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# A PAIR OF COMPLEMENTARY EXTREMUM PRINCIPLES FOR FREE SURFACE FLOW IN FLUID MECHANICS VARIATIONAL PRINCIPLE

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## Abstract

By variational analysis, a couple of variational principles for dam spillway problem were testified a couple of maximum principle and minimum principle, and by variational analysis, furthermore, this couple of maximum and minimum principles was proved to be a pair of complementary extremum principles. So it not only provides a more rigorous theoretical foundation for their application, but also contributes to find the FEM solutions of problems in fluid mechanics.

## Nomenclature

| | |
|---|---|
| $\Omega$ | Moment function |
| $Y$ | Azimuthal angle function |
| $\Psi$ | Stream function |



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| | |
|---|---|
| $U$ | Velocity in $x$ direction (m/s) |
| $p$ | Pressure (Pa) |
| $g$ | Gravity acceleration $(\mathrm{m/s^2})$ |

**Superscript**

| | |
|---|---|
| $\wedge$ | The variable corresponding to stationary value functional |

**Subscript**

| | |
|---|---|
| $pr$ | The known variable |
| $b$ | Bottom |
| $h$ | Free surface |

## 1. Introduction

Extremum principle is a very important property to equations or some process. If some equations or process could be testified to have extremum principle, on the basis of the nature of extremum principle, then some important properties of the equations or process which provide a great important mode of thinking to solve and analyze the sort of problems, could be known.

For example, by applying the energy extremum principle with bond conditions in [1], the practical scope of directivity synthesis in the theory of key array sound field is broaden. The optimum *N* impact transition (at any time) for space was studied in [2], the integral of accompanying system in the process of engine ignition was derived, the procedures of the curve field for space extreme value were produced.

The relations between the maximum critical thickness and the metalloid content in amorphous alloys of the Fi-Si-B system were studied, the analysis and discussion of the hierarchical structure and formation principle for noncrystalline state alloy went into details in [3]. The generation laws for mud-rock flow home and abroad were studied in [4], the special phenomenon for the natural calamity-extreme value was raised, which provides the theoretical and technological guide for people to understand the mud-rock flow deeply.

The functional extreme value in variational principle is very important to the numerical calculation like the sort of problem. For instance, if the variational $\delta J(\Phi) = 0$ of function $J(\Phi)$ is minimum principle [5], $\delta^2 J(\Phi)$ is the positive (negative), the positive (negative) of function is closely related to the positive (negative) of coefficient matrix (rigidity matrix) for a set of algebraic equations gained from discrete finite element, the latter plays key role in solving the algebraic problems and it also is the fundamental condition required for a lot of highly efficient algebraic solutions. In addition, the extremum principle plays a key role in analyzing the error of numerical calculating result and the convergence of equations.

The analysis of variational principle on the problems of overflow over dam went into details in literature [6], the different forms of variational principles were inferred. A pair of extremum principles was derived by regarding a couple of equations acquired in literature [6] as known conditions and they were a pair of complementary extremum principles.

## 2. Basic Equations

Basic equations [7] for two dimension incompressible and non-viscous fluid flow:

Continuity equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1}$$

Momentum equations ($y$ direction)

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g - \frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{2}$$

After Von Mises streamline coordinate was introduced and coordinate conversions were done, formulas (1) and (2) were changed:

$$\frac{\partial}{\partial\xi}\left(\frac{1}{u}\right) - \frac{\partial}{\partial\Psi}\left(\frac{v}{u}\right) = 0, \tag{3}$$

$$\frac{\partial v}{\partial\xi} + \frac{\partial p'}{\partial\Psi} = 0. \tag{4}$$

Here $\xi = x$ or $\xi = f(x)$, $\Psi = \Psi(x, y)$ is called *stream function* and is defined by $\frac{\partial\Psi}{\partial x} = -v$, $\frac{\partial\Psi}{\partial y} = u$. Because $p' = gy + \frac{p}{\rho}$, it is equal to the total of the potential energy and pressure energy, so

