac{\alpha}{\sqrt{\varepsilon}}\right) j^2$, $\wedge_0 = \max\{\lambda_j^{\pm} \mid j \leq N_0\}$. When $j > N_0 + 1$, eigenvalues are complex,

$$\Re\lambda_j^{\pm} = \frac{1}{2}\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right), \tag{3.11}$$

therefore, there exists $N_1 \geq N_0 + 1$ so that $\Re\lambda_j^{\pm} > N_0$, $j \geq N_1$.

Assuming $N \geq N_1$, (3.9) is right. The point spectrum of the operator A can be divided into two parts σ_1 and σ_2,

$$\sigma_1 = \{\lambda_j^{\pm} \mid j \le N\}, \quad \sigma_2 = \{\lambda_j^{\pm} \mid j > N+1\}. \tag{3.12}$$

Assume the corresponding subspaces are

$$X_1 = span\{\lambda_j^{\pm} \mid j \le N\}, \quad X_2 = span\{\lambda_j^{\pm} \mid j > N+1\}. \tag{3.13}$$

Thereon exists j so that $\lambda_j^- \in \sigma_1$ and $\lambda_j^+ \in \sigma_2$. There cannot be $U_j^- \in X_1$ and $U_j^{\pm} \notin X_2$. Therefore, X_1 and X_2 are the orthogonal subspaces of X, with the help of (2.24) and (3.11),

$$\begin{aligned}\Re(\lambda_{N+1}^- - \lambda_N^+) &= \frac{1}{2}\left[\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}}(N+1)^2\right] - \frac{1}{2}\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}}N^2\right)\\ &= \frac{1}{2}\sqrt{\varepsilon} + \frac{1}{2}\cdot\frac{\alpha}{\sqrt{\varepsilon}}(N+1)^2 - \frac{1}{2}\sqrt{\varepsilon} - \frac{1}{2}\cdot\frac{\alpha}{\sqrt{\varepsilon}}N^2\\ &= \frac{1}{2}\cdot\frac{\alpha}{\sqrt{\varepsilon}}[(N+1)^2 - N^2],\end{aligned} \tag{3.14}$$

therefore, with the help of (3.9), A satisfies the spectral gap condition (2.26).

□

Theorem 3.2. *Assume* $\frac{\alpha}{\sqrt{\varepsilon}} \ge 2$, l *the Lipschitz constant of* $g(u)$.

(1) *Assume* $\frac{\alpha}{\sqrt{\varepsilon}} > 2$, $N_1 \in N$ *sufficiently big, when* $N \ge N_1$, *there are some inequations,*

$$(2N+1)\left(\frac{\alpha}{\sqrt{\varepsilon}} - \sqrt{\frac{\alpha^2}{\varepsilon} - 4}\right) \ge \frac{8l}{\zeta} + 1,$$

$$(\sqrt{R(N)} - \sqrt{R(N+1)}) + (2N+1)\sqrt{\frac{\alpha^2}{\varepsilon} - 4} \le 1,$$

$$R(N) = \left(\frac{\alpha^2}{\varepsilon} - 4\right)N^2 + 2\alpha N^2 + \varepsilon, \quad \zeta = \min\left\{\frac{\alpha}{2\sqrt{\varepsilon}} - 1, \frac{\alpha}{\sqrt{\varepsilon}} - 2\right\}.$$

(2) *Assume* $\frac{\alpha}{\sqrt{\varepsilon}} = 2,$ $N_1 \in N$ *sufficiently big*, *when* $N \geq N_1,$ *there is an inequation*,

$$2(2N+1) - 2\varepsilon^{\frac{1}{4}} > 8l,$$

therefore, *in* (1) *or* (2), *the operator A satisfies the spectral gap condition* (2.26).

