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WEYL TYPE THEOREM AND SPECTRUM FOR k-QUASI-$*$-CLASS A OPERATORS

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Abstract

Let T be a k-quasi-$*$-class A operator on a complex Hilbert space $\mathcal{H}$ if $T^{*k}(|T^2| - |T^*|^2)T^k \geq 0$, where k is a natural number. In this paper, we prove that the spectral mapping theorem for Weyl spectrum holds for k-quasi-$*$-class A operators. Also, we prove that the Weyl

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type theorems holds for k-quasi-$*$-class A. We also prove that if T^* is k-quasi-$*$-class A, then generalized a-Weyl's theorem holds for T. Also, we prove that $\sigma_{jp}(T) - \{0\} = \sigma_{ap}(T) - \{0\}$ holds for k-quasi-$*$-class A operator.

1. Introduction

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. For positive operators A and B, write $A \geq B$ if $A - B \geq 0$. If A and B are invertible and positive operators, then it is well known that $A \geq B$ implies that $\log A \geq \log B$. However [2], $\log A \geq \log B$ does not necessarily imply $A \geq B$. A result due to Ando [6] states that for invertible positive operators A and B, $\log A \geq \log B$ if and only if $A^r \geq (A^{\frac{r}{2}} B^r A^{\frac{r}{2}})^{\frac{1}{2}}$ for all $r \geq 0$. For an operator T, let $U|T|$ denote the polar decomposition of T, where U is a partially isometric operator, $|T|$ is a positive square root of T^*T and $\ker(T) = \ker(U) = \ker(|T|)$, where $\ker(S)$ denotes the kernel of operator S.

An operator $T \in B(\mathcal{H})$ is positive, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and posinormal if there exists a positive $\lambda \in B(\mathcal{H})$ such that $TT^* = T^*\lambda T$. Here λ is called *interrupter* of T. In other words, an operator T is called *posinormal* if $TT^* \leq c^2 T^* T$, where T^* is the adjoint of T and $c > 0$ [15]. An operator T is said to be *heminormal* if T is hyponormal and T^*T commutes with TT^*. An operator T is said to be *p-posinormal* if $(TT^*)^p \leq c^2 (T^*T)^p$ for some $c > 0$. It is clear that 1-posinormal is posinormal. An operator T is said to be *p-hyponormal*, for $p \in (0, 1)$, if $(T^*T)^p \geq (TT^*)^p$. A 1-hyponormal operator is hyponormal which has been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [30]. Furuta et al. [19] have characterized class A operator as follows. An operator T belongs to class A if and only if $(T^{*}|T|T)^{\frac{1}{2}} \geq T^{*}T$.

An operator T is said to be *paranormal* if $\| T^{2}x \| \geq \| Tx \|^{2}$ and $*$-*paranormal* if $\| T^{2}x \| \geq \| T^{*}x \|^{2}$ for all unit vector $x \in \mathcal{H}$. Recently, Duggal et al. [17] have considered the new class of operators: an operator $T \in B(\mathcal{H})$ belongs to $*$-class A if $|T^{2}| \geq |T^{*}|^{2}$. The authors of [28] have extended $*$-class A operators to quasi-$*$-class A operators. An operator $T \in B(\mathcal{H})$ is said to be *quasi-$*$-class A* if $T^{*}|T^{2}|T \geq T^{*}|T^{*}|^{2}T$ and *quasi-$*$-paranormal* if $\| T^{*}Tx \|^{2} \leq \| T^{3}x \| \| Tx \|$ for all $x \in \mathcal{H}$. As a further generalization, Mecheri [25] has introduced the class of k-quasi-$*$-class A operators. An operator T is said to be *k-quasi-$*$-class A operator* on a complex Hilbert space $\mathcal{H}$ if $T^{*k}(|T^{2}| - |T^{*}|^{2})T^{k} \geq 0$, where k is a natural number.

An operator T is called *normal* if $T^{*}T = TT^{*}$ and (p, k)-*quasihyponormal* if $T^{*^{k}}((T^{*}T)^{p} - (TT^{*})^{p})T^{k} \geq 0 \quad (0 < p \leq 1, k \in \mathbb{N})$. Aluthge [1], Campbell and Gupta [11] and Arora and Arora [3] introduced p-hyponormal, p-quasihyponormal and k-quasihyponormal operators, respectively.

Aluthge [1] studied p-hyponormal operators for $0 < p \leq 1$. In particular, he defined the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ which is called the *Aluthge transformation* and the operator $\widetilde{\widetilde{T}} = |\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}$, where $\widetilde{T} = \widetilde{U}|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$. An operator T is said to be *w-hyponormal* if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^{*}|$,

$$p\text{-hyponormal} \subset p\text{-posinormal} \subset (p,\, k)\text{-quasiposinormal},$$

$$p\text{-hyponormal} \subset p\text{-quasihyponormal}$$

$$\subset (p,\, k)\text{-quasihyponormal} \subset (p,\, k)\text{-quasiposinormal}$$

and

$$\text{hyponormal} \subset k\text{-quasihyponormal} \subset (p,\, k)\text{-quasihyponormal}$$

$$\subset (p,\, k)\text{-quasiposinormal}$$

for a positive integer *k* and a positive number $0 < p \leq 1$.

