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CANONICAL EXTENSIONS AND DIRECT SUMS OF FUZZY PARTIALLY ORDERED SETS

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Abstract

Fuzzy relations are used to create partially ordered sets and lattices contained in this paper. Extensions of partially ordered sets and lattices are examined. The extensions studied include what are called canonical extensions and direct sums. Among the results is one showing that every cover function for a finitary distributive lattice is also a cover function for some nonfinitary distributive lattice. This result shows the necessity of the condition "finitary" that is used in an earlier result to prove the isomorphism of any two finitary distributive lattices which share a common cover function.

1. Introduction

A crisp set X means a collection of objects as usually defined. The

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cardinality of any crisp set X will be denoted, as usual, by $|X|$. The set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ consists of the nonnegative integers where $\mathbb{N}$ denotes the set of natural numbers.

Another kind of set appearing in this paper is the fuzzy set. For this we need a nonempty crisp set X. Then, a fuzzy subset of the crisp set X is a function from X into the unit interval $[0, 1]$. A fuzzy relation on the crisp set X is a fuzzy subset of the Cartesian product $X \times X$. The constructions used in this paper to develop nonfinitary distributive lattices use fuzzy relations. Two examples of fuzzy relations are as follows. Let R be the relation of divisibility defined on $\mathbb{N}$, where mRn means "m divides n" denoted by $m|n$. One way of fuzzifying this crisp relation is by defining the fuzzy relation

$$\mu_d(m, n) = \begin{cases} 1, & \text{if } m|n, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

This particular fuzzification is simply another way of restating the given crisp relation since given any m and n in $\mathbb{N}$ either $m|n$ or $m \nmid n$. Consider the expression

$$T(n, m) = \frac{n/m}{(n/m) + \dfrac{(n/m) - 1}{n/m}}. \tag{2}$$

We can create a more general fuzzy relation such that the fuzzy relation has more values than just 0 and 1 by defining

$$\mu_D(m, n) = \begin{cases} T(n, m), & \text{if } m|n, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

In fact, if m_1, m_2, n_1, and n_2 are in $\mathbb{N}$ such that $m_1|n_1$ and $m_2|n_2$, where $1 < n_1/m_1 < n_2/m_2$, then

$$\mu_D(m_1, n_1) < \mu_D(m_2, n_2).$$

This shows that the fuzzy relation given by equation (3) gives the strength of divisibility and has the capability of comparing the quotients of such divisions. For example

$$0.8 = \mu_D(2, 4) < \mu_D(3, 9) = 0.\overline{81}.$$

See Zadeh [13], Mordeson and Nair [9] and Venugopalan [12] for more information on fuzzy sets and relations.

A special kind of fuzzy relation called a *fuzzy partial ordering* will be defined in the next section. Throughout this paper, a fuzzy partially ordered set (poset) will mean a crisp set X together with a fuzzy partial ordering μ defined on X. Such a poset is denoted by (X, μ). Poset will always mean a fuzzy poset.

The concepts of cover function and fuzzy lattice are defined in Section 4. Lattice will always mean a fuzzy lattice. A distributive lattice with a minimum element is finitary if every interval $[a, b]$ is finite. Let f be a function defined on the set $\mathbb{N}_0$. Then f is said to be a *cover function* for a finitary distributive lattice if every element of the lattice with $n \in \mathbb{N}_0$ lower covers has $f(n)$ upper covers [7], [10], [11]. We shall extend the definition of a cover function for a distributive lattice by eliminating the finitary condition.

Creating an extension of a fuzzy poset or a fuzzy lattice is the task of finding a crisp set Y and a fuzzy partial ordering θ such that (Y, θ) is a structure of the same kind as (X, μ), and (X, μ) is embeddable in (Y, θ). The search for such a set Y leads us to consider the fuzzy singletons e_r defined by equation (4). A commonly known fuzzy subset of X is the characteristic function of the crisp set X. Others are the fuzzy singletons of X, where a fuzzy singleton of X is a function of the form e_r $(r \in [0, 1])$ defined by

$$e_r(x) = \begin{cases} r, & \text{if } x = e, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Fuzzy subsets and singletons are also discussed in [3], [4], [5] and [8].

