Advances in Fuzzy Sets and Systems
Volume 14, Number 1, 2013, Pages 53-69
Published Online: April 2013
Available online at http://pphmj.com/journals/afss.htm
Published by Pushpa Publishing House, Allahabad, INDIA

ON FUZZY RETRACTS OF FUZZY CLOSED FLAT ROBERTSON-WALKER SPACE

A. E. El-Ahmady[1] **and A. S. Al-Luhaybi**[2]

[1,2]Mathematics Department
Faculty of Science
Taibah University
Madinah, Saudi Arabia

[1]Mathematics Department
Faculty of Science
Tanta University
Tanta, Egypt

Abstract

Our aim in the present paper is to introduce and study certain types of fuzzy retractions of fuzzy closed flat Robertson-Walker space $\tilde{W}^4$ model. Some kinds of fuzzy deformation retracts of $\tilde{W}^4$ model are obtained. The relations between the fuzzy folding and the fuzzy deformation retracts of $\tilde{W}^4$ model are deduced. Types of fuzzy minimal retractions of $\tilde{W}^4$ model are also presented. New types of homotopy maps are deduced. New types of conditional fuzzy folding of $\tilde{W}^4$ model are presented. Some commutative diagrams are obtained.

2010 Mathematics Subject Classification: 53A35, 51H05, 58C05, 51F10, 58B34.
Keywords and phrases: fuzzy retraction, fuzzy deformation retract, fuzzy folding, fuzzy closed flat Robertson-Walker space $\tilde{W}^4$ model.
Communicated by K. K. Azad
Received November 10, 2012; Revised March 4, 2013

Introduction and Background

Robertson-Walker space represents one of the most intriguing and emblematic discoveries in the history of geometry. Although if it was introduced for a purely geometrical purpose, they came into prominence in many branches of mathematics and physics. This association with applied science and geometry generated synergistic effect: applied science gave relevance to Robertson-Walker space and Robertson-Walker space allowed formalizing practical problems [13-15, 18, 19, 23]. As is well known, the theory of retractions is always one of the interesting topics in Euclidian and non-Euclidian spaces and it has been investigated from the various viewpoints by many branches of topology and differential geometry [2-5, 7, 8, 11, 16, 17, 21]. There are many diverse applications of certain phenomena for which it is impossible to get relevant data. It may not be possible to measure essential parameters of a process such as the temperature inside molten glass or the homogeneity of a mixture inside some tanks. The required measurement scale may not exist at all, such as in the case of evaluation of offensive smells, evaluating the taste of foods or medical diagnoses by touching [1-3, 6, 9-12, 20, 22]. A fuzzy manifold is manifold which has a physical character. This character is represented by the density function μ, where $\mu \in [0, 1]$ [7, 8].

A fuzzy subset $(\underset{\sim}{A}, \mu)$ of a fuzzy manifold $(\underset{\sim}{M}, \mu)$ is called a *fuzzy retraction* of $(\underset{\sim}{M}, \mu)$ if there exists a continuous map $\tilde{r} : (\underset{\sim}{M}, \mu) \to (\underset{\sim}{A}, \mu)$ such that $\tilde{r}(a, \mu(a)) = (a, \mu(a)), \forall a \in \underset{\sim}{A}, \mu \in [0, 1]$ [1-7].

A fuzzy subset $(\underset{\sim}{\overline{M}}, \tilde{\mu})$ of a fuzzy manifold $(\underset{\sim}{M}, \mu)$ is called a *fuzzy deformation retract* if there exist a fuzzy retraction $\tilde{r} : (\underset{\sim}{M}, \mu) \to (\underset{\sim}{\overline{M}}, \tilde{\mu})$ and a fuzzy homotopy $\tilde{F} : (\underset{\sim}{M}, \mu) \times I \to (\underset{\sim}{M}, \mu)$ [2, 6, 9, 11] such that

$$\left.\begin{aligned} \tilde{F}((x, \mu), 0) &= (x, \mu) \\ \tilde{F}((x, \mu), 1) &= \tilde{r}(x, \mu) \end{aligned}\right\}, \quad x \in \underset{\sim}{M},$$

$\tilde{F}((a, \mu), t) = (a, \mu), \ \forall (a, \mu) \in \underset{\sim}{\overline{M}}, \ t \in I, \ \mu \in [0, 1],$ where $\tilde{r}(x, \mu)$ is the retraction. From the beginning of the line a map $\tilde{\mathfrak{F}} : \tilde{W}^4 \to \tilde{W}^4$ is said to be an *isometric folding* of $\tilde{W}^4$ model into itself iff for any piecewise fuzzy geodesic path $\gamma : J \to \tilde{W}^4$ the induced path $\tilde{\mathfrak{F}} \circ \gamma : J \to \tilde{W}^4$ is a piecewise fuzzy geodesic and of the same length as γ, where $J = [0, 1]$. If $\tilde{\mathfrak{F}}$ does not preserve lengths, then $\tilde{\mathfrak{F}}$ is a topological folding of $\tilde{W}^4$ model [1, 4-6, 8, 9].

The isofuzzy folding of $\cup \tilde{M}_i \subseteq \tilde{W}^4$ is a folding $\tilde{f} : \cup \tilde{M}_i \to \cup \tilde{M}_i$ such that $\tilde{f}(\tilde{M}) = \tilde{M}$ and any $\tilde{M}_i$ belong to the upper hypermanifolds $\exists \tilde{M}_j$ down $\tilde{M}$ such that $\mu_i = \mu_j$ for every corresponding points, i.e., $\mu(a_i) = \mu(a_j)$ [6]. See Figure 1:

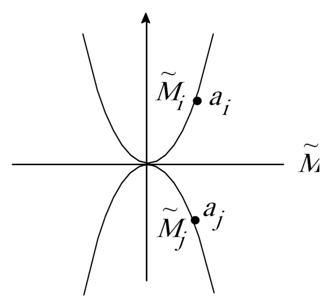

Figure 1

Main Results

Theorem 1. *The retractions of the fuzzy closed flat Robertson-Walker space $\tilde{W}^4$ model are the fuzzy sphere, fuzzy hypersphere, fuzzy circle and fuzzy minimal manifolds.*