If $T \in B(\mathcal{H})$, then we shall write $N(T)$ and $R(T)$ for the null space and the range of *T*, respectively. Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of *T*, respectively. An operator *T* is called *Fredholm* if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim \mathcal{H}/R(T) < \infty$. Moreover, if $i(T) = \alpha(T) - \beta(T) = 0$, then *T* is called *Weyl*. The essential spectrum $\sigma_e(T)$ and the Weyl $\sigma_W(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

respectively. It is known that $\sigma_e(T) \subset \sigma_W(T) \subset \sigma_e(T) \cup \operatorname{acc} \sigma(T)$, where we write $\operatorname{acc} K$ for the set of all accumulation points of $K \subset \mathbb{C}$. If we write $\operatorname{iso} K = K \backslash \operatorname{acc} K$, then we let

$$\pi_{00}(T) = \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

We say that Weyl's theorem holds for *T* if

$$\sigma(T) \backslash \sigma_W(T) = \pi_{00}(T).$$

Let $\sigma_p(T)$ denote the point spectrum of *T*, i.e., the set of its eigenvalues. Let $\sigma_{jp}(T)$ denote the joint point spectrum of *T*. We note that $\lambda \in \sigma_{jp}(T)$ if and only if there exists a non-zero vector *x* such that $Tx = \lambda x$, $T^* x = \overline{\lambda} x$. It is evident that $\sigma_{jp}(T) \subset \sigma_p(T)$. It is well known that if T is normal, then $\sigma_{jp}(T) = \sigma_p(T)$. Let $T = U|T|$ be the polar decomposition of T and $\lambda = |\lambda| e^{i\theta}$ be the complex number, $|\lambda| > 0$, $|e^{i\theta}| = 1$. Then $\lambda \in \sigma_{jp}(T)$ if and only if there exists a non-zero vector x such that $Ux = e^{i\theta}$, $|T|x = |\lambda|x$. Let $\sigma_{ap}(T)$ denote the approximate point spectrum of T, i.e., the set of all complex numbers λ which satisfy the following condition: there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\lim_n \|(T-\lambda)x_n\| = 0$. It is evident that $\sigma_p(T) \subset \sigma_{ap}(T)$. Let $\sigma_{jap}(T)$ be the joint approximate point spectrum of T. Then $\lambda \in \sigma_{jap}(T)$ if and only if there exists a sequence $\{x_n\}$ of unit vectors such that $\lim_{n\to\infty}\|(T-\lambda)x_n\| = \lim_{n\to\infty}\|(T^* - \overline{\lambda})x_n\|$ $= 0$. It is evident that $\sigma_{jap}(T) \subset \sigma_{ap}(T)$ for all $T \in B(\mathcal{H})$. It is well known that for a normal operator T, $\sigma_{jap}(T) = \sigma_{ap}(T) = \sigma(T)$.

In [29], Weyl proved that Weyl's theorem holds for Hermitian operators. Weyl's theorem has been extended from Hermitian operators to hyponormal operators [13], algebraically hyponormal operators [21], p-hyponormal operators [12] and algebraically p-hyponormal operators [16]. More generally, Berkani investigated generalized Weyl's theorem which extends Weyl's theorem, and proved that generalized Weyl's theorem holds for hyponormal operators [7-9]. In a recent paper [24], the author showed that generalized Weyl's theorem holds for (p, k)-quasihyponormal operators. Recently, Cao et al. [10] proved Weyl type theorems for p-hyponormal operators. In this paper, we prove that Weyl type theorems holds for k-quasi-$*$-class A operators. Especially, we prove that if T^* is k-quasi-$*$-class A, then generalized a-Weyl's theorem holds for T.

2. Weyl's Theorem for k-quasi-$*$-class A Operators

Salah Mecheri has introduced k-quasi-$*$-class A operators and has proved many interesting properties of it.

Lemma 2.1 [25, Lemma 2.1, Theorem 2.3]**.** (1) *Let* $T \in B(\mathcal{H})$ *be k-quasi-*$*$*-class A operator and suppose the range of* T^k *is not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathcal{H} = \overline{ran(T^k)} \oplus \ker(T^{*k})$, *where* T_1 *is* $*$*-class A operator,* $T_3^k = 0$ *and* $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) *Let* $T \in B(\mathcal{H})$ *be a k-quasi-*$*$*-class A operator and let M be a closed T-invariant subspace of* $\mathcal{H}$. *Then the restriction* $T|_M$ *of T to M is a k-quasi-*$*$*-class A operator*.

Lemma 2.2 [25, Theorem 2.5]**.** *Let* $T \in B(\mathcal{H})$ *be a k-quasi-*$*$*-class A operator*. *If* $\lambda \neq 0$ *and* $(T - \lambda)x = 0$ *for some* $x \in \mathcal{H}$, *then* $(T - \lambda)^* x = 0$.