Energy equations

$$p' + \frac{1}{2}(u^2 + v^2) = B. \tag{5}$$

On the basis of formulas (3) and (4), azimuthal angle function and moment function were introduced as follows:

$$\frac{\partial Y}{\partial\xi} = Y_\xi = \frac{v}{u}, \frac{\partial Y}{\partial\Psi} = Y_\Psi = u^{-1}, \tag{6}$$

$$\frac{\partial\Omega}{\partial\xi} = \Omega_\xi = p', \frac{\partial\Omega}{\partial\Psi} = \Omega_\Psi = -v. \tag{7}$$

By doing the variational principle for formula (3), the variational principle for dam overflow could be gotten as follows $\delta I_A = 0$:

$$I_A(\Omega) = \iint_A u d\xi\Psi + J_A = \iint_A \sqrt{2(B - \Omega_\xi) - (\Omega_\Psi)^2}\, d\xi d\Psi + J_A. \tag{8}$$

Here

$$J_A = \int_{C_1}\left[\frac{1}{u_{pr}}\vec{i}_\Psi + \left(\frac{v}{u}\right)_{pr}\vec{i}_\xi\right]\Omega d\vec{s} + \int_{\text{botton}} tg\beta_{pr}\Omega d\xi$$

$$-\frac{1}{g}\int_{\text{water surface}}\left(\frac{\partial^2\Omega}{\partial\xi^2}\right)\Omega d\xi,$$

$C_1$ is the entrance section.

By doing the variational principle for formula (4), another variational principle for dam overflow could be gotten as follows, $\delta I_B = 0$, and formula (5) was regarded as bound condition

$$I_B(Y) = \iint_A (p' + u^2 + v^2)\frac{1}{u}\, d\xi d\Psi + J_B. \tag{9}$$

Here $J_B = \int_{C_1}[p'_{pr}\vec{i}_\Psi + v_{pr}\vec{i}_\xi]\cdot \vec{n}\overline{Y}ds + \int_{\text{water surface}}\left(\frac{p_a}{\rho} + gh\right)\overline{Y}d\xi,$ $C_1$ is the entrance section.

Now it was known by demonstration that: formula (8) is maximum (minimum) principle, formula (9) is minimum (maximum) principle, and formulas (8) and (9) are a pair of complementary extremum principles.

## 3. Demonstration of Extremum Principle

The variational principle for formula (8) was done in order to show formula (8) is extremum principle, so

$$\delta I_A(\Omega) = \iint_A \frac{-\delta(\Omega_\xi) - \Omega_\Psi \delta(\Omega_\Psi)}{\sqrt{2(B - \Omega_\xi) - (\Omega_\Psi)^2}}\, d\xi d\Psi + \delta J_A$$

$$= \iint_A \frac{\delta p' + v\delta v}{\sqrt{2(B - \Omega_\xi) - (\Omega_\Psi)^2}}\, d\xi d\Psi + \delta J_A.$$

The second variational principle for formula (8) was done, so

$\delta(\delta I_A(\Omega))$

$$= \iint_A \frac{(\delta v)^2\sqrt{2(B - \Omega_\xi) - (\Omega_\Psi)^2} + \dfrac{(\delta p' + v\delta v)^2}{\sqrt{2(B - \Omega_\xi) - (\Omega_\Psi)^2}}}{[2(B - \Omega_\xi) - (\Omega_\Psi)^2]}\, d\xi d\Psi + \delta(\delta J_A)$$

$$= -\iint_A \frac{u^2(\delta v)^2 + (\delta p' + v\delta v)^2}{u^3}\, d\xi d\Psi + \delta J_A.$$

Because $\delta J_A = \oint_C \delta\left[\frac{d\Psi}{u_{pr}} + \left(\frac{v}{u}\right)_{pr} d\xi\right]\delta\Omega = 0$, if $u > 0$, $\delta(\delta I_A(\Omega)) < 0$, so formula (8) is maximum principle; if $u < 0$, $\delta(\delta I_A(\Omega)) > 0$, so formula (8) is minimum principle.