Proof. Consider the fuzzy $\tilde{W}^4$ model with metric

$$d\ell^2(\mu) = d\chi^2(\mu) + \sin^2\chi(\mu)(d\theta^2(\mu) + \sin^2\theta(\mu)d\phi^2(\mu)), \ \mu \in [0, 1]. \quad (1)$$

The coordinates of fuzzy $\tilde{W}^4$ model are

$$\tilde{x}_1 = \sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu), \quad \tilde{x}_3 = \sin\chi(\mu)\cos\theta(\mu),$$

$$\tilde{x}_2 = \sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu), \quad \tilde{x}_4 = \cos\chi(\mu), \tag{2}$$

where the ranges are $0 \le \chi \le \infty$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

Now, we use Lagrangian equations

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \dot{\varphi}_i(\mu)}\right) - \frac{\partial T}{\partial \varphi_i(\mu)} = 0, \quad i = 1, 2, 3, 4$$

to find a geodesic which is a subset of $\tilde{W}^4$ model. Since

$$\tilde{T} = \frac{1}{2}\{\dot{\chi}^2(\mu) + \sin^2\chi(\mu)(\dot{\theta}^2(\mu) + \sin^2\theta(\mu)\,\dot{\phi}^2(\mu))\},$$

the Lagrangian equations for $\tilde{W}^4$ model are

$$\frac{\tilde{d}}{\widetilde{ds}}(\dot{\chi}(\mu)) - (\sin\chi(\mu)\cos\chi(\mu)(\dot{\theta}^2(\mu) + \sin^2\theta(\mu)\,\dot{\phi}^2(\mu))) = 0, \tag{3}$$

$$\frac{\tilde{d}}{\widetilde{ds}}(\sin^2\chi(\mu)\,\dot{\theta}(\mu)) - (\sin^2\chi(\mu)\sin\theta(\mu)\cos\theta(\mu)\,\dot{\phi}^2(\mu)) = 0, \tag{4}$$

$$\frac{\tilde{d}}{\widetilde{ds}}(\sin^2\chi(\mu)\sin^2\theta(\mu)\phi(\dot{\mu})) = 0, \tag{5}$$

$$\begin{aligned}
-(\dot{\chi}(\mu)) + (&\sin\chi(\mu)\cos\chi(\mu)\,\dot{\theta}^2(\mu) + \sin^2\chi(\mu)\,\dot{\theta}(\mu) \\
&+ \sin\chi(\mu)\cos\chi(\mu)\sin^2\theta(\mu)\,\dot{\phi}^2(\mu) \\
&+ \sin^2\chi(\mu)\sin\theta(\mu)\cos\theta(\mu)\,\dot{\phi}^2(\mu) \\
&+ \sin^2\chi(\mu)\sin^2\theta(\mu)\phi(\dot{\mu})) = 0.
\end{aligned} \tag{6}$$

From equation (5), we obtain $\sin^2\chi(\mu)\sin^2\theta(\mu)\phi(\dot{\mu}) = \text{constant}$, say β_1, if $\beta_1 = 0$, then we obtain the following cases:

If $\phi = 0$, hence we obtain the following fuzzy geodesic retraction sphere $\tilde{S}_1^2$, $\tilde{X}_1^2 + \tilde{X}_3^2 + \tilde{X}_4^2 = 1$ with $\tilde{X}_2 = 0$. Also, if $\phi = \frac{\pi}{6}$ or $\frac{\pi}{4}$ and $\frac{\pi}{3}$, then we have the retractions fuzzy spheres $\tilde{S}_1^3$, $\tilde{S}_2^3$ and $\tilde{S}_3^3$ in $\tilde{W}^4$ model. Again, if $\phi = \frac{\pi}{2}$, hence we obtain the geodesic retraction fuzzy sphere $\tilde{S}_2^2$ in $\tilde{W}^4$ model given by

$$(0, \sin\chi(\mu)\sin\theta(\mu), \sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)).$$

Also, if $\phi = \pi$, then the geodesic retraction fuzzy sphere $\tilde{S}_3^2$ in $\tilde{W}^4$ model is defined as

$$(\sin\chi(\mu)\sin\theta(\mu), 0, \sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)).$$

Now, if $\theta = 0$, hence we obtain the following coordinate of geodesic fuzzy circle $\tilde{S}_1^1$ in $\tilde{W}^4$ model given by $(0, 0, \sin\chi(\mu), \cos\chi(\mu))$. Also, if $\theta = \frac{\pi}{6}$, then we have the retraction fuzzy sphere $\tilde{S}_4^3$ in $\tilde{W}^4$ model defined as

$$\left(\frac{1}{2}\sin\chi(\mu)\cos(\mu), \frac{1}{2}\sin\chi(\mu)\sin(\mu), \frac{\sqrt{3}}{2}\sin\chi(\mu), \cos\chi(\mu)\right).$$

Again, if $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{3}$, then we have the retraction unit spheres $\tilde{S}_5^3$, $\tilde{S}_6^3$. Now, if $\theta = \frac{\pi}{2}$, hence we obtain the following coordinate of fuzzy sphere $\tilde{S}_4^2$ in $\tilde{W}^4$ model given by

$$(\sin\chi(\mu)\cos(\mu), \sin\chi(\mu)\sin(\mu), 0, \cos\chi(\mu)).$$

Again, if $\chi = 0$, then we get the following minimal fuzzy geodesic retraction $\breve{W}^0(0, 0, 0, 1)$ in $\tilde{W}^4$ model. In a special case, if $\chi = \frac{\pi}{2}$, hence we get the coordinate of fuzzy geodesic sphere $\tilde{S}_5^2$ in $\tilde{W}^4$ model which is represented by $(\sin \theta(\mu) \cos(\mu), \sin \theta(\mu) \sin(\mu), \cos \theta(\mu), 0)$. Also, if $\phi = 90$, and $\chi = 90$, then we obtain the retraction fuzzy circle $\tilde{S}_2^1 = (0, \sin \theta(\mu), \cos \theta(\mu), 0)$.