The collection of all fuzzy singletons of the crisp set X is denoted by $\mathcal{FS}(X)$. A fuzzy partial ordering $\tilde{\mu}$ is defined on $\mathcal{FS}(X)$, where $\tilde{\mu}$ depends on the relation μ defined on X and $(\mathcal{FS}(X), \tilde{\mu})$ is a structure of the same kind as (X, μ). Besides, (X, μ) is embeddable in $(\mathcal{FS}(X), \tilde{\mu})$. For a fixed $r \in [0, 1]$, let $X_{(r)}$ denote the set of all fuzzy singletons e_r of X. The sets $X_{(r)}$ are crisp subsets of $\mathcal{FS}(X)$ and for each $r \in (0, 1]$ we have $|X_{(r)}| = |X|$.

Section 2 will examine in detail the structure and extensions of fuzzy posets. The extension $(\mathcal{FS}(X), \tilde{\mu})$ is called the *canonical extension* of (X, μ). If (X, μ) is a finitary distributive lattice, then $(\mathcal{FS}(X), \tilde{\mu})$ is a nonfinitary distributive lattice which is complete whenever (X, μ) is complete. Cover functions for finitary distributive lattices were investigated in [7], [10] and [11]. It will be shown in Section 4 that every cover function for a finitary distributive lattice (X, μ) is also a cover function for the nonfinitary distributive lattice $(\mathcal{FS}(X), \tilde{\mu})$. This will show that the condition 'finitary' cannot be dropped in ([10, Exercise 22(a), p. 157]) regarding the isomorphism of any two finitary distributive lattices which share a given cover function *f*. Another class of extensions of posets and lattices examined are called *direct sums* of posets and lattices. The definition of the direct sum of a family $\{(X_\alpha, \mu_\alpha)\}$ of posets is given in Definition 3.2. Proposition 3.2 shows that any two such direct sums are equal up to isomorphism. While the direct sum may be a lattice when the posets are all lattices, the direct sum of a family of lattices need not always be a lattice. This is shown in Example 4.2. Relationships between cover functions of distributive lattices and those of their direct sums are examined. Finally, a commutative diagram connecting the two types of extensions-canonical extensions and direct sums, is shown in Figure 1. Theorem 4.6 shows that a given poset (X, μ) is amenable to an infinite process of extensions.

Fuzzy subsets and fuzzy singletons are instances of what are generally known as *L*-subsets and *L*-singletons where *L* is a complete Heyting algebra [8]. As a result of the various ways of choosing *L*, a given poset (X, μ) is amenable to many forms of extensions one of which is $(\mathcal{FS}(X), \tilde{\mu})$ when $L = [0, 1]$.

2. Partially Ordered Sets

In this section, we discuss fuzzy relations and fuzzy partially ordered sets and their extensions.

Definition 2.1. A fuzzy ordering on X is defined as a fuzzy relation μ on X, where $x \leq y$ if and only if $\mu(x, y) > 0$ for x and y in X.

Among the fuzzy relations discussed by Zadeh in [13] are similarity relations and fuzzy partial orderings. A fuzzy partial ordering μ on a set X is defined as follows.

Definition 2.2. A fuzzy partial ordering on a crisp set X is a fuzzy relation μ defined on X satisfying the properties:

(1) $\mu(x, x) = 1 \ \forall x \in X$ (reflexivity),

(2) $\mu(x, y) > 0$ and $\mu(y, x) > 0 \Rightarrow x = y \ \forall x, y \in X$ (antisymmetry),

(3) $\mu(x, z) \geq \max_y \{\min\{\mu(x, y), \mu(y, z)\}\}$ for $x, y, z \in X$ (transitivity).

We can show that such a fuzzy partial ordering on a set X or a similarity relation on X is extendable to $\mathcal{FS}(X)$. If ψ is any similarity relation on X, then it is easy to verify that

$$\tilde{\psi}(x_r, y_s) = \begin{cases} \psi(x, y), & \text{if } \psi(x, y) > 0 \text{ and } s = r, \\ 0, & \text{otherwise} \end{cases}$$

is an extension of ψ to $\mathcal{FS}(X)$. This gives an example of an extension of Zadeh's concept of a similarity relation to $\mathcal{FS}(X)$. An extension of a fuzzy partial ordering μ on X to $\mathcal{FS}(X)$ is shown in Proposition 2.2.

Proposition 2.1. *The fuzzy relation* μ_D *given by equation* (3) *is a fuzzy partial ordering on* $\mathbb{N}$.