Lemma 2.3 [25, Theorem 2.2]**.** *Let* $T \in B(\mathcal{H})$ *be a k-quasi-*$*$*-class A operator*. *Then T has Bishop's property* (β) (*i.e., if* $f_n(z)$ *is analytic on D and* $(T - z) f_n(z) \to 0$ *uniformly on each compact subset of D, then* $f_n(z) \to 0$ *uniformly on each compact subset of D*), *the single valued extension property and Dunford property* (*C*).

Proposition 2.4. *Weyl's theorem holds for k-quasi-*$*$*-class A operator T, i.e.,* $\sigma(T) \backslash \sigma_W(T) = \pi_{00}(T)$.

Proof. Let $\lambda \in \sigma(T) \backslash \sigma_W(T)$. Then $T - \lambda$ is Weyl and not invertible. If λ is an interior point of $\sigma(T)$, then there exists an open set G such that $\lambda \in G \subset \sigma(T) \backslash \sigma_W(T)$. Hence $\dim N(T - \mu) > 0$ for all $\mu \in G$ and T does not have the single valued extension property by [18, Theorem 9]. This is a contradiction. Hence λ is a boundary point of $\sigma(T)$, and hence an isolated point of $\sigma(T)$ by [14, Theorem XI 6.8]. Thus, $\lambda \in \pi_{00}(T)$.

Let $\lambda \in \pi_{00}(T)$ and E_λ be the Riesz idempotent for λ of T. Then $0 < \dim N(T - \lambda) < \infty$,

$$T = T|_{E_\lambda \mathcal{H}} \oplus T|_{(I - E_\lambda)\mathcal{H}}$$

and

$$\sigma(T|_{E_\lambda \mathcal{H}}) = \{\lambda\}, \quad \sigma(T|_{(I - E_\lambda)\mathcal{H}}) = \sigma(T) \backslash \{\lambda\}.$$

We remark $T|_{E_\lambda \mathcal{H}}$ is k-quasi-$*$-class A operator by Lemma 2.1.

If $\lambda \neq 0$, then $T|_{E_\lambda \mathcal{H}} = \{\lambda\}$ by [25]. Hence $E_\lambda \mathcal{H} \subset N(T - \lambda)$ and E_λ is of finite rank. Since $(T - \lambda)|_{(I - E_\lambda)\mathcal{H}}$ is invertible, $T - \lambda = 0|_{E_\lambda \mathcal{H}} \oplus (T - \lambda)|_{(I - E_\lambda)\mathcal{H}}$ is Weyl. Hence $\lambda \in \sigma(T) \backslash \sigma_W(T)$.

If $\lambda = 0$, then $(T|_{E_0 \mathcal{H}})^k = 0$ by [25]. Hence $E_0 \mathcal{H} \subset N(T^k)$ and

$$\dim E_0 \mathcal{H} \leq \dim N(T^k) \leq k \dim N(T) < \infty.$$

Then $T|_{E_\lambda \mathcal{H}}$ is compact. Since $T|_{(I - E_0)}$ is invertible, $\lambda \in \sigma(T) \backslash \sigma_W(T)$ by [14, Proposition XI 6.9]. □

Theorem 2.5. *If T is an n-multicyclic k-quasi-$*$-class A operator, then the restriction T_1 of T on $\overline{ran(T^k)}$ is also an n-multicyclic operator.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{ran(T^k)} \oplus \ker(T^{*k})$. Since $\sigma(T_1) \subset \sigma(T)$ by Lemma 2.1, $\mathcal{R}(\sigma(T)) \subset \mathcal{R}(\sigma(T_1))$. By hypothesis, there exist n vectors $x_1, \ldots, x_n \in \mathcal{H}$ such that

$$\mathcal{H} = \bigvee g(T) x_i \,|\, i = 1, 2, \ldots, n \text{ and } g \in \mathcal{R}(\sigma(T)).$$

Now let $y_i = T^k x_i$, $i = 1, \ldots, n$. Then we have the following:

$$\bigvee g(T_1)y_i \mid i = 1, 2, \ldots, n,\ g \in \mathcal{R}(\sigma(T_1))$$

$$\supset \bigvee g(T_1)y_i \mid i = 1, \ldots, n,\ g \in \mathcal{R}(\sigma(T))$$

$$= \bigvee g(T)T^k x_i \mid i = 1, 2, \ldots, n,\ g \in \mathcal{R}(\sigma(T))$$

$$= \bigvee T^k g(T)x_i \mid i = 1, 2, \ldots, n,\ g \in \mathcal{R}(\sigma(T))$$

$$= \overline{ran(T^k)}$$

and $y_1, \ldots, y_n$ are n-multicyclic vectors of T_1. □

Lemma 2.6 [22, Theorem 6]**.** *For given operators $A, B, C \in B(\mathcal{H})$, there is equality $\sigma_W(A) \cup \sigma_W(B) = \sigma_W(M_c \cup \mathfrak{G})$, where $M_c = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\mathfrak{G}$ is the union of certain of the holes in $\sigma_W(M_c)$ which happen to be subsets of $\sigma_W(A) \cap \sigma_W(B)$.*

The following theorem shows that the spectral mapping theorem for Weyl spectrum holds for k-quasi-$*$-class A operators.