In this position, we present some cases of fuzzy deformation retracts of $\tilde{W}^4$ model. The fuzzy deformation retract of $\tilde{W}^4$ model is

$$\tilde{\eta} : \{\tilde{W}^4 - \tilde{\beta}\} \times I \to \{\tilde{W}^4 - \tilde{\beta}\},$$

where $\{\tilde{W}^4 - \tilde{\beta}\}$ is open in $\tilde{W}^4$ model and I is the closed interval $[0, 1]$, be presented as

$$\begin{aligned} \tilde{\eta}(x, h) : \{&(\sin \chi(\mu) \sin \theta(\mu) \cos \phi(\mu), \sin \chi(\mu) \sin \theta(\mu) \sin \phi(\mu), \\ &\sin \chi(\mu) \cos \theta(\mu), \cos \chi(\mu)) - \tilde{\beta}\} \times I \\ &\to \{(\sin \chi(\mu) \sin \theta(\mu) \cos \phi(\mu), \sin \chi(\mu) \sin \theta(\mu) \sin \phi(\mu), \\ &\sin \chi(\mu) \cos \theta(\mu), \cos \chi(\mu)) - \tilde{\beta}\}. \end{aligned}$$

The fuzzy deformation retract of $\tilde{W}^4$ model into the minimal fuzzy geodesic $\breve{W}^0(0, 0, 0, 1)$ is

$$\begin{aligned} \tilde{\eta}(m, h) = (1 + h)\{&(\sin \chi(\mu) \sin \theta(\mu) \cos \phi(\mu), \sin \chi(\mu) \sin \theta(\mu) \sin \phi(\mu), \\ &\sin \chi(\mu) \cos \theta(\mu), \cos \chi(\mu)) - \tilde{\beta}\} \\ &+ \tan \frac{\pi h}{4} \{0, 0, 0, 1\}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\eta}(m, 0) = \{&(\sin \chi(\mu) \sin \theta(\mu) \cos \phi(\mu), \sin \chi(\mu) \sin \theta(\mu) \sin \phi(\mu), \\ &\sin \chi(\mu) \cos \theta(\mu), \cos \chi(\mu)) - \tilde{\beta}\} \end{aligned}$$

and

$$\tilde{\eta}(m, 1) = \{0, 0, 0, 1\}.$$

The fuzzy deformation retract of $\tilde{W}^4$ model into the fuzzy circle $\tilde{S}_1^1$ is

$$\check{\eta}(m, h) = (1 - 2h + h^2)\{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu),$$

$$\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu), \sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)) - \tilde{\beta}\}$$

$$+ (h^3 - 2h + 2h^2)\{0, 0, \sin\chi(\mu), \cos\chi(\mu)\}.$$

The fuzzy deformation retract of $\tilde{W}^4$ model into the fuzzy sphere $\tilde{S}_3^2$ is

$$\tilde{\eta}(m, h) = \cos\frac{\pi h}{2}\{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu),$$

$$\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu), \sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)) - \tilde{\beta}\}$$

$$+ \sin\frac{\pi h}{2}\{\sin\chi(\mu)\sin\theta(\mu), 0, \sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)\}.$$

Now, we are going to discuss the fuzzy folding $\tilde{\mathfrak{I}}$ of $\tilde{W}^4$ model. Let $\tilde{\mathfrak{I}} : \tilde{W}^4 \to \tilde{W}^4$, where

$$\tilde{\mathfrak{I}}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (\tilde{x}_1, |\tilde{x}_2|, \tilde{x}_3, \tilde{x}_4). \tag{7}$$

An isometric fuzzy folding of $\tilde{W}^4$ model into itself may be defined by

$$\tilde{\mathfrak{I}} : \{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu), \sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu),$$

$$\sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)) - \tilde{\beta}\}$$

$$\to \{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu), |\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu)|,$$

$$\sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)) - \tilde{\beta}\}.$$

The fuzzy deformation retract of the folded $\tilde{\mathfrak{I}}(\tilde{W}^4)$ into the minimal fuzzy geodesic $\breve{W}^0(0, 0, 0, 1)$ is

$$\tilde{\eta}_{\mathfrak{I}} : \{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu), |\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu)|,$$

$$\sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)) - \tilde{\beta}\} \times I$$

$$\to \{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu), |\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu)|,$$

$$\sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)) - \tilde{\beta}\}$$

with

$$\tilde{\eta}_{\mathfrak{I}}(m, h) = (1+h)\{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu), |\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu)|,$$

$$\sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)) - \tilde{\beta}\} + \tan\frac{\pi h}{4}\{0, 0, 0, 1\}.$$

The fuzzy deformation retract of the folded $\tilde{\mathfrak{I}}(\tilde{W}^4)$ into the folded fuzzy circle $\tilde{S}_1^1$ is

$$\tilde{\eta}_{\mathfrak{I}}(m, h) = (1+h)\{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu),$$

$$|\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu)|,$$

$$\sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)) - \tilde{\beta}\}$$

$$+ (h^3 - 2h + 2h^2)\{0, 0, \sin\chi(\mu), \cos\chi(\mu)\}.$$

The fuzzy deformation retract of the folded $\mathfrak{I}(\tilde{W}^4)$ into the folded fuzzy sphere $\tilde{S}_3^2$ is

$$\tilde{\eta}_{\mathfrak{I}}(m, h) = (1+h)\{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu),$$

$$|\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu)|, \sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)) - \tilde{\beta}\}$$

$$+ \sin\frac{\pi h}{2}\{\sin\chi(\mu)\sin\theta(\mu), 0, \sin\chi(\mu)\cos\theta(\mu), \cos\chi(\mu)\}.$$

Then the following theorem has been proved.

