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THE ECCENTRIC DIGRAPH OF (n, t)-KITE GRAPH

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Abstract

Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The distance from vertex u to vertex v in G is the length of the shortest path from vertex u to v. The eccentricity $e(u)$ of a vertex u is the maximum distance of u to any other vertices of G. A vertex v is an eccentric vertex of vertex u if the distance from u to v is equal to $e(u)$. The eccentric digraph $ED(G)$ of a graph G is the digraph that has the same set of vertices as G, and there is an arc (directed edge) from u to v in $ED(G)$ if and only if v is an eccentric vertex of u in G.

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In this paper, we determine the eccentric digraph of an (n, t)-kite graph.

I. Introduction

The notations and terminologies mostly follow that of Chartrand and Oellermann [2] and Gallian [3]. Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The distance from vertex u to vertex v in G, denoted by $d(u, v)$, is the length of the shortest path from vertex u to v. If there is no path joining vertex u and vertex v, then $d(u, v) = \infty$. The eccentricity of vertex u in graph G is the maximum distance from vertex u to any other vertices in G, denoted by $e(u)$, and so $e(u) = \max\{d(u, v) | v \in V(G)\}$. Radius of a graph G, denoted by $rad(G)$, is the minimum eccentricity of every vertex in G. The diameter of a graph G, denoted by $diam(G)$, is the maximum eccentricity of every vertex in G. If $e(u) = rad(G)$, then vertex u is called *central vertex*. Center of a graph G, denoted by $cen(G)$, is an induced subgraph formed from central vertices of G. Vertex v is an eccentric vertex from u if $d(u, v) = e(u)$. The eccentric digraph $ED(G)$ of a graph G is a graph that has the same set of vertices as G, $V(ED(G)) = V(G)$, and there is an arc (directed edge) joining vertex u to v if v is an eccentric vertex from u. An arc of a digraph D joining vertex u to v and vertex v to u is called a *symmetric arc*.

Boland and Miller [1] proposed an open problem to find the eccentric digraph of various classes of graphs. Some authors have investigated the problem of finding the eccentric digraph. For example, Gimbert et al. [4] found the characterisation of the eccentric digraphs while Wang and Sun [7] found the eccentric digraphs of the digraphs. Kusmayadi and Sudibyo [5] also found the eccentric digraph of friendship graph and firecracker graph. In this paper, we tackle the open problem proposed by Boland and Miller [1]. In particular, we determine the eccentric digraph of an (n, t)-kite graph.

II. The Eccentric Digraph of (n, t)-kite Graph

According to Wallis [6], an (n, t)-kite graph, or $k_{n,t}$ for short, consists of a cycle of length *n* with a *t*-edge path (the tail) attached to one vertex. We assume that the (n, t)-kite graph has vertex set $V(k_{n,t}) = \{x_1, x_2, ..., x_t, y_0, y_1, ..., y_{n-1}\}$. The (n, t)-kite graph can be described as in Figure 1.

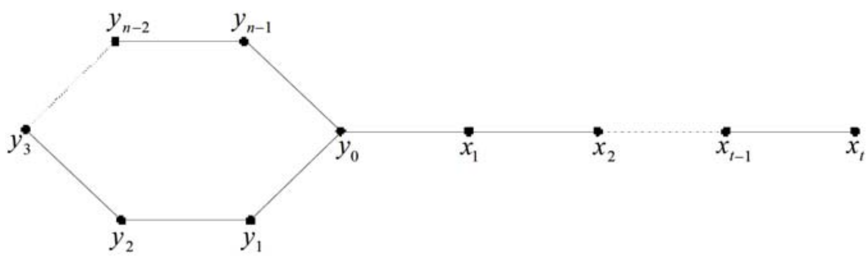

Figure 1. The (n, t)-kite graph $k_{n,t}$.

The following results are the eccentric digraphs of (n, t)-kite graph. We divide into some cases according to the different values of *n* and *t*.

Theorem 1. *Let* $k_{n,t}$ *be* (n, t)*-kite graph, for* $t = \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 3$, $t \geq 1$. *Then the eccentric digraphs* $ED(k_{n,t})$ *are* 5*-partite digraph* $F_{1,1,2,t-1,n-3}$, *for n odd, and* 5*-partite digraph* $F_{1,1,1,t-1,n-2}$, *for n even.*

Proof. We consider two cases according to the values of *n*.

Case 1. *n* odd.

