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ARC-SMOOTH CONTINUUM X ADMITS A WHITNEY MAP FOR $C(X)$ IFF IT IS METRIZABLE

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Abstract

Let X be a non-metric continuum, and $C(X)$ be the hyperspace of subcontinua of X. It is known that there is no Whitney map on the hyperspace 2^X for non-metrizable Hausdorff compact spaces X. On the other hand, there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$. In particular, locally connected or rim-metrizable continuum admits a Whitney map if and only if it is metrizable. In this paper we will show that an arc-smooth continuum X admits a Whitney map for $C(X)$ if and only if it is metrizable.

1. Introduction

Introduction contains some basic definitions, results and notations. An external characterization of non-metric continua which admit a Whitney map is given in Section 2 (Theorem 2.1). In Section 3 we shall prove the main results of this paper, Theorems 3.4 and 3.6.

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$. The cardinality

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of a set A is denoted by card(A). We shall use the notion of inverse system as in [2, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

For a compact space X we denote by 2^X the hyperspace of all nonempty closed subsets of X equipped with the Vietoris topology. $C(X)$ and $X(n)$, where n is a positive integer, stand for the sets of all connected members of 2^X and of all nonempty subsets consisting of at most n points, respectively, both considered as subspaces of 2^X, see [4].

For a mapping $f : X \to Y$ define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [9, 5.10] 2^f is continuous, $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f \mid C(X)$ is denoted by $C(f)$.

If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system, then an element $\{x_a\}$ of the Cartesian product $\prod\{X_\alpha : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the *limit* of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [2, p. 135].

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim \mathbf{X} \to X_a$, for $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n), 2^{p_{ab}} \mid X_b(n), A\}$ form inverse systems.

Lemma 1.1 [4, Lemma 2]**.** *Let* $X = \lim \mathbf{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\mathbf{X})$ *and* $X(n) = \lim \mathbf{X}(n)$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is σ-*directed* if for each sequence $a_1, a_2, \ldots, a_k, \ldots$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

In the next we will use the following expanding theorem of non-metric compact spaces into σ-directed inverse system of compact metric spaces.

Theorem 1.2 [7, Theorem 1.8]**.** *Let X be a compact Hausdorff space such that $w(X) \geq \aleph_1$. There exists a σ-directed inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *of metric compact X_a such that X is homeomorphic to* $\lim \mathbf{X}$.

2. Whitney Map and Hereditarily Irreducible Mappings

The notion of an irreducible mapping was introduced by Whyburn [11, p. 162]. If X is a continuum, a surjection $f : X \to Y$ is *irreducible* provided no proper subcontinuum of X maps onto all of Y under f.

A mapping $f : X \to Y$ is said to be *hereditarily irreducible* [10, p. 204, (1.212.3)] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

A mapping $f : X \to Y$ is *light* (*zero-dimensional*) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [2, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger that one $(\dim f^{-1}(y) \leq 0)$. Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

Every hereditarily irreducible mapping is light. If $f : X \to Y$ is monotone and hereditarily irreducible, then f is one-to-one.

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [10, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

(a) if $\{A\}, \{B\} \in \Lambda$ such that $A \subset B, A \neq B$, then $g(\{A\}) < g(\{B\})$ and

(b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([10, pp. 24-26], [3, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [1]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [1]. Moreover, if X is a non-metrizable locally connected or a rim-metrizable continuum, then X admits no Whitney map for $C(X)$ [6, Theorems 8 and 11].

The following external characterization of non-metric continua which admit a Whitney map was proved in [7, Theorem 2.3].

Theorem 2.1. *Let X be a non-metric continuum. Then X admits a Whitney map for $C(X)$ if and only if for each σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of continua which admit Whitney maps for $C(X_a)$ and $X = \lim \mathbf{X}$ there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible.*

Now we shall prove the metrizability of $C(X)\setminus X(1)$ if X is an arcwise continuum which admits a Whitney map for $C(X)$.