To prove formula (9) is extremum principle, formula (9) was dealt with in the same way as that for formula (8). So the variational principle of formula (9) is

$$\delta I_B(Y) = \iint_A \frac{(\delta p' + 2u\delta u + 2v\delta v)u - (p' + u^2 + v^2)\delta u}{u^2} d\xi d\Psi + \delta J_B.$$

The second variational principle for formula (9) was done, so

$$\begin{aligned}\delta(\delta I_B(Y)) = \iint_A \{&[\delta(\delta p') + 2(\delta u)^2 u + (\delta p' + 2u\delta u + 2v\delta v)\delta u \\ &- (\delta p' + 2u\delta u + 2v\delta v)\delta u]u^2 - [(\delta p' + 2u\delta u + 2v\delta v)u \\ &- (p' + u^2 + v^2)\delta u]2u\delta u\}\frac{1}{u^4} d\xi d\Psi + \delta(\delta J_B).\end{aligned}$$

Because $p' = B - \frac{1}{2}(u^2 + v^2)$, so $\delta p' = -u\delta u - v\delta v$, because $p'$ is the function for the partial derivative of function $Y$, so

$$\delta(\delta p') = -(\delta u)^2 - (\delta v)^2.$$

The concrete compositions of $\delta p'$ and $\delta(\delta p')$ were introduced in the formula $\delta(\delta I_B(Y))$ above and the formula $\delta(\delta I_B(Y))$ was rearranged, so

$$\delta(\delta I_B(Y)) = \iint_A \frac{(u\delta v - v\delta u)^2}{u^3} d\xi d\Psi + \delta(\delta J_B).$$

Because

$$\delta J_B = \oint_C [v_{pr} d\Psi - p'_{pr} d\xi]dY, \quad \delta(\delta J_B) = \oint_C \delta[v_{pr} d\Psi - p'_{pr} d\xi]dY = 0,$$

if $u > 0$, $\delta(\delta I_B(Y)) > 0$, formula (9) is minimum principle; if $u < 0$, $\delta(\delta I_B(Y)) < 0$, formula (9) is minimum principle.

## 4. Demonstration of Complementary Extremum Principle

To testify formulas (8) and (9) are a pair of complementary extremum principles, at first the relations between formulas (8) and (9) were testified. So Lagrange multipliers $\mu_i$ were introduced in formula (8), and the bound condition (7) was merged into formula (8), so

$$I'(\Omega,\ \mu_1,\ \mu_2) = \iint [u + \mu_1(\Omega_\xi - p') + \mu_2(\Omega_\Psi + v)] d\xi d\Psi + J_A$$

$$= \iint \left( u - \frac{\partial \mu_1}{\partial \xi} \Omega - \mu_1 p' - \frac{\partial \mu_2}{\partial \Psi} \Omega + \mu_2 v \right) d\xi d\Psi + J_b + J_A$$

$$= \iint \left( u - \frac{\partial \mu_1}{\partial \xi} \Omega - \mu_1 p' - \frac{\partial \mu_2}{\partial \Psi} \Omega + \mu_2 v \right) d\xi d\Psi + J_b'. \quad (10)$$

Here $J_b = \oint (\mu_2 \Omega d\xi - \mu_1 \Omega d\Psi)$, $J_b' = J_b + J_A$.

The variation of formula (10) was done and rearranged, so

$$\delta I' = \iint \left[ -\delta\Omega \left( \frac{\partial \mu_1}{\partial \xi} + \frac{\partial \mu_2}{\partial \Psi} \right) + \delta\mu_1(\Omega_\xi - p') + \delta\mu_2(v + \Omega_\Psi) \right.$$

$$\left. + \ \delta u + \mu_2 \delta v - \mu_1 \delta p' \right] d\xi d\Psi + \delta I_b''$$