Theorem 2.7. *If T is k-quasi-$*$-class A operator, then $f(\sigma_W(T)) = \sigma_W(f(T))$ for any analytic function f on a neighborhood of $\sigma(T)$.*

Proof. We need only to prove that $\sigma_W(p(T)) = p(\sigma_W(T))$ for any polynomial p. Since T has the matrix representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 is $*$-class A operator and $T_3^k = 0$, and the spectral mapping theorem for Weyl spectrum holds for $*$-class A operator, it follows that

$$\begin{aligned} \sigma_W(p(T)) &= \sigma_W(p(T_1)) \cup \sigma_W(p(T_3)) \\ &= p(\sigma_W(T_1)) \cup p(\sigma_W(T_3)) \\ &= p(\sigma_W(T_1)) \cup \sigma_W(T_3) \\ &= p(\sigma_W(T)). \end{aligned}$$

It was known [22] if A and B are isoloid and if Weyl's theorem holds for A and B, then Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Leftrightarrow \sigma_W \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_W(A) \cup \sigma_W(B)$.

We know that the "spectral picture" [26] of the operator $T \in B(\mathcal{H})$, denoted by $SP(T)$, consists of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes.

In general, Weyl's theorem does not hold for operator matrix $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ even though Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Lee showed that (see [23]) following lemma:

Lemma 2.8. *If either $SP(A)$ or $SP(B)$ has no pseudoholes and if A is an isoloid operator for which Weyl's theorem holds, then for every $C \in B(\mathcal{H})$, Weyl's theorem holds for* $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Leftrightarrow \sigma_W \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

The following corollary is result from the above lemma.

Corollary 2.9. *Weyl's theorem holds for every k-quasi-$*$-class A operator.*

Proof. Let $T \in B(\mathcal{H})$ be a k-quasi-$*$-class A operator. Then by Lemma 2.1, T has the following matrix representation: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{ran(T^k)} \oplus \ker(T^{*k})$, where T_1 is $*$-class A, and T_3 is nilpotent operator. Therefore, Weyl's theorem holds for $\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$ because Weyl's theorem holds for $*$-class A operator and nilpotent operator and both $*$-class

A operator and nilpotent operator are isoloid. Hence by Lemma 2.8, Weyl's theorem holds for $\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ because $SP(T_3)$ has no pseudoholes. $\square$

3. Generalized a-Weyl's Theorem for k-quasi-$*$-class A Operators

More generally, Aiena and Berkani investigated B-Fredholm theory as follows [4, 7-9]. An operator T is called *B-Fredholm* if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator

$$T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$$

is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\alpha(T_{[n]}) = \dim N(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim R(T^n)/R(T_{[n]}) < \infty.$ Similarly, a B-Fredholm operator T is called *B-Weyl* if $i(T_{[n]}) = 0.$ The following result is due to Berkani and Sarih [9].

Proposition 3.1. *Let* $T \in B(\mathcal{H})$.

(1) *If* $R(T^n)$ *is closed and* $T_{[n]}$ *is Fredholm, then* $R(T^m)$ *is closed and* $T_{[m]}$ *is Fredholm for every* $m \geq n.$ *Moreover,* $ind\, T_{[m]} = ind\, T_{[n]} = ind\, T.$

(2) *An operator T is B-Fredholm* (B-*Weyl*) *if and only if there exist T-invariant subspaces M and N such that* $T = T|_M \oplus T|_N,$ *where* $T|_M$ *is Fredholm* (*Weyl*) *and* $T|_N$ *is nilpotent*.

The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\} \subset \sigma_W(T).$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \backslash \sigma_{BW}(T) = E(T),$$

where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that if the generalized

Weyl's theorem holds for T, then so does Weyl's theorem [8]. Recently, in [7], Berkani and Arroud showed that if T is hyponormal, then generalized Weyl's theorem holds for T.

Proposition 3.2. *Generalized Weyl's theorem holds for k-quasi-$*$-class A operator T.*

Proof. Let $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl and not invertible. Then

$$T - \lambda = (T - \lambda)|_M \oplus (T - \lambda)|_N,$$

where $(T - \lambda)|_M$ is Weyl and $(T - \lambda)|_N$ is nilpotent by Proposition 3.1. The case $M = \{0\}$ or $N = \{0\}$ is easy, so we may assume $M \neq \{0\}$ and $N \neq \{0\}$.

First, we assume $\lambda \in \sigma(T|_M)$. In this case, $T|_M$ is k-quasi-$*$-class A by Lemma 2.1 and

$$\lambda \in \sigma(T|_M)\backslash\sigma_W(T|_M) = \pi_{00}(T|_M)$$

by Proposition 2.4. Hence λ is an isolated point of $\sigma(T|_M)$ and an eigenvalue of $T|_M$. Hence λ is an eigenvalue of T. On the other hand, $(T - \lambda)|_N$ is nilpotent, so λ is an isolated point of $\sigma(T)$. Hence $\lambda \in E(T)$.