Theorem 2. *Under the defined fuzzy folding and any folding homeomorphic to this type of folding, the fuzzy deformation retract of the folded fuzzy closed flat Robertson-Walker space* $\tilde{\mathfrak{I}}(\tilde{W}^4)$ *into the folded fuzzy geodesics is the same as the fuzzy deformation retract of the fuzzy closed flat Robertson-Walker space* $\tilde{W}^4$ *into the fuzzy geodesics.*

Now, let the fuzzy folding be defined by

$$\tilde{\mathfrak{I}}^* : \tilde{W}^4 \to \tilde{W}^4,$$

where

$$\tilde{\mathfrak{I}}^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, |\tilde{x}_4|). \tag{8}$$

The isometric folded $\tilde{\mathfrak{I}}^*(\tilde{W}^4)$ is

$$\overline{R} = \{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu), \sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu),$$
$$\sin\chi(\mu)\cos\theta(\mu), |\cos\chi(\mu)|) - \tilde{\beta}\}.$$

The fuzzy deformation retract of the folded $\tilde{\mathfrak{I}}^*(\tilde{W}^4)$ into the folded fuzzy circle $\tilde{\mathfrak{I}}^*(\tilde{S}_1^1)$ is

$$\tilde{\eta}_{\mathfrak{I}}(m, h) = (1+h)\{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu),$$
$$\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu), \sin\chi(\mu)\cos\theta(\mu), |\cos\chi(\mu)|) - \tilde{\beta}\}$$
$$+ (h^3 - 2h + 2h^2)\{0, 0, \sin\chi(\mu), |\cos\chi(\mu)|\}.$$

The fuzzy deformation retract of the folded $\tilde{\mathfrak{I}}^*(\tilde{W}^4)$ into the folded fuzzy sphere $\tilde{\mathfrak{I}}^*(\tilde{S}_3^2)$ is

$$\tilde{\eta}_{\mathfrak{I}}(m, h) = (1+h)\{(\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu),$$
$$\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu), \sin\chi(\mu)\cos\theta(\mu), |\cos\chi(\mu)|) - \tilde{\beta}\}$$
$$+ \sin\frac{\pi h}{2}\{\sin\chi(\mu)\sin\theta(\mu), 0, \sin\chi(\mu)\cos\theta(\mu), |\cos\chi(\mu)|\}.$$

Then the following theorem has been proved.

Theorem 3. *Under the defined fuzzy folding and any folding homeomorphic to this type of folding, the fuzzy deformation retract of the folded fuzzy closed flat Robertson-Walker space into the folded fuzzy geodesics is different from the fuzzy deformation retract of the fuzzy closed flat Robertson-Walker space into the fuzzy geodesics***.**

Corollary 1. *The relations between the fuzzy retractions and the limits of the fuzzy folding of* $\tilde{W}^4$ *model are discussed from the following commutative diagram*:

$$\begin{tikzcd}
\{\tilde{W}^4 - \tilde{\beta}\} \arrow[r, "\tilde{r}_1"] \arrow[d, dash, "\lim_{m\to\infty} \widetilde{\mathrm{f}_\mathrm{m}}"'] & \tilde{S}_1^{\,2} \arrow[d, "\lim_{m\to\infty} \widetilde{\mathrm{f}_{\mathrm{m}+1}}"] \\
\tilde{S}_1^{\,2} \arrow[r, "\tilde{r}_2"] & \tilde{S}_1^1
\end{tikzcd}$$

Corollary 2. *Consider the* $\tilde{W}^4$ *model and* $\tilde{\beta} \in \tilde{W}^4$, *then the open fuzzy retraction of* $\{\tilde{W}^4 - \tilde{\beta}\}$ *induced two chains of open fuzzy retractions of the two open fuzzy systems of* $\bigcup\{\overline{\tilde{W}^4} - \overline{\tilde{\beta}}\}$ *and* $\bigcap\{\underline{\tilde{W}^4} - \underline{\tilde{\beta}}\}$.

Corollary 3. *The relations between the fuzzy retraction and the fuzzy deformation retract of* $\tilde{W}^4$ *model are discussed from the following commutative diagram*:

$$\begin{tikzcd}
\{\tilde{W}^4 - \tilde{\beta}\} \times I \arrow[r, "\widetilde{D.R}"] \arrow[d, "\tilde{r}_1"'] & \{\tilde{S}_1^{\,2} - \beta\} \arrow[d, "\tilde{r}_2"] \\
\{\tilde{S}_1^{\,2} - \tilde{\beta}\} \times I \arrow[r, "\widetilde{\mathrm{D.R}}"] & \tilde{W}^0
\end{tikzcd}$$

Theorem 4. *Any fuzzy folding of fuzzy hypersphere* $\tilde{S}^3_\alpha \subset \tilde{W}^4$ *into fuzzy hypersphere* $\tilde{S}^3_\beta \subset \tilde{W}^4$ *induces fuzzy folding of fuzzy open ball* $\tilde{B}(\widetilde{\pi\alpha})$ *into fuzzy open ball* $\tilde{B}(\widetilde{\pi\beta})$ *from* $\tilde{T}_{P_\alpha}(\tilde{S}^3_\alpha) \to \tilde{T}_{P_\beta}(\tilde{S}^3_\beta)$.

Proof. Let $\tilde{\mathfrak{F}} : \tilde{S}^3_\alpha \subset \tilde{W}^4 \to \tilde{S}^3_\beta \subset \tilde{W}^4$. Then there is an induced fuzzy folding from the fuzzy open ball $\tilde{B}(\widetilde{\pi\alpha})$ into the fuzzy open ball $\tilde{B}(\widetilde{\pi\beta})$ defined as $\tilde{\mathfrak{F}}^* : \tilde{B}(\widetilde{\pi\alpha}) \to \tilde{B}(\widetilde{\pi\beta})$ such that $\widetilde{\exp}^{-1} : \tilde{S}^3_\alpha \subset \tilde{W}^4 \to \tilde{B}(\widetilde{\pi\alpha})$ and $\widetilde{\exp}^{-1} : \tilde{S}^3_\beta \subset \tilde{W}^4 \to \tilde{B}(\widetilde{\pi\beta})$ such that the following diagram is commutative:

$$\begin{tikzcd}
\tilde{S}^3_\alpha \subset \tilde{W}^4 \arrow[r, "\tilde{\mathfrak{F}}"] \arrow[d, "\widetilde{exp}^{-1}"'] & \tilde{S}^3_\beta \subset \tilde{W}^4 \arrow[d, "\widetilde{exp}^{-1}"] \\
\tilde{B}(\widetilde{\pi\alpha}) \arrow[r, "\tilde{\mathfrak{F}}^*"] & \tilde{B}(\widetilde{\pi\beta})
\end{tikzcd}$$

i.e., $\tilde{\mathfrak{F}}^* \circ \widetilde{\exp}^{-1} = \widetilde{\exp}^{-1} \circ \tilde{\mathfrak{F}}$.