$\delta p' = -u\delta u - v\delta v$ was introduced in the formula $\delta I'$ above, so

$$\delta I' = \iint \left[ -\delta\Omega \left( \frac{\partial \mu_1}{\partial \xi} + \frac{\partial \mu_2}{\partial \Psi} \right) + \delta\mu_1(\Omega_\xi - p') + \delta\mu_2(v + \Omega_\Psi) \right.$$

$$\left. + \ \delta u(1 + \mu_1 u) + \delta v(\mu_2 + \mu_1 v) \right] d\xi d\Psi + \delta I_b''.$$

Because $\delta I' = 0,$ so the formulas below were gotten:

$$\left.\begin{aligned} \delta\mu_1 : \Omega_\xi - p' = 0 \\ \delta\mu_2 : v + \Omega_\Psi = 0 \end{aligned}\right\}, \tag{11}$$

$$\left.\begin{aligned} \delta u : 1 + \mu_1 u = 0 \\ \delta v : \mu_2 + \mu_1 v = 0 \end{aligned}\right\}, \tag{12}$$

$$\delta\Omega : \frac{\partial\mu_1}{\partial\xi} + \frac{\partial\mu_2}{\partial\Psi} = 0. \tag{13}$$

Based on formula (12), two Lagrange multipliers were identified

$$\mu_1 = -\frac{1}{u},\ \mu_2 = \frac{v}{u}. \tag{14}$$

Formula (13) could be written as the formula below

$$\frac{\partial}{\partial\xi}\left(\frac{1}{u}\right) + \frac{\partial}{\partial\Psi}\left(\frac{v}{u}\right) = 0. \tag{15}$$

Formula (14) was introduced in formula (10), and formulas (15) and (11) were regarded as variational bounds, or formulas (6) and (7) were regarded as bounds, so

$$I' = \iint_A u d\xi d\Psi + J_A = \iint_A \left(u + \frac{p'}{u} + \frac{v^2}{u}\right) d\xi d\Psi + J_b + J_A. \tag{16}$$

Here $J_b = \int_C \left[\frac{v}{u}\Omega d\xi + \frac{1}{u}\Omega d\Psi\right] = \int_C \left(\frac{v}{u}\vec{i}_\xi + \frac{1}{u}\vec{i}_\Psi\right) \cdot \Omega d\vec{s}.$

Formula (9) from formula (8) was

$$\begin{aligned} \Delta I &= I_A(\Omega) - I_B(Y) \\ &= \iint_A u d\xi d\Psi + J_A - \iint_A (p' + u^2 + v^2)\frac{1}{u} d\xi d\Psi - J_B. \end{aligned} \tag{17}$$

Formula (16) was introduced in formula (17), so

$$\Delta I = J_b + J_A - J_B. \tag{18}$$

By conversion of equations it was known that if there was extremum solution, $J_A$ could be changed into

$$\hat{J}_A = \oint_C \left( \frac{1}{u} \vec{i}_\Psi + \frac{v}{u} \vec{i}_\xi \right)_{pr} \cdot \Omega d\vec{s}$$

$$= \oint_C \left[ \frac{1}{u_{pr}} \Omega d\Psi + \left( \frac{v}{u} \right)_{pr} \Omega d\xi \right]$$

$$= -\iint \left[ \frac{\partial}{\partial \xi} \left( \frac{1}{u} \Omega \right) - \frac{\partial}{\partial \Psi} \left( \frac{v}{u} \Omega \right) \right]_{ex} d\Psi d\xi$$

$$= -\iint \left\{ \Omega \left[ \frac{\partial}{\partial \xi} \left( \frac{1}{u} \right) - \frac{\partial}{\partial \Psi} \left( \frac{v}{u} \right) \right] + \left[ \frac{1}{u} \frac{\partial \Omega}{\partial \xi} - \frac{v}{u} \frac{\partial \Omega}{\partial \Psi} \right] \right\}_{ex} d\Psi d\xi$$

$$= -\iint \left\{ \Omega \left[ \frac{\partial Y_\Psi}{\partial \xi} - \frac{\partial Y_\xi}{\partial \Psi} \right] + \left[ \frac{p'}{u} + \frac{v^2}{u} \right] \right\}_{ex} d\Psi d\xi$$

$$= -\iint \left\{ \Omega [Y_{\Psi\xi} - Y_{\xi\Psi}] + \left[ \frac{p'}{u} + \frac{v^2}{u} \right] \right\}_{ex} d\Psi d\xi.$$

