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ON THE SECOND KIND TWISTED (h, q)-EULER POLYNOMIALS

C. S. Ryoo

Department of Mathematics
Hannam University
Daejeon 306-791, Korea

Abstract

In this paper, we introduce and study the second kind twisted (h, q)-Euler numbers $E_{n,q,w}^{(h)}$ and polynomials $E_{n,q,w}^{(h)}(x)$.

1. Introduction

Many mathematicians have studied Euler numbers and Euler polynomials. Euler polynomials possess many interesting properties and are arising in many areas of mathematics and physics. In this paper, we introduce the second kind twisted (h, q)-Euler numbers and polynomials. Throughout this paper, we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the

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normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, then we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$,

$$[x]_q = [x : q] = \frac{1-q^x}{1-q}, \quad \text{cf. [1, 2, 3, 4, 5, 6].}$$

For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

Kim defined the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0 \leq x < p^N} g(x)(-1)^x \quad \text{cf. [1, 2].} \tag{1.1}$$

From (1.1), we obtain

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_{-1}(x) I_{-1} = (-1)^n \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \tag{1.2}$$

2. The Second Kind Twisted (h, q)-Euler Numbers and Polynomials

Our primary aim in this section is to define the second kind twisted (h, q)-Euler numbers $E_{n,q,w}^{(h)}$ and polynomials $E_{n,q,w}^{(h)}(x)$ and to investigate their properties. In this section, we assume that $h \in \mathbb{Z}$. Let $T_p = \bigcup_{N \geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{w \mid w^{p^N} = 1\}$ is the cyclic group of order p^N. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function

$x \mapsto w^x$. In (1.2), if we take $g(x) = \phi_w(x) q^{hx} e^{(2x+1)t}$, then we easily see that

$$\int_{\mathbb{Z}_p} \phi_w(x) q^{hx} e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t}{wq^h e^{2t} + 1}.$$

Let us define the second kind twisted (h, q)-Euler numbers $E_{n,q,w}$ and polynomials $E_{n,q,w}(x)$ as follows:

$$\int_{\mathbb{Z}_p} \phi_w(y) q^{hy} e^{(2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)} \frac{t^n}{n!}, \tag{2.1}$$

$$\int_{\mathbb{Z}_p} \phi_w(y) q^{hy} e^{(2y+1+x)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we obtain the following Witt's formula.

Theorem 1. *For $w \in T_p$ and $h \in \mathbb{Z}$, we have*

$$\int_{\mathbb{Z}_p} \phi_w(x) q^{hx} (2x+1)^n d\mu_{-1}(x) = E_{n,q,w}^{(h)},$$

$$\int_{\mathbb{Z}_p} \phi_w(y) q^{hy} (2y+1+x)^n d\mu_{-1}(y) = E_{n,q,w}^{(h)}(x).$$

Let q be a complex number with $|q| < 1$ and w be the p^Nth root of unity. By the meaning of (1.3) and (1.4), let us define the second kind twisted (h, q)-Euler numbers $E_{n,q,w}^{(h)}$ and polynomials $E_{n,q,w}^{(h)}(x)$ as follows:

$$\frac{2e^t}{wq^h e^{2t} + 1} = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)} \frac{t^n}{n!}, \tag{2.3}$$

$$\frac{2e^t}{wq^h e^{2t} + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{2.4}$$

By above definition, we obtain

$$\sum_{l=0}^{\infty} E_{l,q,w}^{(h)}(x)\frac{t^l}{l!} = \frac{2e^t}{wq^h e^{2t}+1} e^{xt}$$

$$= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} E_{n,q,w}^{(h)} \frac{t^n}{n!} x^{l-n} \frac{t^{l-n}}{(l-n)!}\right)$$

$$= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} \binom{l}{n} E_{n,q,w}^{(h)} x^{l-n}\right)\frac{t^l}{l!}.$$

By using comparing coefficients $\frac{t^l}{l!}$, we have the following theorem.

Theorem 2. *For any positive integer n, we have*

$$E_{n,q,w}^{(h)}(x) = \sum_{k=0}^{n}\binom{n}{k} E_{k,q,w}^{(h)} x^{n-k}.$$

Over five decades ago, Carlitz [1] defined q-extensions of the classical Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ and proved properties analogues to those satisfied by B_n and $B_n(x)$. Carlitz's q-Bernoulli numbers $\beta_n = \beta_{n,q}$ can be determined inductively by [1],

$$\beta_0 = 1, \quad q(q\beta+1)^k - \beta_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention about replacing β^k by β_k. For the second kind twisted (h, q)-Euler numbers, we obtain the following theorem.