Proof. We can divide three steps to proof:

(1) Let

$$Z_0 = \left\{1 \leq j \leq N \,|\, \frac{\alpha}{\sqrt{\varepsilon}} > 2\right\}, \quad Z_1 = \left\{j \in N \,|\, 0 < \frac{\alpha}{\sqrt{\varepsilon}} < 2\right\}. \tag{3.15}$$

If $j \in Z_0,$ $\lambda^{\pm} \in R$; if $j \in Z_1,$ $\lambda_j^{\pm}$ are complex. And if $j \in Z_0,$

$$0 < \lambda_1^- < \cdots < \lambda_{N_0+1}^- < \frac{1}{2\sqrt{\varepsilon}} < \lambda_{N_0+1}^+ < \cdots < \lambda_1^+, \tag{3.16}$$

$N_0 = \sup Z_0,$ $\Re\lambda_j^{\pm} = \frac{1}{2}\left(\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} j^2\right),$ $\forall j \in N_0.$

If $N_0 \geq N,$ let

$$\sigma_1 = \{\lambda_k^- \,|\, 1 \leq k \leq N\}, \quad \sigma_2 = \{\lambda_k^+, \lambda_j^{\pm} \,|\, 1 \leq k \leq N \leq j\}. \tag{3.17}$$

(2) We consider the corresponding decomposition of X,

$$X_1 = span\{U_k^- \,|\, 1 \leq k \leq N\},$$

$$X_2 = span\{U_k^+, U_j^{\pm} \,|\, 1 \leq k \leq N \leq j\}. \tag{3.18}$$

There is an equivalent inner product $((U, V))_X$ in X so that X_1 and X_2 are orthogonal.

Let

$$\begin{cases} X_1 = X_C \oplus X_R, \\ X_C = span\{U_1^k, \ldots, U_N^k\}, \\ X_R = span\{U_j^{\pm} \mid j \geq N+1\} \end{cases} \tag{3.19}$$

and $X_N = X_1 \oplus X_C$.

We define functions $\Phi : X_N \to R$ and $\Psi : X_R \to R$,

$$\Phi(U, V) = \sqrt{\varepsilon}(\nabla u, \nabla \overline{u}) + \left(\frac{\alpha}{\sqrt{\varepsilon}} - 1\right)(\Delta u, \Delta \overline{y}) + (\overline{z}, (-\Delta)u)$$

$$+ (\overline{v}, (-\Delta)y) + (\overline{z}, v), \tag{3.20}$$

$$\Psi(U, V) = \frac{\alpha}{2\sqrt{\varepsilon}}(\Delta u, \Delta \overline{y}) + (\overline{z}, (-\Delta)u) + (\overline{v}, (-\Delta)y) + (\overline{z}, v), \tag{3.21}$$

$U = (u, v), V = (y, z) \in X_N$ and X_R.

Letting $U = (u, v) \in X_N$, we can get

$$\Phi(U, U) = \sqrt{\varepsilon}(\nabla u, \nabla \overline{u}) + \left(\frac{\alpha}{\sqrt{\varepsilon}}\right)(\Delta u, \Delta \overline{u}) + (\overline{v}, (-\Delta)u) + (\overline{v}, (-\Delta)y) + (\overline{v}, v)$$

$$\geq \sqrt{\varepsilon}\| u \|_1^2 + \left(\frac{\alpha}{\sqrt{\varepsilon}} - 2\right)\| u \|_2^2, \tag{3.22}$$

let $\frac{\alpha}{\sqrt{\varepsilon}} \geq 2$, $\Phi(U, U) \geq 0$, $\forall U \in X_N$, that is to say, Φ is positive definite.

Similarly, for $U = (u, v) \in X_R$,

$$\Psi(U, U) = \frac{\alpha}{2\sqrt{\varepsilon}}(\Delta u, \Delta \overline{u}) + (\overline{v}, (-\Delta)u) + (\overline{v}, (-\Delta)y) + (\overline{v}, v)$$

$$\geq \left(\frac{\alpha}{2\sqrt{\varepsilon}} - 1\right)\| u \|_2^2, \tag{3.23}$$

for $\frac{\alpha}{\sqrt{\varepsilon}} \geq 2$, therefore, $\frac{\alpha}{2\sqrt{\varepsilon}} - 1 \geq 0$, that is to say, $\Psi(U, U) \geq 0$.