Theorem 5. *Any fuzzy retraction of fuzzy open hypersphere* $(\tilde{S}^3_\alpha - \tilde{\lambda}) \subset \tilde{W}^4$ *into fuzzy open hypersphere* $(\tilde{S}^3_\beta - \tilde{\lambda}) \subset \tilde{W}^4$, $\tilde{\beta} \subset \tilde{\alpha}$ *induces fuzzy retraction of fuzzy open ball* $\tilde{B}(\widetilde{\pi\alpha})$ *into fuzzy open ball* $\tilde{B}(\widetilde{\pi\beta})$.

Proof. Let $\tilde{r}$ be a fuzzy retraction map, $\tilde{r} : (\tilde{S}^3_\alpha - \tilde{\lambda}) \subset \tilde{W}^4 \to (\tilde{S}^3_\beta - \tilde{\lambda}) \subset \tilde{W}^4$, where $(\tilde{S}^3_\alpha - \tilde{\lambda})$ and $(\tilde{S}^3_\beta - \tilde{\lambda})$, $\tilde{\beta} \subset \tilde{\alpha}$. Also, let $\widetilde{\exp}^{-1}$: $(\tilde{S}^3_\alpha - \tilde{\lambda}) \subset \tilde{W}^4 \to \tilde{B}(\widetilde{\pi\alpha} - \grave{\tilde{\lambda}})$ and $\widetilde{\exp}^{-1} : (\tilde{S}^3_\beta - \tilde{\lambda}) \subset \tilde{W}^4 \to \tilde{B}(\widetilde{\pi\beta} - \grave{\tilde{\lambda}})$ such that $\tilde{B}(\widetilde{\pi\alpha} - \grave{\tilde{\lambda}}) \supset \tilde{B}(\widetilde{\pi\beta} - \grave{\tilde{\lambda}})$. Then we have the fuzzy retraction $\tilde{r}^* : \tilde{B}(\widetilde{\pi\alpha} - \grave{\tilde{\lambda}}) \to \tilde{B}(\widetilde{\pi\beta} - \grave{\tilde{\lambda}})$ such that $\tilde{r}^* \circ \widetilde{\exp}^{-1} = \widetilde{\exp}^{-1} \circ \tilde{r}$,

$$\begin{tikzcd}
(\tilde{S}^3_\alpha - \tilde{\leftthreetimes}) \arrow[r, "\tilde{r}"] \arrow[d, "\widetilde{exp}^{-1}"'] & (\tilde{S}^3_\beta - \tilde{\leftthreetimes}) \arrow[d, "\widetilde{exp}^{-1}"] \\
\tilde{B}(\widetilde{\pi\alpha} - \tilde{\leftthreetimes}) \arrow[r, "\tilde{r}^*"] & \tilde{B}(\widetilde{\pi\beta} - \tilde{\leftthreetimes})
\end{tikzcd}$$

Theorem 6. *Any fuzzy retraction* $\tilde{r}_1 : \tilde{T}_p\{\tilde{W}^4 - \tilde{\beta}\} \to \tilde{T}_q(\tilde{S}^3)$ *induces fuzzy retraction* $\tilde{r}_2 : \{\tilde{W}^4 - \tilde{\beta}\} \to \tilde{S}^3$ *such that* $\tilde{r}_2 \circ \widetilde{\exp}_p = \widetilde{\exp}_q \circ \tilde{r}_1$.

Proof. Let $\tilde{r}_1 : \tilde{T}_p\{\tilde{W}^4 - \tilde{\beta}\} \to \tilde{T}_q(\tilde{S}^3)$ be a fuzzy retraction of $\tilde{T}_p\{\tilde{W}^4 - \tilde{\beta}\}$ into $\tilde{T}_q(\tilde{S}^3)$. Also, let $\widetilde{\exp}_p : \tilde{T}_p\{\tilde{W}^4 - \tilde{\beta}\} \to \{\tilde{W}^4 - \tilde{\beta}\}$ and $\widetilde{\exp}_q : \tilde{T}_q(\tilde{S}^3) \to (\tilde{S}^3)$. Then we have $\tilde{r}_2 : \{\tilde{W}^4 - \tilde{\beta}\} \to \tilde{S}^3$ such that $\tilde{r}_2 \circ \widetilde{\exp}_p = \widetilde{\exp}_q \circ \tilde{r}_1$,

$$\begin{tikzcd}
\tilde{T}_p\{\tilde{W}^4 - \tilde{\beta}\} \arrow[r, "\tilde{r}_1"] \arrow[d, "\widetilde{exp}_p"'] & \tilde{T}_q\left(\tilde{S}_2{}^3\right) \arrow[d, "\widetilde{exp}_q"] \\
\{\tilde{W}^4 - \tilde{\beta}\} \arrow[r, "\tilde{r}_2"] & \tilde{S}_1{}^3
\end{tikzcd}$$

Theorem 7. *Any fuzzy retraction* $\tilde{r} : \{\tilde{W}^4 - \tilde{\beta}\} \to \tilde{S}_1^2$, *then the map* $\tilde{r}_1 : \tilde{T}_p\{\tilde{W}^4 - \tilde{\beta}\} \to \tilde{T}_q(\tilde{S}_1^3)$ *induces by the inverse fuzzy exponential map.*