Because $Y$ can be secondarily differentiated, so $Y_{\xi\Psi} = Y_{\Psi\xi}$, and the formula $\hat{J}_A$ above could be written as $\hat{J}_A = -\iint \frac{\hat{p}' + \hat{v}^2}{\hat{u}} d\Psi d\xi.$

According to the same way as above, it could be testified

$$\hat{J}_B = -\iint \frac{\hat{p}' + \hat{v}^2}{\hat{u}} d\Psi d\xi.$$

So, for extremum solution, formula (18) could be written as

$$\Delta \hat{I} = \hat{J}_b. \tag{19}$$

To testify formulas (8) and (9) are a pair of complementary extremum principles, only work to do was to testify $\Delta \hat{I}$ has a defined numerical value. Here, it was studied through the problem of dam overflow.

According to Figure 1, it is known that $C$ is composed of two parts, $C = C_1 + C_2$, $C_1$ are the boundaries of inlet and outlet, $C_2$ are the boundaries of bottom and free surface.

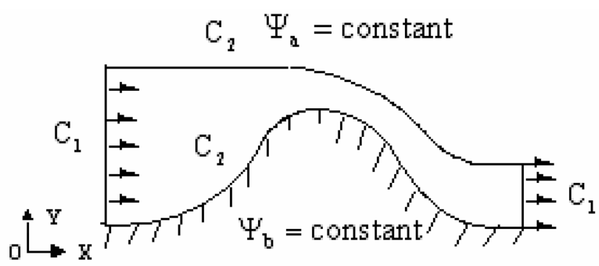


**Figure 1.** Schedule of the overflow over dam.

Formula (19) is developed on boundary $C$, so

$$\Delta\hat{I} = \hat{J}_b = \int_{C_1} \frac{1}{u_{pr}} \hat{\Omega} d\Psi + \int_{C_2} \left(\frac{v}{u}\right)_{pr} \hat{\Omega} d\xi.$$

There is indeterminate variable $\hat{\Omega}$ in the integral term of $C_1$ and $C_2$ for the above formula $\Delta\hat{I}$, if the number of the variable $\hat{\Omega}$ could be determined, finally $\Delta\hat{I}$ could be determined to be a finite numerical value, so the numerical value of $\hat{J}_b$ is determined.

The determination of the $\Delta\hat{I}$ value could be illustrated in the light of the properties of definite integral-theorem of mean. Because the functions in the integral term of formula $\Delta\hat{I}$ above are continuous function, and independent variables have defined upper limit and lower limit, so the numerical value difference between the upper limit and lower limit of the arguments is multiplied by the special numerical value of functions in the integral term that is determined by the numerical value in the integral district, the result of integral for formula above is gotten. The formula below could be gotten as

$$[\hat{I}_A(\Omega)]_{ex} = [\hat{I}_B(Y)]_{ex} + I_b. \tag{20}$$

It is complementary principle to testify.

## 5. Conclusions

(1) By variational analysis, a pair of functions in fluid mechanics was testified to be extremum principle, through further variational analysis, the extremum principle was testified to be a pair of complementary principles.

(2) Because formulas (8) and (9) were testified to be extremum principle, the positive (negative) of function $\delta^2 J(\Phi)$ could be determined, and the positive (negative) of coefficient matrix (rigidity matrix) for algebraic equations gained from separating finite element could be determined further, the latter plays key role in solving the algebraic problems and it also is the fundamental condition required for a lot of highly efficient algebraic solutions.

(3) A theoretical foundation for analyzing the error of numerical calculating result and the convergence of equations was laid because formulas (8) and (9) were testified to be the testified extremum principle.

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