There is an inner product in X,

$$((U, V))_X = \Phi(P_N U, P_N V) + \Psi(P_R U, P_R V), \tag{3.24}$$

P_N and P_R are the projections of X_N and X_R, equation (3.24) is called

$$((U, V))_X = \Phi(U, V) + \Psi(U, V).$$

With the help of inner product of equation (3.24) in X, X_1 and X_2 are orthogonal, in fact, only when X_1 and X_2 are orthogonal, we can prove

$$((U_j^-, U_j^+))_X = 0 \quad (\text{if } U_j^- \in X_N, U_j^+ \in X_C),$$

since $U_j^{\pm} = (u_j, -\lambda_j^{\pm} u_j)$, we have

$$\Phi(U_j^-, U_j^+) = \sqrt{\varepsilon}(\nabla u_j, \nabla \overline{u_j}) + \left(\frac{\alpha}{\sqrt{\varepsilon}} - 1\right)(\Delta u_j, \Delta \overline{u_j}) + (\overline{-\lambda_j^+ u_j}, (-\Delta) u_j)$$

$$+ (\overline{-\lambda_j^- u_j}, (-\Delta) u_j) + (\overline{-\lambda_j^+ u_j}, -\lambda_j^- u_j)$$

$$= \sqrt{\varepsilon} \| u_j \|_1^2 + \left(\frac{\alpha}{\sqrt{\varepsilon}} - 1\right) \| u_j \|_2^2 - (\lambda_j^- + \lambda_j^+) \| u_j \|_1^2$$

$$+ \lambda_j^- \cdot \lambda_j^+ | u_j |^2, \tag{3.25}$$

we combine $| u_j |^2 = 1$, $\| u_j \|_1^2, \| u_j \|_2^2 = j^4$, $\lambda_j^- + \lambda_j^+ = \sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}} \cdot j^2$, $\lambda_j^- \cdot \lambda_j^+ = j^4$, therefore,

$$\Phi(U_j^-, U_j^+) = 0,$$

thus, $((U_j^-, U_j^+))_X = \Phi(u_j^-, u_j^+) = 0.$

(3) We define the norm $|\| \cdot \||_X$ of X in equation (3.24), we need to prove the spectral gap condition (2.26).

First, we estimate the Lipschitz constant l_F of $F(u) = (0, g(u))$. Assume $g : H^2 \to H$, $P_1 : X \to X_1$, $P_2 : X \to X_2$ are orthogonal projections, if

$U = (u, v) \in X$, $U_1 = (u_1, u_2) = P_1U$, $U_2 = (u_2, v_2) = P_2U$, therefore,

$$P_1u = u, \quad P_2u = u_2,$$

$$|\| U \||_X^2 = \Phi(P_1U, P_1U) + \Psi(P_2U, P_2U), \tag{3.26}$$

and

$$\Phi(P_1U, P_1U) = \sqrt{\varepsilon}(\nabla u_1, \nabla \bar{u}_1) + \left(\frac{\alpha}{\sqrt{\varepsilon}} - 1\right)(\Delta u_1, \Delta \bar{u}_1)$$

$$+ (\bar{v}_1, (-\Delta)u_1) + (\bar{v}_1, (-\Delta)u_1) + (\bar{v}_1, v_1)$$

$$\geq \left(\frac{\alpha}{\sqrt{\varepsilon}} - 2\right)\| u_1 \|_2^2,$$

$$\Psi(P_2U, P_2U) = \frac{\alpha}{2\sqrt{\varepsilon}}(\Delta u_2, \Delta \bar{u}_2) + (\bar{v}_2, (-\Delta)u_2) + (\bar{v}_2, (-\Delta)u_2) + (\bar{v}_2, v_2)$$

$$\geq \left(\frac{\alpha}{2\sqrt{\varepsilon}} - 1\right)\| u_2 \|_2^2,$$

let $\zeta = \min\left\{\frac{\alpha}{\sqrt{\varepsilon}} - 2, \frac{\alpha}{2\sqrt{\varepsilon}} - 1\right\}$. Then we combine (2.26):

$$|\| U \||_X^2 = \Phi(P_1U, P_1U) + \Psi(P_2U, P_2U) \geq \zeta \| u \|_2^2. \tag{3.27}$$

Let $U = (u, \bar{u})$, $V = (v, \bar{v}) \in X$,

$$|\| F(U) - F(V) \||_X = \| g(u) - g(v) \| \leq \frac{1}{\zeta}\| U - V \|_X, \tag{3.28}$$

therefore, $l_F \leq \frac{1}{\zeta}$, let $\zeta = \min\left\{\frac{\alpha}{\sqrt{\varepsilon}} - 2, \frac{\alpha}{2\sqrt{\varepsilon}} - 1\right\}$.