Proof. Let $\tilde{r} : \{\tilde{W}^4 - \tilde{\beta}\} \to (\tilde{S}_1^2)$ be a fuzzy retraction of $\{\tilde{W}^4 - \tilde{\beta}\}$ onto $\tilde{S}_1^2$. Also, let $\widetilde{\exp}_p^{-1} : \{\tilde{W}^4 - \tilde{\beta}\} \to \tilde{T}_p(\tilde{W}^4)$ and $\widetilde{\exp}_q^{-1} : (\tilde{S}_2^3) \to \tilde{T}_q(\tilde{S}_1^1)$. Then we have the fuzzy retraction $\tilde{r}_1 : \tilde{T}_p\{\tilde{W}^4 - \tilde{\beta}\} \to \tilde{T}_p(\tilde{S}_1^1)$ such that $\tilde{r}_1 \circ \widetilde{\exp}_p^{-1} = \widetilde{\exp}_q^{-1} \circ \tilde{r}$,

$$\begin{tikzcd}
\{\widetilde{W}^4 - \tilde{\beta}\} \arrow[r, "\tilde{r}"] \arrow[d, "\widetilde{exp}^{-1}{}_n"'] & \tilde{S}_1{}^2 \arrow[d, "\widetilde{exp}^{-1}{}_q"] \\
\tilde{T}_p(\{\widetilde{W}^4 - \tilde{\beta}\}) \arrow[r, "\tilde{r}_1"] & \tilde{T}_q(\tilde{S}_1{}^1)
\end{tikzcd}$$

Theorem 8. *Let* $\tilde{S}^3 \subset W^4$ *be a fuzzy hypersphere in* $\tilde{W}^4$ *model which is homeomorpic to* $\tilde{D}^2 \subset R^3$, *and* $\tilde{r}_1 : \{\tilde{S}^3 - \tilde{\beta}\} \to \tilde{S}^2$ *be a fuzzy retraction. Then there is an induced fuzzy retraction* $\tilde{r}_2 : \{\tilde{D}^2 - \tilde{\beta}\} \to \tilde{D}^1$ *such that the following diagram is commutative*:

$$\begin{tikzcd}
\{\widetilde{S^3} - \tilde{\beta}\} \subset W^4 \arrow[r, "\widetilde{P_1}"] \arrow[d, "\widetilde{r_1}"'] & \{\tilde{D}^2 - \tilde{\beta}\} \subset R^3 \arrow[d, "\widetilde{r_2}"] \\
\widetilde{S^2} \subset W^4 \arrow[r, "\widetilde{P_2}"] & \widetilde{D^1} \subset R^3
\end{tikzcd}$$

Proof. Let $\tilde{r}_1 : \{\tilde{S}^3 - \tilde{\beta}\} \to \tilde{S}^2$, where $\tilde{S}^3 = (\sin\chi(\mu)\sin\theta(\mu)\cos\phi(\mu),$ $\sin\chi(\mu)\sin\theta(\mu)\sin\phi(\mu),\ \sin\chi(\mu)\cos\theta(\mu),\ \cos\chi(\mu)),$ $\tilde{r}_2 : \{\tilde{D}^2 - \tilde{\beta}\} \to \tilde{D}^1,$ where $\tilde{D}^2 = (\sin\theta(\mu)\cos\phi(\mu),\ \sin\theta(\mu)\sin\phi(\mu),\ \cos\theta(\mu),\ 0),$ using the homeomorphism maps $\tilde{P}_1 : \{\tilde{S}^3 - \tilde{\beta}\} \to \tilde{S}^2 \subset W^4 \to \{\tilde{D}^2 - \tilde{\beta}\} \subset R^3$ and $\tilde{P}_2 : \tilde{S}^2 \subset W^4 \to \tilde{D}^1 \subset R^3.$ This proves that the diagram is commutative.

Corollary 4. *Under the above conditions in Theorem* 8, *the following diagrams are commutative*:

$$\begin{tikzcd}
\{\overline{\widetilde{S^3}} - \overline{\tilde{\beta}}\} \subset W^4 \arrow[r, "\overline{\widetilde{P_1}}"] \arrow[d, "\overline{\widetilde{r_1}}"'] & \{\overline{\widetilde{D^2}} - \overline{\tilde{\beta}}\} \subset R^3 \arrow[d, "\overline{\widetilde{r_2}}"] \\
\overline{\widetilde{S^2}} \subset W^4 \arrow[r, "\overline{\widetilde{P_2}}"] & \overline{\widetilde{D^1}} \subset R^3
\end{tikzcd}$$

$$\begin{tikzcd}
\{\underline{\widetilde{S^3}} - \underline{\tilde{\beta}}\} \subset W^4 \arrow[r, "\underline{\widetilde{P_1}}"] \arrow[d, "\underline{\widetilde{r_1}}"'] & \{\underline{\widetilde{D}^2} - \underline{\tilde{\beta}}\} \subset R^3 \arrow[d, "\underline{\widetilde{r_2}}"] \\
\underline{\widetilde{S^2}} \subset W^4 \arrow[r, "\underline{\widetilde{P_2}}"] & \underline{\widetilde{D}^1} \subset R^3
\end{tikzcd}$$

Theorem 9. *Let* $\tilde{S}^3 \subset W^4$ *be a fuzzy hypersphere in* $\tilde{W}^4$ *model which is homeomorphic to* $\tilde{D}^2 \subset R^3$, *and* $\widetilde{\lim_{n\to\infty} r_n} : \{\tilde{S}^3 - \tilde{\beta}\} \to \tilde{S}^2$. *Then there is an induced limit fuzzy retraction* $\widetilde{\lim_{n+1\to\infty} r_{n+1}} : \{\tilde{D}^2 - \tilde{\beta}\} \to \tilde{D}^1$ *such that the following diagram is commutative*:

$$\begin{tikzcd}
\{\widetilde{S^3} - \tilde{\beta}\} \subset W^4 \arrow[r, "\widetilde{P_1}"] \arrow[d, "\widetilde{\lim_{n\to\infty} r_n}"'] & \{\tilde{D}^2 - \tilde{\beta}\} \subset R^3 \arrow[d, "\widetilde{\lim_{n+1\to\infty} r_{n+1}}"] \\
\widetilde{S^2} \subset W^4 \arrow[r, "\widetilde{P_2}"] & \widetilde{D^1} \subset R^3
\end{tikzcd}$$

Proof. Let $\widetilde{\lim_{n\to\infty} r_n} : \{\tilde{S}^3 - \tilde{\beta}\} \to \tilde{S}^2$, $\widetilde{\lim_{n+1\to\infty} r_{n+1}} : \{\tilde{D}^2 - \tilde{\beta}\} \to \tilde{D}^1$, using the homeomorphism maps $\tilde{P}_1 : \{\tilde{S}^3 - \tilde{\beta}\} \subset W^4 \to \{\tilde{D}^2 - \tilde{\beta}\} \subset R^3$ and $\tilde{P}_2 : \tilde{S}^2 \subset W^4 \to \tilde{D}^1 \subset R^3$. This proves that the diagram is commutative.