Theorem 2.2. *If an arcwise connected continuum X admits a Whitney map for $C(X)$, then $w(C(X)\setminus X(1)) \leq \aleph_0$.*

Proof. If X is metrizable, then X admits a Whitney map for $C(X)$ and $C(X)$ is metrizable. Conversely, let X admits a Whitney map for $C(X)$. From Theorem 1.2 it follows that there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua and surjective bonding mapping such that X is homeomorphic to $\lim \mathbf{X}$. Consider inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit is $C(X)$ (Lemma 1.1). From Theorem 2.1 it follows that the projections p_a are hereditarily irreducible and $C(p_a)$ are light. If $C(p_a)$ are one-to-one, then we have a homeomorphism $C(p_a)$ of $C(X)$ onto $C(p_a)(X)$. Since $C(p_a)(X)$ is a subspace of a metric space $C(X_a)$, $C(X)$ is metrizable. It follows that X is metrizable since X is homeomorphic to $X(1)$. Suppose that $C(p_a)$ is not one-to-one. Then there exists a continuum C_a in X_a and two continua C, D in X such that $p_a(C) = p_a(D) = C_a$. It is impossible that $C \subset D$ or $D \subset C$ since p_a is hereditarily irreducible. Otherwise, if $C \cap D \neq \emptyset$, then for a continuum $Y = C \cup D$ we have that C and D are subcontinua of Y and $p_a(Y) = p_a(C) = p_a(D) = C_a$ which is impossible since p_a is hereditarily irreducible. We infer that $C \cap D = \emptyset$. There exists an arc E with endpoints in C and D, respectively. Moreover, we may assume that $E \cap C \neq C$ and $E \cap D \neq D$. Now $p_a(E \cup D) = p_a(E)$ which is impossible since p_a is hereditarily irreducible. Furthermore, $C(p_a)^{-1}(X_a(1)) = X(1)$ since from the hereditarily irreducibility of p_a it follows that no non-degenerate subcontinuum of X maps under p_a onto a point. Let $Y_a = C(p_a)(C(X))$. We infer that $C(p_a)^{-1}[Y_a \setminus X_a(1)] = C(X) \setminus X(1)$. It follows that the restriction $P_a = C(p_a) | (C(X) \setminus X(1))$ is one-to-one and closed [2, Proposition 2.1.4]. Hence, P_a is a homeomorphism [2, Proposition 1.4.18, p. 54] and $C(X) \setminus X(1)$ is metrizable. Moreover, $w(C(X) \setminus X(1)) \leq \aleph_0$ since Y_a as a compact metrizable space is separable and, consequently, second-countable [2, p. 320].

In the sequel we shall use the following result [12, p. 173, Problem 23C].

Theorem 2.3. *The following are all equivalent, for locally compact metric space X:*

(a) *X is separable.*

(b) $X = \bigcup_{n=1}^{\infty} K_n$, *where* K_n *is compact and* $K_n \subset Int\, K_{n+1}$.

(c) *The one point compactification* X^* [12, p. 136] *of X is metrizable.*

3. Arc-Smooth Continuum X Admits a Whitney Map for $C(X)$ iff it is Metrizable

An *arc-structure on a continuum X* [5] is a function $A : X \times X \to C(X)$

such that for $x \neq y$ in X, the set $A(x, y)$ is a generalized arc from x to y and such that the following metric-like conditions are satisfied for all x, y and z in X;

(a) $A(x, x) = \{x\}$,

(b) $A(x, y) = A(y, x)$ and

(c) $A(x, z) \subseteq A(x, y) \cup A(y, z)$ with equality prevailing whenever y belongs to $A(x, z)$.

Lemma 3.1. *If X is a continuum with an arc-structure A, then the function A is two-to-one.*

Proof. Let y be a point in $A(X \times X)$. Then y is an arc L in X with endpoints z and u. It is clear that $A^{-1}(x) = \{(z, u), (u, z)\}$. Hence, A is two-to-one.

The pair (X, A) is *arc-smooth at point p in X* if the induced function $A_p : X \to C(X)$ defined by $A_p(x) = A(p, x)$ is continuous. The pair (X, A) is *arc-smooth* if there exists a point in X at which (X, A) arc-smooth.