If

$$\lambda_{N+1}^{-} - \lambda_N^{-} > \frac{1}{\zeta}, \tag{3.29}$$

thus, the spectral gap condition (2.26) is valid. With the help of (3.8), there can be

$$\lambda_{N+1}^{-} - \lambda_{N}^{-} = \frac{\left[\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}}(N+1)^2\right] - \sqrt{\left[\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}}(N+1)^2\right]^2 - 4(N+1)^4}}{2}$$

$$- \frac{\left[\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}}N^2\right] - \sqrt{\left[\sqrt{\varepsilon} + \frac{\alpha}{\sqrt{\varepsilon}}N^2\right]^2 - 4(N)^4}}{2}$$

$$= \frac{1}{2}\left[\sqrt{R(N)} - \sqrt{R(N+1)} + \frac{\alpha}{\sqrt{\varepsilon}}(2N+1)\right], \tag{3.30}$$

we let $R(N) = \left(\frac{\alpha^2}{\varepsilon} - 4\right)N^4 + 2\alpha N^2 + \varepsilon.$

And

$$\lim_{N \to +\infty}\left[\sqrt{R(N)} - \sqrt{R(N+1)} + (2N+1)\sqrt{\frac{\alpha^2}{\varepsilon} - 4}\right] = 0. \tag{3.31}$$

In fact, for (3.31), we let

$$R'(N) = 1 + \frac{2\alpha}{\left(\frac{\alpha^2}{\varepsilon} - 4\right)N^2} + \frac{\varepsilon}{\left(\frac{\alpha^2}{\varepsilon} - 4\right)N^4},$$

therefore,

$$\sqrt{R(N)} - \sqrt{R(N+1)} + (2N+1)\sqrt{\frac{\alpha^2}{\varepsilon} - 4}$$

$$= \sqrt{\frac{\alpha^2}{\varepsilon} - 4}\left[(N+1)^2 - N^2 + \sqrt{N^4 + \frac{2\alpha N^2}{\frac{\alpha^2}{\varepsilon} - 4} + \frac{\varepsilon}{\frac{\alpha^2}{\varepsilon} - 4}}\right.$$

$$+\sqrt{(N+1)^4+\frac{2\alpha(N+1)^2}{\frac{\alpha^2}{\varepsilon}-4}+\frac{\varepsilon}{\frac{\alpha^2}{\varepsilon}-4}}\;\Bigg]$$

$$=\sqrt{\frac{\alpha^2}{\varepsilon}-4}\,[(N+1)^2(1-\sqrt{R'(N)})-N^2(1-\sqrt{R'(N)})], \tag{3.32}$$

for (3.32),

$$\lim_{N\to+\infty} N^2(1-\sqrt{R'(N)})=\frac{\alpha}{4-\frac{\alpha^2}{\varepsilon}},$$

therefore, (3.31) is proved.

For $N_1 > 0$, when $N > N_1$, the spectral gap inequation reads

$$\lambda_{N+1}^- - \lambda_N^- \geq \frac{1}{2}\left[(2N+1)\left(\frac{\alpha}{\sqrt{\varepsilon}}-\sqrt{\frac{\alpha^2}{\varepsilon}-4}\right)\right] \geq 4l,$$

therefore, when $\frac{\alpha}{\sqrt{\varepsilon}} > 2$, Theorem 3.2 is proved.

Furthermore, we will consider the case when $\frac{\alpha}{\sqrt{\varepsilon}} = 2$. Therefore, the definition of Ψ is modified in equation (3.21), and increase $(u, \bar{y})$, then estimate (3.23) is replaced by

$$\Psi(U, U) \geq |u|^2, \tag{3.33}$$

when $\alpha = 2\sqrt{\varepsilon}$, inequation (3.33) is valid. In turn, instead of (3.27) and (3.28), the estimates

$$|\|U\||_X \geq \|u\|_2, \quad l_F \leq l, \tag{3.34}$$

so that the spectral gap condition (3.29) reads

$$\lambda_{N+1}^{-} - \lambda_{N}^{-} = \frac{1}{2}[\sqrt{4\sqrt{\varepsilon}N^2 + \varepsilon} - \sqrt{4\sqrt{\varepsilon}(N+1)^2 + \varepsilon} + 2(2N+1)] > 4l, \quad (3.35)$$

$$\lim_{N \to +\infty} [\sqrt{4\sqrt{\varepsilon}N^2 + \varepsilon} - \sqrt{4\sqrt{\varepsilon}(N+1)^2 + \varepsilon}] = -2\varepsilon^{\frac{1}{4}}. \quad (3.36)$$