Corollary 5. *Under the above conditions in Theorem* 9, *the following diagrams are commutative*:

$$\begin{tikzcd}
\{\overline{\widetilde{S^3}} - \overline{\tilde{\beta}}\} \subset W^4 \arrow[r, "\overline{\widetilde{P_1}}"] \arrow[d, "\overline{\widetilde{\lim_{n\to\infty} r_n}}"'] & \{\overline{\widetilde{D^2}} - \overline{\tilde{\beta}}\} \subset R^3 \arrow[d, "\overline{\widetilde{\lim_{n+1\to\infty} r_{n+1}}}"] \\
\overline{\widetilde{S^2}} \subset W^4 \arrow[r, "\overline{\widetilde{P_2}}"] & \overline{\widetilde{D^1}} \subset R^3
\end{tikzcd}$$

$$\begin{tikzcd}
\{\underline{\widetilde{S}^3} - \underline{\widetilde{\beta}}\} \subset W^4 \arrow[r, "\underline{\widetilde{P}_1}"] \arrow[d, "\underline{\widetilde{\lim_{n\to\infty} r_n}}"'] & \{\underline{\widetilde{D}^2} - \underline{\widetilde{\beta}}\} \subset R^3 \arrow[d, "\underline{\widetilde{\lim_{n+1\to\infty} r_{n+1}}}"] \\
\underline{\widetilde{S}^2} \subset W^4 \arrow[r, "\underline{\widetilde{P}_2}"] & \underline{\widetilde{D}^1} \subset R^3
\end{tikzcd}$$

i.e., $\overline{\widetilde{\lim_{n+1\to\infty} r_{n+1}}} \circ \overline{\widetilde{p_1}} = \overline{\widetilde{p_2}} \circ \overline{\widetilde{\lim_{n\to\infty} r_n}}$ *and* $\underline{\widetilde{\lim_{n+1\to\infty} r_{n+1}}} \circ \underline{\widetilde{p_1}} = \underline{\widetilde{p_2}} \circ \underline{\widetilde{\lim_{n\to\infty} r_n}}.$

Theorem 10. *If the fuzzy deformation retract of the fuzzy hypersphere* $\widetilde{S}^3 \subset \widetilde{W}^4$ *is* $\widetilde{D} : (\widetilde{S}^3 - \{\widetilde{\beta}\} \times I) \to \widetilde{S}^3$, *the fuzzy retraction of* $\widetilde{S}^3 \subset \widetilde{W}^4$ *is* $\widetilde{r} : (\widetilde{S}^3 - \{\widetilde{\beta}\}) \to \widetilde{S}^2$, $\widetilde{S}^2 \subset \widetilde{S}^3$, *and the limit of the fuzzy folding of* S^3 *is* $\lim_{m\to\infty} \widetilde{f}_m : \widetilde{S}^3 \to \widetilde{S}^2$, *then there are induces fuzzy retraction, limit of the fuzzy folding, and the fuzzy deformation retract such that the following diagram is commutative.*

Proof. Let the fuzzy deformation retract is $\widetilde{D}_1 : (\widetilde{S}^3 - \{\widetilde{\beta}\}) \times I \to (\widetilde{S}^3 - \{\widetilde{\beta}\})$, the fuzzy retraction of $(\widetilde{S}^3 - \{\widetilde{\beta}\}) \times I$ is defined by $\widetilde{r}_1 : (\widetilde{S}^3 - \{\widetilde{\beta}\} \times I) \to (\widetilde{S}^2 - \{\widetilde{\beta}\}) \times I$, $\lim_{m\to\infty} \widetilde{f}_m : D_1(\widetilde{S}^3 - \{\widetilde{\beta}\} \times I) \to (\widetilde{S}^2 - \{\widetilde{\beta}\})$, the fuzzy retraction of $(\widetilde{S}^3 - \{\widetilde{\beta}\} \times I)$ is $\widetilde{r}_2 : \widetilde{r}_1(\widetilde{S}^3 - \{\widetilde{\beta}\} \times I) \to \widetilde{S}^2 - \{\widetilde{\beta}\} \times I$, the limit of fuzzy folding of $\lim_{m\to\infty} \widetilde{f}_m(\widetilde{S}^3 - \{\widetilde{\beta}\})$ is $\lim_{m\to\infty} \widetilde{f}_{m+1} : \lim_{m\to\infty} \widetilde{f}_m(\widetilde{D}_1(\widetilde{S}^3 - \{\widetilde{\beta}\} \times I)) \to (\widetilde{S}^1 - \{\widetilde{\beta}\})$, and $\widetilde{D}_2 : \widetilde{r}_2(\widetilde{r}_1(\widetilde{S}^3 - \{\widetilde{\beta}\} \times I)) \to (\widetilde{S}^1 - \{\widetilde{\beta}\})$. Hence the following diagram is commutative:

$$\begin{tikzcd}
(\widetilde{S}^3 - \{\widetilde{\beta}\} \times I) \arrow[r, "\widetilde{r}_1"] \arrow[d, "\widetilde{D}_1"'] & (\widetilde{S}^2 - \{\widetilde{\beta}\} \times I) \arrow[r, "\widetilde{r}_2"] & (\widetilde{S}^1 - \{\widetilde{\beta}\} \times I) \arrow[d, "\widetilde{D}_2"] \\
\widetilde{S}^3 - \{\widetilde{\beta}\} \arrow[r, "\lim_{m\to\infty} \widetilde{f}_m"'] & \widetilde{S}^2 - \{\widetilde{\beta}\} \arrow[r, "\mathrm{im}_{m\to\infty} \widetilde{f}_{m+1}"'] & \widetilde{S}^1 - \{\widetilde{\beta}\}
\end{tikzcd}$$

i.e., $\widetilde{D}_2 \circ \widetilde{r}_2 \circ \widetilde{r}_1(\widetilde{S}^3 \times I) = \lim_{m\to\infty} \widetilde{f}_{m+1} \circ \lim_{m\to\infty} \widetilde{f}_m \circ \widetilde{D}_1.$