By determining the eccentricity and eccentric vertex for each vertex of the (n, t)-kite graph, it is easy to check that the arcs are from vertex y_0 to the vertex x_t and y_i for $i = \frac{n-1}{2}, \frac{n+1}{2}$. Also, the arcs are from vertex y_i to the vertex x_t for $i \in [1, n]$. In addition, the arcs are from vertex x_j to the vertex y_i for $j \in [1, t]$ and $i = \frac{n-1}{2}, \frac{n+1}{2}$. Now, the arcs are from vertex

x_t to the vertex y_i for $i = \frac{n-1}{2}, \frac{n+1}{2}$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into five subsets of vertices $V_1 = \{x_t\}$, $V_2 = \left\{y_{\frac{n-1}{2}}, y_{\frac{n+1}{2}}\right\}$, $V_3 = \{y_0\}$, $V_4 = \{x_1, x_2, ..., x_{t-1}\}$ and

$$V_5 = \left\{y_1, y_2, ..., y_{\frac{n-3}{2}}, y_{\frac{n+3}{2}}, y_{\frac{n+5}{2}}, ..., y_{n-1}\right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_2, while all arcs from vertices of V_5 are incident to the vertices of V_1, and all arcs from vertices of V_3 are incident to the vertices of V_1 and V_2. The arcs from V_1 and V_2 are symmetric arcs. Hence the eccentric digraph of $k_{n,t}$ is a 5-partite digraph $F_{1,1,2,t-1,n-3}$.

Case 2. *n* even.

By determining the eccentricity and vertex eccentric for each vertex of the (n, t)-kite graph, it is easy to check that the arcs are from vertex y_0 to the vertex x_t and to the vertex $y_{\frac{n}{2}}$. Also, the arcs are from vertex y_i to the vertex x_t for $i \in [0, n]$. In addition, the arcs are from vertex x_j to the vertex y_i for $j \in [1, t]$ and $i = \frac{n-1}{2}, \frac{n+1}{2}$. Now, the arcs are from vertex x_t to the vertex $y_{\frac{n}{2}}$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into five subsets of vertices $V_1 = \{x_t\}$, $V_2 = \left\{y_{\frac{n}{2}}\right\}$, $V_3 = \{y_0\}$, $V_4 = \{x_1, x_2, ..., x_{t-1}\}$ and

$$V_5 = \left\{y_1, y_2, ..., y_{\frac{n-2}{2}}, y_{\frac{n+2}{2}}, y_{\frac{n+4}{2}}, ..., y_{n-1}\right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_2, while all arcs from vertices of V_5 are incident to the vertices of V_1, and all arcs from vertices of V_3 are incident to the vertices of V_1 and V_2. The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 5-partite digraph $F_{1,1,1,t-1,n-2}$. □

For simplicity, let $z = \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor + t}{2} \right\rfloor$ for $t > \frac{n}{2}$.

Theorem 2. *Let* $k_{n,t}$ *be* (n, t)*-kite graph, for* $t > \frac{n}{2}$, $n \geq 3$. *Then the eccentric digraph* $ED(k_{n,t})$ *is*

a. 5-*partite digraph* $F_{1,1,1,z-1,n+t-z-2}$ *for n even and* $\frac{n}{2} + t$ *even,*

b. 4-*partite digraph* $F_{1,1,z,n+t-z-2}$ *for n even and* $\frac{n}{2} + t$ *odd,*

c. 5-*partite digraph* $F_{1,2,1,z-1,n+t-z-3}$ *for n odd and* $\left\lfloor \frac{n}{2} \right\rfloor + t$ *even,*

d. 4-*partite digraph* $F_{1,2,z,n+t-z-3}$ *for n odd and* $\left\lfloor \frac{n}{2} \right\rfloor + t$ *odd.*

Proof. We consider four cases according to the values of *n* and *t*.

Case 1. *n* even and $\frac{n}{2} + t$ even.

By determining the eccentricity and eccentric vertex for each vertex of the (n, t)-kite graph, we observe that the arcs are from vertex y_i to the vertex x_t for $i \in [0, n-1]$. Also, the arcs are from vertex x_i to the vertex $y_{\frac{n}{2}}$ for $i \in [t-z, t]$. In addition, the arcs are from vertex x_i to the vertex

x_t for $i \in [i, t-z]$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into five subsets of vertices $V_1 = \{x_t\}$, $V_2 = \left\{y_{\frac{n}{2}}\right\}$, $V_3 = \{x_{t-z}\}$, $V_4 = \{x_{t+1-z}, ..., x_{t-1}\}$ and

$$V_5 = \left\{y_0, ..., y_{\frac{n-2}{2}}, y_{\frac{n+2}{2}}, ..., y_{n-1}, x_1, ..., x_{t-z-1}\right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_2, while all arcs from vertices of V_5 are incident to the vertices of V_1, and all arcs from vertices of V_3 are incident to the vertices of V_1 and V_2. The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 5-partite digraph $F_{1,1,1,z-1,n+t-z-2}$.

Case 2. n even and $\frac{n}{2} + t$ odd.

By determining the eccentricity and eccentric vertex for each vertex of the (n, t)-kite graph, we observe that the arcs are from vertex y_i to the vertex x_t for $i \in [0, n-1]$. Also, the arcs are from vertex x_i to the vertex $y_{\frac{n}{2}}$ for $i \in [t-z, t]$. In addition, the arcs are from vertex x_i to the vertex x_t for $i \in [i, t-z-1]$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into four subsets of vertices $V_1 = \{x_t\}$, $V_2 = \left\{y_{\frac{n}{2}}\right\}$, $V_3 = \{x_{t-z}, ..., x_{t-1}\}$ and

$$V_4 = \left\{y_0, ..., y_{\frac{n-2}{2}}, y_{\frac{n+2}{2}}, ..., y_{n-1}, x_1, ..., x_{t-z-1}\right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_1, while all arcs from vertices of V_3 are incident to the vertices of V_2. The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 4-partite digraph $F_{1, 1, z, n+t-z-2}$.