Second, we assume $\lambda \notin \sigma(T|_M)$. In this case, $(T - \lambda)|_N$ is nilpotent, and λ is an eigenvalue of $T|_N$ and T. Since $(T - \lambda)|_M$ is invertible, λ is an isolated point of $\sigma(T)$. Hence $\lambda \in E(T)$.

Conversely, let $\lambda \in E(T)$. Since λ is an isolated point of $\sigma(T)$,

$$T - \lambda = (T - \lambda)|_{E_\lambda \mathcal{H}} \oplus (T - \lambda)|_{(I - E_\lambda)\mathcal{H}},$$

where E_λ denotes the Riesz idempotent for λ of T. Then $(T - \lambda)|_{E_\lambda \mathcal{H}}$ is k-quasi-$*$-class A by Lemma 2.1 and $\sigma(T|_{E_\lambda \mathcal{H}}) = \{\lambda\}$.

If $\lambda \neq 0$, then $T|_{E_\lambda \mathcal{H}} = \{\lambda\}$ by [25]. Hence

$$T - \lambda = 0|_{E_\lambda \mathcal{H}} \oplus (T - \lambda)|_{(I - E_\lambda)\mathcal{H}}.$$

Since $(T - \lambda)|_{(I - E_\lambda)\mathcal{H}}$ is invertible, $T - \lambda$ is B-Weyl by Proposition 3.1. Hence $\lambda \in \sigma(T) \backslash \sigma_{BW}(T)$.

If $\lambda = 0$, then $(T|_{E_\lambda \mathcal{H}})^k = 0$ by [25]. Hence $\lambda \in \sigma(T) \backslash \sigma_{BW}(T)$ by Proposition 3.1. □

Theorem 3.3. *If T^* is k-quasi-$*$-class A, then Weyl's theorem holds for T.*

Proof. Proposition 3.2 implies that

$$\sigma(T^*) \backslash \sigma_{BW}(T^*) = E(T^*).$$

It is obvious that

$$[\sigma(T^*) \backslash \sigma_{BW}(T^*)]^*, = \sigma(T) \backslash \sigma_{BW}(T)$$

hence we have to prove

$$(E(T^*)^*) = E(T).$$

Let $\lambda^* \in E(T^*)$. Then λ is an isolated point of $\sigma(T)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*. If $\lambda^* \neq 0$, then F_{λ^*} is self-adjoint,

$$\{0\} \neq F_{\lambda^*}\mathcal{H} = N((T - \lambda)^*) = N(T - \lambda)$$

by [25]. Hence $\lambda \in E(T)$. If $\lambda^* = 0$, then $T^*|_{F_0}$ is (p, k)-quasiposinormal by Lemma 2.1 and $(T^*|_{F_0 \mathcal{H}})^k = 0$ by [25]. Hence $T^{*^k} F_0 = 0$. Let $E_0 = F_0^*$ be the Riesz idempotent for 0 of T. Then $T^k E_0 = (T^{*^k} F_0)^* = 0$. Hence $T|_{E_0 \mathcal{H}}$ is nilpotent. Thus, $\lambda = 0 \in E(T)$.

Conversely, let $\lambda \in E(T)$. Then λ^* is an isolated point of $\sigma(T^*)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*. If $\lambda \neq 0$, then F_{λ^*} is self-adjoint and

$$\{0\} \neq F_{\lambda^*}\mathcal{H} = N((T-\lambda)^*) = N(T-\lambda)$$

by [25]. Hence $\lambda^* \in E(T^*)$. Let $\lambda = 0$. Since $T^*|_{F_0}\mathcal{H}$ is (p, k)-quasiposinormal and $\sigma(T^*|_{F_0}\mathcal{H}) = \{0\}$, we have $(T^*|_{F_0}\mathcal{H})^k = 0$ by [25]. This implies that $T^*|_{F_0}\mathcal{H}$ is nilpotent. Thus, $\lambda^* = 0 \in E(T^*)$. □

Next, we investigate a-Weyl's theorem [4].

We define $T \in SF_+^-$ if $R(T)$ is closed, $\dim N(T) < \infty$ and $\operatorname{ind} T \leq 0$. Let $\pi_{00}^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim\ker(T-\lambda) < \infty$. Let $\sigma_{SF_+^-}(T) = \{\lambda \mid T-\lambda \notin SF_+^-\} \subset \sigma_W(T)$.

We say that a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T).$$

Rakocevic [27, Corollary 2.5] proved that if a-Weyl's theorem holds for T, then Weyl's theorem holds for T.

Theorem 3.4. *If T^* is k-quasi-$*$-class A, then a-Weyl's theorem holds for T.*

Proof. Since T^* has the single valued extension property by Lemma 2.3, we have $\sigma(T) = \sigma_a(T)$ and $\pi_{00}(T) = \pi_{00}^a(T)$ [4, Corollary 2.45].