Corollary 6. *Under the above conditions in Theorem* 10, *the following diagrams are commutative*:

$$\begin{tikzcd}
(\overline{\tilde{S}^3} - \{\overline{\tilde{\beta}}\} \times I) \arrow[r, "\overline{\widetilde{r_1}}"] \arrow[d, "\overline{\widetilde{D_1}}"'] & \overline{\widetilde{S^2}} - \{\overline{\tilde{\beta}}\} \times I \arrow[r, "\overline{\widetilde{r_2}}"] & \overline{\tilde{S}^1} - \{\overline{\tilde{\beta}}\} \times I \arrow[d, "\overline{\widetilde{D_2}}"] \\
\widetilde{S^3} - \{\overline{\tilde{\beta}}\} \arrow[r, "\lim_{m\to\infty} \overline{\tilde{f}_m}"'] & \overline{\widetilde{S^2}} - \{\overline{\tilde{\beta}}\} \arrow[r, "\lim_{m\to\infty} \overline{\widetilde{f_{m+1}}}"'] & \overline{\widetilde{S^1}} - \{\overline{\tilde{\beta}}\}
\end{tikzcd}$$

$$\begin{tikzcd}
(\underline{\tilde{S}^3} - \{\underline{\tilde{\beta}}\} \times I) \arrow[r, "\underline{\widetilde{r_1}}"] \arrow[d, "\underline{\widetilde{D_1}}"'] & \underline{\widetilde{S^2}} - \{\underline{\tilde{\beta}}\} \times I \arrow[r, "\underline{\widetilde{r_2}}"] & \underline{\tilde{S}^2} - \{\underline{\tilde{\beta}}\} \times I \arrow[d, "\underline{\widehat{D}}"] \\
\underline{\widetilde{S^3}} - \{\underline{\tilde{\beta}}\} \arrow[r, "\lim_{m\to\infty} \underline{\widetilde{f_m}}"'] & \underline{\widetilde{S^2}} - \{\underline{\tilde{\beta}}\} \arrow[r, "\lim_{m\to\infty} \underline{\widetilde{f_{m+1}}}"'] & \underline{\widetilde{S^1}} - \{\underline{\tilde{\beta}}\}
\end{tikzcd}$$

References

[1] A. E. El-Ahmady, The variation of the density on chaotic spheres in chaotic space-like Minkowski space time, Chaos Solitons Fractals 31 (2007), 1272-1278.

[2] A. E. El-Ahmady, Folding of fuzzy hypertori and their retractions, Proc. Math. Phys. Soc. 85(1) (2007), 1-10.

[3] A. E. El-Ahmady, Limits of fuzzy retractions of fuzzy hyperspheres and their foldings, Tamkang J. Math. 37(1) (2006), 47-55.

[4] A. E. El-Ahmady and H. Rafat, Retraction of chaotic Ricci space, Chaos Solitons Fractals 41 (2009), 394-400.

[5] A. E. El-Ahmady and H. Rafat, A calculation of geodesics in chaotic flat space and its folding, Chaos Solitons Fractals 30 (2006), 836-844.

[6] A. E. El-Ahmady and A. El-Araby, On fuzzy spheres in fuzzy Minkowski space, Nuovo Cimento 125B(10) (2010), 1153-1160.

[7] A. E. El-Ahmady, Retraction of chaotic black hole, J. Fuzzy Math. 19(4) (2011), 833-838.

[8] A. E. El-Ahmady and H. Rafat, Fractal retraction of chaotic Narlikar and Karmarkar space, Adv. Stud. Contemp. Math. 15(2) (2007), 259-268.

[9] A. E. El-Ahmady, Fuzzy Lobachevskian space and its folding, J. Fuzzy Math. 12(2) (2004), 609-614.

[10] A. E. El-Ahmady, Fuzzy folding of fuzzy horocycle, Circolo Matematico di Palermo Serie II, Tomo L III, (2004), 443-450.

[11] A. E. El-Ahmady and H. M. Shamara, Fuzzy deformation retract of fuzzy horospheres, Indian J. Pure Appl. Math. 32(10) (2001), 1501-1506.

[12] A. P. Balachandran, S. Kurkcuoglu and S. Vaidya, Lectures on Fuzzy and Fuzzy Susy Physics, New Jersey, London, New York, 2007.

[13] Bernd Schmeikal, Primordial Space - The Metric Case, New York, 2010.

[14] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni and P. Zampetti, Geometry of Minkowski Space-time, Birkhäuser, Verlag, Basel, Boston, Berlin, 2011.

[15] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti and P. Zampetti, The Mathematics of Minkowski Space-time, Birkhäuser, Verlag, Basel, Boston, Berlin, 2008.

[16] G. L. Naber, Topology, Geometry and Gauge Fields: Foundations, Texts in Applied Mathematics 25, Springer-Verlag, New York, Berlin, 2011.

[17] G. L. Naber, Topology, Geometry and Gauge Fields: Interactions, Applied Mathematics Sciences, Springer-Verlag, New York, Berlin, 2011.

[18] J. B. Hartle, Gravity: An Introduction to Einstein’s General Relativity, Addison-Wesley, New York, 2003.

[19] J. B. Griffiths and J. Podolsky, Exact Space-times in Einstein’s General Relativity, Cambridge University Press, Cambridge, New York, 2009.

[20] L. A. Zadeh, Fuzzy Sets and their Application to Cognitive and Decision, Academic Press, New York, 1975.

[21] M. Reid and Balazs Szendroi, Topology and Geometry, Cambridge University Press, Cambridge, New York, 2005.

[22] N. Palaniappan, Fuzzy Topology, London, New York, 2002.

[23] N. Straumann, General Relativity with Application to Astrophysics, Springer-Verlag, New York, Heidelberg, Berlin, 2004.