Case 3. n odd and $\left\lfloor \frac{n}{2} \right\rfloor + t$ even.

By determining the eccentricity and eccentric vertex for each vertex of the (n, t)-kite graph, it is easy to check that the arcs are from vertex y_i to the vertex x_t for $i \in [0, n-1]$. Also, the arcs are from vertex x_i to the vertex y_j for every $i \in [t-z, t]$ and $j = \frac{n-1}{2}, \frac{n+1}{2}$. In addition, the arcs are from vertex x_i to the vertex x_t for $i \in [i, t-z]$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into five subsets of vertices $V_1 = \{x_t\}$, $V_2 = \left\{y_{\frac{n-1}{2}}, y_{\frac{n+1}{2}}\right\}$, $V_3 = \{x_{t-z}\}$, $V_4 = \{x_{t+1-z}, ..., x_{t-1}\}$ and

$$V_5 = \left\{y_0, ..., y_{\frac{n-3}{2}}, y_{\frac{n+3}{2}}, ..., y_{n-1}, x_1, ..., x_{t-z-1}\right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_2, while all arcs from vertices of V_5 are incident to the vertices of V_1, and all arcs from vertices of V_3 are incident to the vertices of V_1 and V_2. The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 5-partite digraph $F_{1, 2, 1, z-1, n+t-z-3}$.

Case 4. n odd and $\left\lfloor \frac{n}{2} \right\rfloor + t$ odd.

Proof. By determining the eccentricity and eccentric vertex for each vertex of the (n, t)-kite graph, we observe that the arcs are from vertex y_i to the vertex x_t for $i \in [0, n-1]$. Also, the arcs are from vertex x_i to the vertex y_j for $i \in [t-z, t]$ and $j = \frac{n-1}{2}, \frac{n+1}{2}$. In addition, the arcs are from vertex x_i to the vertex x_t for $i \in [i, t-z-1]$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into four subsets of vertices $V_1 = \{x_t\}$, $V_2 = \left\{y_{\frac{n-1}{2}}, y_{\frac{n+1}{2}}\right\}$, $V_3 = \{x_{t-z}, ..., x_{t-1}\}$ and $V_4 = \left\{y_0, ..., y_{\frac{n-3}{2}}, y_{\frac{n+3}{2}}, ..., y_{n-1}, x_1, ..., x_{t-z-1}\right\}$. All arcs from vertices of V_4 are incident to the vertices of V_1, while all arcs from vertices of V_3 are incident to the vertices of V_2. The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 4-partite digraph $F_{1,2,z,n+t-z-3}$. □

Theorem 3. *Let* $k_{n,t}$ *be* (n, t)*-kite graph, for* $t < \frac{n}{2}$, $n \geq 3$. *Then the eccentric digraph of* $k_{n,t}$, $ED(k_{n,t})$ *is a digraph having vertex set* $V(k_{n,t})$ *and the arc set are*

$$\left\{\overrightarrow{x_i y_j},\ i \in [1, t],\ j = \frac{n-1}{2}, \frac{n+1}{2}\right\}$$

$$\cup \left\{\overrightarrow{y_l y_m},\ l \in \left[0, \frac{n-2t-1}{2}\right],\ m = \frac{n+2l-1}{2}, \frac{n+2l+1}{2}\right\}$$

$$\cup \left\{\overrightarrow{y_p x_t},\ p \in \left[\frac{n-2t-1}{2}, \frac{n+2t+1}{2}\right]\right\}$$

$$\cup \left\{ \overrightarrow{y_q y_r},\ q \in \left[\frac{n + 2t + 1}{2},\ n - 1 \right], \right.$$

$$\left. r = \left(\frac{n + 2p - 1}{2} \right) \bmod n,\ \left(\frac{n + 2p + 1}{2} \right) \bmod n \right\} \ \textit{for } n \textit{ odd} \tag{1}$$

and

$$\left\{ \overrightarrow{x_i y_{\frac{n}{2}}},\ i \in [1,\ t] \right\}$$

$$\cup \left\{ \overrightarrow{y_j y_l},\ j \in \left[0,\ \frac{n - 2t}{2} \right],\ l = \frac{n + 2j}{2} \right\} \cup \left\{ \overrightarrow{y_m x_t},\ m = \frac{n - 2t}{2},\ \frac{n + 2t}{2} \right\}$$

$$\cup \left\{ \overrightarrow{y_p y_q},\ p \in \left[\frac{n + 2t}{2},\ n - 1 \right],\ q = \left(\frac{n + 2p}{2} \right) \bmod n \right\} \ \textit{for } n \textit{ even.} \tag{2}$$

Proof. By determining the eccentricity and eccentric vertex for each vertex of the (n, t)-kite graph, it is easy to check that the arcs are as stated in (1) for n odd and (2) for n even. □

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