Let $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. If λ is an interior point of $\sigma_a(T)$, then there exists an open set G such that $\lambda \in G \subset \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Since T^* has the single valued extension property, $\operatorname{ind}(T-\mu)^* \leq 0$ for all $\mu \in \mathbb{C}$ by [4, Corollary 3.19]. Let $\mu \in G$. Then $T-\mu \in SF_+^-$ and $\operatorname{ind}(T-\mu) = 0$. On the other hand, $R(T-\mu)$ is closed, $T-\mu$ is not invertible and $0 < \dim N(T-\mu) < \infty$. Hence $0 < \dim N((T-\mu)^*) < \infty$ and T^* does not have a single valued extension property by [18, Theorem 9]. This is a contradiction. Hence we may assume that λ is a boundary point of $\sigma(T)$. Since $T-\lambda \in SF_+^-$, λ is an isolated point of $\sigma(T)$ by [14, Theorem XI 6.8]. Thus, $\lambda \in \pi_{00}(T) = \pi_{00}^a(T)$.

Conversely, $\lambda \in \pi_{00}^a(T) = \pi_{00}(T)$. Then λ^* is an isolated point of $\sigma(T^*)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*. If $\lambda^* \neq 0$, then F_{λ^*} is self-adjoint and

$$F_{\lambda^*}\mathcal{H} = N((T-\lambda)^*) = N(T-\lambda)$$

by [25]. Since $\dim N(T-\lambda) < \infty$, F_{λ^*} is compact. We decompose

$$(T-\lambda)^* = 0|_{F_{\lambda^*}\mathcal{H}} \oplus (T-\lambda)^*|_{(I-F_{\lambda^*})\mathcal{H}}.$$

Then $(T-\lambda)^*|_{(I-F_{\lambda^*})\mathcal{H}}$ is invertible and

$$T-\lambda = 0|_{F_{\lambda^*}\mathcal{H}} \oplus (T-\lambda)|_{(I-F_{\lambda^*})\mathcal{H}}.$$

Hence $R(T-\lambda) = (I-F_{\lambda^*})\mathcal{H}$ is closed and $\operatorname{ind}(T-\lambda) = 0$. Thus, $\lambda \in \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$.

If $\lambda^* = 0$, then

$$T^{*^k}|_{F_0\mathcal{H}} = (T^*|_{F_0\mathcal{H}})^k = 0$$

by [25]. Since $E_0 = F_0^*$ is the Riesz idempotent for 0 of T and $T^k E_0 = (T^{*^k} F_0)^* = 0,$ we have $E_0 \mathcal{H} \subset N(T^k).$ Then

$$\dim E_0 \mathcal{H} \leq \dim N(T^k) \leq k \dim N(T) < \infty.$$

This implies E_0 is compact. We decompose

$$T = T|_{E_0 \mathcal{H}} \oplus T|_{(I - E_0)\mathcal{H}}.$$

Since $T|_{(I - E_0)\mathcal{H}}$ is invertible, $R(T) = R(T|_{E_0 \mathcal{H}}) \oplus (I - E_0)\mathcal{H}$ is closed, $N(T) \subset E_0 \mathcal{H}$ and $\operatorname{ind} T = 0.$ Thus, $0 \in \sigma_a(T) \backslash \sigma_{SF_+^-}(T).$ □

Next, we investigate generalized a-Weyl's theorem [4].

We define $T \in SBF_+^-$ if there exists a positive integer n such that $R(T^n)$ is closed, $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) = \dim N(T) \cap R(T^n) < \infty$) and $0 \geq \operatorname{ind} T_{[n]}$ $(= \operatorname{ind} T)$ [9]. We define $\sigma_{SBF_+^-}(T) = \{\lambda \,|\, T - \lambda \notin SBF_+^-\} \subset \sigma_{SF_+^-}(T).$ Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim \ker(T - \lambda).$ We say that generalized a-Weyl's theorem holds for T if

$$\sigma_a(T) \backslash \sigma_{SBF_+^-}(T) = E^a(T).$$

Berkani and Koliha [8] proved that if generalized a-Weyl's theorem holds for T, then a-Weyl's theorem holds for T.

Theorem 3.5. *If T^* is k-quasi-$*$-class A, then generalized a-Weyl's theorem holds for T.*

Proof. Since T^* has the single valued extension property by Lemma 2.3, we have $\sigma(T) = \sigma_a(T)$, $\pi_{00}(T) = \pi_{00}^a(T)$ and $E(T) = E^a(T)$ [4, Corollary 2.45].