□

3.2. Non-existence

Furthermore, we consider the non-existence of inertial manifolds of equation (2.2), when $\alpha = 0$, we assume that it satisfies the spectral gap condition. With the help of (3.8),

$$\lambda_j^{\pm} = \frac{\sqrt{\varepsilon} \pm \sqrt{\varepsilon - 4j^4}}{2}, \quad (3.37)$$

when ε is sufficiently small for $j \geq 1$, $\varepsilon - 4j^4 < 0$, therefore,

$$\lambda_j^{\pm} \in C, \quad (3.38)$$

thus, equation (2.2) does not satisfy the spectral gap condition and inertial manifolds of equation (2.2) are not existent.

With the help of every theorem, we all know that

Theorem 3.3. *Assuming* ε *is sufficiently small*, *there is a positive integer* N, $F(u)$ *of equation* (2.3) *satisfies the Lipschitz condition*, *A satisfies the spectral gap condition*, *therefore*, *equation* (2.3) *has s inertial manifold* $\mu \subset X$,

$$\mu = graph(\Phi) = \{\varsigma + \Phi(\varsigma) | \varsigma \in X_1\}.$$

$\Phi : X_1 \to X_2$ *is a Lipschitz continuous function and* $graph(\Phi)$ means *diagram of* Φ.

References

[1] A. Novick-Cohen, On the viscous Cahn-Hilliard equation, J. Ball, ed., Material Instabilities in Continuum Mechanics and Related Mathematical Problems, Oxford Univ. Press, New York, 1988.

[2] J. W. Cahn and E. Hilliard, Free energy of a nonuniform system, I, interfacial free energy, J. Chem. Phys. 28 (1958), 258-367.

[3] S. M. Zheng and M. L. Albert, Exponential attractors and inertial manifolds for singular perturbations of the Cahn-Hilliard equation, Nonlinear Anal. 54 (2004), 842-877.

[4] Nguyen Thieu Huy, Inertial manifolds for semi-linear parabolic equations in admissible space, J. Math. Anal. Appl. 386 (2012), 894-909.

[5] W. K. Wang and Z. G. Wu, Optimal decay rate of solutions for Cahn-Hilliard equation with inertial term in multi-dimensions, J. Math. Anal. Appl. 387 (2012), 349-358.

[6] J. Z. Wu, P. Zhao and G. G. Lin, An inertial manifold of the damped Boussinesq equation, Journal of Yunnan University 32 (2010), 310-314.

[7] G. G. Lin, Nonlinear evolution equations, Yunnan University, Kunming, 2011.

[8] Nail H. Ibragimov, Differential Equations and Mathematical Physics Problems, Higher Education Press, Beijing, 2010.

[9] L. An, Global dynamics of the viscone Cahn-Hilliard equations, J. Lanzhou Univ. 36(5) (2000), 17-23.

[10] E. Bonetti, W. Dreyer and G. Schimperna, Global solutions to a generalized Cahn-Hilliard equation with viscosity, Adv. Differential Equations 8(2) (2003), 231-256.

[11] P. Rybka and K.-H. Hoffmann, Convergence of solutions to Cahn-Hilliard equation, Comm. Partial Differential Equations 24(5-6) (1999), 1055-1077.

[12] C. M. Elliott and A. M. Stuart, The viscous Cahn-Hilliard equation, II, J. Differential Equations 128(2) (1996), 387-414.

[13] A. Novick-Cohen and L. Peletier, Steady states of the one-dimensional Cahn-Hilliard equation, Proc. Roy. Soc. Edinburgh Sect. A 123(6) (1993), 1071-1098.

[14] Guoguang Lin and Murong Jiang, An inertial manifold of the generalized Ginzburg-Landau equations, Proceedings of the Conference on Nonlinear Partial Differential Equations and Applications, Guo Boling and Yang Dadi, eds., World Scientific, 1998, pp. 122-128.

[15] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations, Springer-Verlag, New York, 1989.