Lemma 3.2. *For each point $p \in X$, the function $A_p : X \to C(X)$ is one-to-one.*

We say that a continuum X is *uniquely arcwise connected* provided for every pair of x, y, $x \neq y$, there exists a unique arc in X with endpoints x and y, respectively. This means that if X is uniquely arcwise connected, then there exists an arc-structure on X.

Lemma 3.3. *Let X be a continuum with an arc-structure A. If (X, A) is arc-smooth at point $p \in X$, then $A_p : X \to C(X)$ is a homeomorphism onto $A_p(X) = \{A_p(x) : x \in X\} \subset C(X)$.*

Proof. Now we have a continuous and one-to-one mapping $A_p : X \to C(X)$.

Now we are ready to prove the main results of this paper.

Theorem 3.4. *If X is an arc-smooth continuum, then X admits a Whitney map for $C(X)$ if and only if X is metrizable.*

Proof. It is known that if X is metrizable, then X admits a Whitney map for $C(X)$. Suppose that X is non-metrizable and that there exists a Whitney map for $C(X)$. Let X be arc-smooth at point p. By Lemma 3.3 X is homeomorphic to $A_p(X) = \{A_p(x) : x \in X\} \subset C(X)$. It is clear that $A_p(X) \setminus \{p\} \subset C(X) \setminus X(1)$. We infer that $A_p(X) \setminus \{p\}$ is metrizable since $C(X) \setminus X(1)$ is metrizable (Theorem 2.2). Hence, $X \setminus p$ is metrizable since it is homeomorphic to $A_p(X) \setminus \{p\}$. Moreover, $X \setminus p$ is separable since $w(C(X) \setminus X(1)) \leq \aleph_0$. Furthermore, X is the one point compactification of $X \setminus p$. Finally, from Theorem 2.3 it follows that X is metrizable, a contradiction.

Theorem 3.5. *Let a continuum X be a countable union of its arc-smooth subcontinua X_i, $i \in \mathbb{N}$. If X admits a Whitney map for $C(X)$, then X is metrizable.*

Proof. Let $\mu : C(X) \to \mathbb{R}$ be a Whitney map for $C(X)$. It is clear that each restriction $\mu | X_i$, $i \in \mathbb{N}$, is a Whitney map for $C(X_i)$. By Theorem 3.4 each X_i is metrizable. Finally, X is metrizable [2, Corollary 3.1.20, p. 171].

An *arboroid* is a hereditarily unicoherent continuum which is arcwise connected by generalized arcs. A metrizable arboroid is a *dendroid*. If X is an arboroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in X with endpoints x and y. If $[x, y]$ is an arc, then $[x, y] \setminus \{x, y\}$ is denoted by (x, y).

A point t of an arboroid X is said to be a *ramification point* of X if t is the only common point of some three arcs such that it is the only common point of any two, and an endpoint of each of them.

A point e of an arboroid X is said to be *endpoint* of X if there exists no arc $[a, b]$ in X such that $x \in [a, b] \setminus \{a, b\}$.

If an arboroid X has only one ramification point t, it is called a *generalized fan* with the top t. A metrizable generalized fan is called a *fan*.

It is clear that if X is an arboroid, then there exists a unique arc-structure A on X since it is uniquely arcwise connected. Moreover, $A_p(X)$ is the set of all arcs in X of the form $[p, x]$.

An arboroid X is *smooth* if there exists a point $p \in X$ such that given any convergent net x_a with $\lim x_a = x$ it follows that $\lim[p, x_a] = [p, x]$.

In [8, p. 112] the set $A_p(X)$ is denoted by $\mathcal{D}(X, p)$ and it is proved that if X is smooth at p, then $\mathcal{D}(X, p)$ is arcwise connected [8, Theorem 4.8]. Moreover, if X is an arcwise connected smooth continuum, then there exists a homeomorphism $h : X \to \mathcal{D}(X, p)$ [8, Theorem 1]. This means that an arboroid is smooth if and only if it is arc-smooth. From Theorem 3.4 it follows the following theorem.

Theorem 3.6. *If a smooth arboroid X admits a Whitney map for $C(X)$, then X is metrizable.*

Corollary 3.7. *A continuum X which is the countable union of smooth arboroid admits a Whitney map for $C(X)$ if and only if it is metrizable.*

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