Let $\lambda_0 \in \sigma_a(T) \backslash \sigma_{SBF_+^-}(T)$. If λ_0 is an interior point of $\sigma_a(T)$, then there exists an open set G such that $\lambda_0 \in G \subset \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$. Let $\lambda \in G$. Then $T - \lambda \in SBF_+^-$, i.e., there exists a positive integer n such that $R((T - \lambda)^n)$ is closed, $\dim N(T_n - \lambda) < \infty$ and $\operatorname{ind}(T - \lambda) = \operatorname{ind}(T_n - \lambda) \leq 0$. Then there exists a positive number ε such that if $0 < |\lambda - \mu| < \varepsilon$, then $T - \mu$ is upper semi-Fredholm, $\operatorname{ind}(T - \mu) = \operatorname{ind}(T - \lambda) \leq 0$ and $\mu \in G$ by [9, Theorem 3.1]. Since T^* has a single valued extension property, $\operatorname{ind}(T - \mu)^* \leq 0$ by [4, Corollary 3.19]. Hence $\operatorname{ind}(T - \mu) = 0$. If $0 = \dim N(T - \mu)$, then $T - \mu$ is invertible. This is a contradiction. Hence $0 < \dim N(T - \mu) < \infty$ and $0 < \dim N((T - \mu)^*) < \infty$. Then T^* does not have the single valued extension property by [18]. This is a contradiction.

Hence we may assume that λ_0 is a boundary point of $\sigma(T)$. Since $T - \lambda_0 \in SBF_+^-$, $T - \lambda_0$ is topologically uniform descent by [9, Proposition 2.5], and λ_0 is an isolated point of $\sigma(T)$ by [20, Corollary 4.9]. We decompose

$$T - \lambda_0 = (T - \lambda_0)|_M \oplus (T - \lambda_0)|_N,$$

where $(T - \lambda_0)|_N$ is nilpotent and $(T - \lambda_0)|_M$ is semi-Fredholm by [9, Theorem 2.6]. If $N = \{0\}$, then

$$\lambda_0 \in \sigma_a(T) \backslash \sigma_{SF_+^-}(T) = \pi_{00}^a(T) = \pi_{00}(T) \subset E(T) = E^a(T)$$

by Theorem 3.4. If $N \neq \{0\}$, then λ_0 is an eigenvalue of $T|_N$ as $T|_N$ is nilpotent. Hence $\lambda_0 \in E(T) = E^a(T)$. Thus, $\sigma_a(T)\backslash\sigma_{SBF_+^-}(T) \subset E^a(T)$.

The converse inclusion is clear because

$$\begin{aligned} E^a(T) &= E(T) \\ &\subset \pi_{00}(T) \\ &= \pi_{00}^a(T) \\ &= \sigma_a(T)\backslash\sigma_{SF_+^-}(T) \\ &\subset \sigma_a(T)\backslash\sigma_{SBF_+^-}(T) \end{aligned}$$

by Theorem 3.4. □

Remark 3.6. (1) If $f(z)$ is an analytic function on $\sigma(T)$, then generalized *a*-Weyl's theorem holds for T (the proof is similar to [10, Theorem 3.3]).

(2) Generalized *a*-Weyl's theorem does not hold for *k*-quasi-∗-class *A* operator T as seen in [5, Example 2.13]. However, if $\ker T \subset \ker T^*$, then generalized *a*-Weyl's theorem holds for T (the proof is similar by [25]).

4. Spectrum of *k*-quasi-∗-class *A* Operators

Corollary 4.1. *If T is k-quasi-∗-class A operator, then* $\sigma_{jp}(T) - \{0\} = \sigma_p(T) - \{0\}$.

Proof. The proof follows from Lemma 2.2. □

Theorem 4.2. *If T is k-quasi-∗-class A operator, then* $\sigma_{jp}(T) - \{0\} = \sigma_{ap}(T) - \{0\}$.

Proof. Let ψ be the representation of Berberian. First, we show that $\psi(T)$ is k-quasi-$*$-class A,

$$(\psi(T))^{*k}[|(\psi(T))^2|-|\psi(T)^*|^2](\psi(T))^k$$

$$= \psi(T^{*k})[|\psi(T^2)|-|\psi(T^*)|^2]\psi(T^k)$$

$$= \psi(T^{*k})[\psi|T^2|-\psi|T^*|^2]\psi(T^k)$$

$$= \psi[T^{*k}(|T^2|-|T^*|^2)T^k].$$

But T is k-quasi-$*$-class A operator, then $T^{*k}(|T^2|-|T^*|^2)T^k \geq 0.$ So

$$\psi[T^{*k}(|T^2|-|T^{*^2}|)T^k] \geq 0.$$

Thus, $\psi(T)$ is k-quasi-$*$-class A operator. Now,

$$\sigma_a(T)-\{0\} = \sigma_a(\psi(T))-\{0\}$$

$$= \sigma_p(\psi(T))-\{0\}$$

$$= \sigma_{jp}(\psi(T))-\{0\} \text{ (by Corollary 4.1)}$$

$$= \sigma_{jap}(T)-\{0\}.$$

□

Corollary 4.3. *If T is an invertible k-quasi-$*$-class A, then*

$$\sigma_{jap}(T) = \sigma_{ap}(T).$$

Definition 4.4 [14, Exercise 2, p. 349]**.** The compression spectrum of T denoted by $\sigma_c(T)$ is

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma_p(T^*)\}.$$

Corollary 4.5. *If T is a k-quasi-$*$-class A operator, then*

$$\sigma(T) - \{0\} = \sigma_c(T) - \{0\}.$$

Proof. Note that for any operator $T \in B(\mathcal{H})$, the equality $\sigma(T) - \{0\} = \sigma_p(T) \cup \sigma_c(T) - \{0\}$ holds. If T is k-quasi-$*$-class A, then Corollary 4.1 implies that $\sigma_{jap}(T) - \{0\} = \sigma_p(T) - \{0\} \subseteq \sigma_c(T) - \{0\}$. Since $\sigma_p(T^*) \subset \sigma_{ap}(T^*)$, the result follows. □

References

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integral Equations Operator Theory 13(3) (1990), 307-315.