Theorem 3. *The second kind twisted (h, q)-Euler numbers $E_{n,q,w}^{(h)}$ are defined, respectively, by*

$$wq^h(E_{q,w}^{(h)}+1)^n + (E_{q,w}^{(h)}-1)^n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

with the usual convention about replacing $(E_{q,w}^{(h)})^n$ by $E_{n,q,w}^{(h)}$ in the binomial expansion.

Proof. From (2.3), we obtain

$$\frac{2}{wq^h e^t + e^{-t}} = \sum_{n=0}^{\infty} E_{n, q, w}^{(h)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} (E_{q, w}^{(h)})^n \frac{t^n}{n!} = e^{E_{q, w}^{(h)} t}$$

which yields

$$2 = (wq^h e^t + e^{-t}) e^{E_{q, w}^{(h)} t} = wq^h e^{(E_{q, w}^{(h)} + 1)t} + e^{(E_{q, w}^{(h)} - 1)t}.$$

Using Taylor expansion of exponential function, we have

$$\begin{aligned} 2 &= \sum_{n=0}^{\infty} \{wq^h (E_{q, w}^{(h)} + 1)^n + (E_{q, w}^{(h)} - 1)^n\} \frac{t^n}{n!} \\ &= wq^h (E_{q, w}^{(h)} + 1)^0 + (E_{q, w}^{(h)} - 1)^0 \\ &\quad + \sum_{n=1}^{\infty} \{wq^h (E_{q, w}^{(h)} + 1)^n + (E_{q, w}^{(h)} - 1)^n\} \frac{t^n}{n!}. \end{aligned}$$

The result follows by comparing the coefficients. □

Because

$$\frac{\partial}{\partial x} F_{q, w}^{(h)}(x, t) = t F_{q, w}^{(h)}(x, t) = \sum_{n=0}^{\infty} \frac{d}{dx} E_{n, q, w}^{(h)}(x) \frac{t^n}{n!},$$

it follows the important relation

$$\frac{d}{dx} E_{n, q, w}^{(h)}(x) = n E_{n-1, q, w}^{(h)}(x).$$

We also obtain the following integral formula:

$$\int_a^b E_{n-1, q, w}^{(h)}(x) dx = \frac{1}{n} (E_{n, q, w}^{(h)}(b) - E_{n, q, w}^{(h)}(a)).$$

Since

$$\begin{aligned} \sum_{l=0}^{\infty} E_{l, q, w}^{(h)}(x + y) \frac{t^l}{l!} &= \frac{2e^t}{wq^h e^{2t} + 1} e^{(x+y)t} \\ &= \sum_{n=0}^{\infty} E_{n, q, w}^{(h)}(x) \frac{t^n}{n!} \sum_{m=0}^{\infty} y^m \frac{t^m}{m!} \end{aligned}$$

$$= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} E_{n,q,w}^{(h)}(x)\frac{t^n}{n!}\, y^{l-n}\,\frac{t^{l-n}}{(l-n)!}\right)$$

$$= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n} E_{n,q,w}^{(h)}(x) y^{l-n}\right)\frac{t^l}{l!},$$

we have the following addition theorem.

Theorem 4. *The second kind twisted Euler polynomial* $E_{n,q,w}^{(h)}(x)$ *satisfies the following relation*:

$$E_{l,q,w}^{(h)}(x+y) = \sum_{n=0}^{l}\binom{l}{n} E_{n,q,w}^{(h)}(x)\, y^{l-n}.$$

It is easy to see that

$$\sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(x)\frac{t^n}{n!} = \frac{2e^t}{wq^h e^{2t}+1}\, e^{xt}$$

$$= \frac{2e^t}{q^{mh}w^m e^{2mt}+1}\, e^{xt}\sum_{a=0}^{m-1}(-1)^a q^{ah} w^a e^{at}$$

$$= \sum_{a=0}^{m-1}(-1)^a q^{ah} w^a\, \frac{2}{q^{mh}w^m e^{mt}+e^{-mt}}\, e^{\left(\frac{2a+x+1-m}{m}\right)(mt)}$$

$$= \sum_{a=0}^{m-1}(-1)^a q^{ah} w^a \sum_{n=0}^{\infty} E_{n,q^m,w^m}^{(h)}\left(\frac{2a+x+1-m}{m}\right)\frac{(mt)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(m^n\sum_{a=0}^{m-1}(-1)^a q^{ah} w^a E_{n,q^m,w^m}^{(h)}\left(\frac{2a+x+1-m}{m}\right)\right)\frac{t^n}{n!}.$$

Hence, we have the below distribution relation.

Theorem 5. *For m an odd positive integer and* $n \in \mathbb{N}$, *we have*

$$E_{n,q,w}(x) = m^n \sum_{i=0}^{m-1}(-1)^i q^{ih} w^i E_{n,q^m,w^m}^{(h)}\left(\frac{2i+x+1-m}{m}\right),\ \textit{for}\ n \geq 0.$$

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