[2] A. Aluthge, Some generalized theorems on p-hyponormal operators, Integral Equations Operator Theory 24(4) (1996), 497-501.

[3] S. C. Arora and P. Arora, On p-quasihyponormal operators for $0 < p < 1$, Yokohama Math. J. 41 (1993), 25-29.

[4] P. Aiena, Fredholm and Local Spectral Theory with Applications to Multipliers, Kluwer Academic Publishers, Dordrecht, Boston, London, 2004.

[5] P. Aiena and P. Pena, Variations on Weyl's theorem, J. Math. Anal. Appl. 324 (2006), 566-579.

[6] T. Ando, On some operator inequalities, Math. Ann. 279(1) (1987), 157-159.

[7] M. Berkani and A. Arroud, Generalized Weyl's theorem and hyponormal operators, J. Aust. Math. Soc. 76 (2004), 291-302.

[8] M. Berkani and J. J. Koliha, Weyl's type theorems for bounded linear operators, Acta. Sci. Math. (Szeged) 69 (2003), 379-391.

[9] M. Berkani and M. Sarih, On semi B-Fredholm operators, Glasgow Math. J. 43 (2001), 457-465.

[10] X. Cao, M. Guo and B. Meng, Weyl type theorems for p-hyponormal and M-hyponormal operators, Studia Math. 163 (2004), 177-188.

[11] S. L. Campbell and B. C. Gupta, On k-quasihyponormal operators, Math. Joponica 23 (1978), 185-189.

[12] M. Cho, M. Ito and S. Oshiro, Weyl's theorem holds for *p*-hyponormal operators, Glasgow Math. J. 39 (1997), 217-220.

[13] L. A. Coburn, Weyl's theorem for nonnormal operators, Michigan Math. J. 13 (1966), 285-288.

[14] J. B. Conway, A Course in Functional Analysis, 2nd ed., Springer-Verlag, New York, 1990.

[15] H. Crawford Rhaly, Posinormal operators, J. Math. Soc. Japan 46 (1994), 587-605.

[16] B. P. Duggal and S. V. Djorjovic, On $*$-paranormal contractions and properties for $*$-class A operators, Linear Algebra Appl. (in press).

[17] B. P. Duggal, I. H. Jeon and I. H. Kim, Weyl's theorem in the class of algebraically *p*-hyponormal operators, Comment. Math. Prace Mat. 40 (2000), 49-56.

[18] J. K. Finch, The single valued extension property on a Banach space, Pacific J. Math. 58 (1975), 61-69.

[19] T. Furuta, M. Ito and T. Yamazaki, A subclass of paranormal including class of log-hyponormal and several related classes, Sci. Math. 1 (1998), 389-403.

[20] S. Grabiner, Uniform ascent and descent of bounded operators, J. Math. Soc. Japan 34 (1982), 317-337.

[21] Y. M. Han and W. Y. Lee, Weyl's theorem holds for algebraically hyponormal operators, Proc. Amer. Math. Soc. 128 (2000), 2291-2296.

[22] W. Y. Lee, Weyl spectra of operator matrices, Proc. Amer. Math. Soc. 129 (2000), 131-138.

[23] W. Y. Lee, Weyl's theorem for operator matrices, Integral Equations Operator Theory 32 (1998), 319-331.

[24] S. Mecheri, Generalized Weyl's theorem for some classes of operators, Kyungpook Math. J. (to appear).

[25] S. Mecheri, Isolated points of spectrum of k-quasi-$*$-class A operators, Studia Math. 208(1) (2012), 87-96.

[26] C. Pearcy, Some Recent Developments in Operator Theory, CBMS 36, AMS, Providence, 1978.

[27] V. Rakocevic, Operators obeying *a*-Weyl's theorem, Rev. Roumaine Math. Pures Appl. 10 (1989), 915-919.

[28] J. L. Shen, F. Zuo and C. S. Yang, On operators satisfying $T^*|T^2|T \geq T^*|T^{*2}|T$, Acta Math. Sin. (Engl. Ser.) 26 (2010), 2109-2116.

[29] H. Weyl, Uber beschrankte quadratische Formen, deren Differenz vollsteig ist, Rend. Circ. Mat. Palermo 27 (1909), 373-392.

[30] D. Xia, Spectral Theory of Hyponormal Operators, Birkhäuser, Verlag, Boston, 1983.