Proof. It is easy to show that μ_D is reflexive and antisymmetric. To show that μ_D is transitive we observe first that if $m = n$ or $n = q$, then

$$\mu_D(m, q) \geq \min\{\mu_D(m, n), \mu_D(n, q)\}$$

is trivially satisfied. Therefore, we shall consider the case where $n = am$ and $q = bn,\ a, b \geq 2.$ Consider the function g on $J = [2, \infty)$ defined by

$$g(t) = \frac{t}{t + \frac{t-1}{t}} = \frac{t^2}{t^2 + t - 1}.$$

(Observe that if $t = n/m,$ then equation (2) becomes $T(n, m) = g(t)$).

We have

$$g'(t) = \frac{t(t-2)}{(t^2 + t - 1)^2} \geq 0 \text{ if } t \geq 2.$$

Thus, g is non-decreasing on J. Hence, if $m, n, q \in \mathbb{N},$ where $n = am$ and $q = bn,\ a, b \in \mathbb{N},$ then we must have

$$g(ab) \geq \max\{g(a), g(b)\}$$

or

$$\mu_D(m, q) \geq \max\{\mu_D(m, n), \mu_D(n, q)\} \geq \min\{\mu_D(m, n), \mu_D(n, q)\}.$$

This shows μ_D is transitive. □

We now note the following definitions.

Definition 2.3. Let (X, μ) be a poset. A subposet of (X, μ) is a poset $(X', \mu'),$ where $X' \subseteq X$ and μ' is the restriction of μ to X'.

Definition 2.4. An isomorphism from the poset (X, μ) and onto the poset (Y, θ) is a bijective function $f : (X, \mu) \to (Y, \theta)$ such that $\forall\, x, x' \in X,\ \mu(x, x') > 0$ if and only if $\theta(f(x), f(x')) > 0.$

Definition 2.5. A poset (Y, θ) is called an *extension* of a poset (X, μ) if there exists a one-to-one function $\sigma : (X, \mu) \to (Y, \theta)$ such that $\forall\, x, x' \in X,$ $\mu(x, x') > 0$ if and only if $\theta(\sigma(x), \sigma(x')) > 0.$ That is, (Y, θ) contains an isomorphic image of (X, μ).

We now state the following proposition whose proof is immediate.

Proposition 2.2. *If* (X, μ) *is a poset, then under the function*

$$\tilde{\mu} : \mathcal{FS}(X) \times \mathcal{FS}(X) \to [0, 1]$$

defined by

$$\tilde{\mu}(x_r, y_s) = \begin{cases} \mu(x, y), & \text{if } \mu(x, y) > 0 \text{ and } s \geq r, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$(\mathcal{FS}(X), \tilde{\mu})$ *is a poset.*

For the remainder of the paper $(\mathcal{FS}(X), \tilde{\mu})$ will be the poset where $\tilde{\mu}$ is defined on $\mathcal{FS}(X)$ by equation (5). In $(\mathcal{FS}(X), \tilde{\mu})$ "$\geq$" is defined by $y_s \geq x_r$ if and only if $y \geq x$ in (X, μ) and $s \geq r$. If $(\mathcal{FS}(X), \tilde{\mu})$ is a poset, then so is $(X_{(r)}, \tilde{\mu}_r)$, where $\tilde{\mu}_r$ is the restriction of $\tilde{\mu}$ to the set $X_{(r)}$.

If we fix $r \in (0, 1]$ and define $\varepsilon : (X, \mu) \to (\mathcal{FS}(X), \tilde{\mu})$ by $\varepsilon(x) = x_r$, then $(\mathcal{FS}(X), \tilde{\mu})$ is an example of an extension of (X, μ). The extension $(\mathcal{FS}(X), \tilde{\mu})$ is called the *canonical extension* of (X, μ), and the embedding ε is called a *canonical embedding* of (X, μ) into $(\mathcal{FS}(X), \tilde{\mu})$. When we have a hierarchy of canonical extensions then $(\mathcal{FS}(X), \tilde{\mu})$ is denoted by $(\mathcal{FS}(X), \tilde{\mu}^{[1]})$. The canonical extension of $(\mathcal{FS}(X), \tilde{\mu}^{[1]})$ is denoted by $(\mathcal{F}^2\mathcal{S}(X), \tilde{\mu}^{[2]})$. In general, the canonical extension of $(\mathcal{FS}^{n-1}(X), \tilde{\mu}^{[n-1]})$ is denoted by $(\mathcal{F}^n\mathcal{S}(X), \tilde{\mu}^{[n]})$.

We now examine analogues for $(\mathcal{FS}(X), \tilde{\mu})$ of certain concepts defined in [7]. Assume (X, μ) is a fuzzy poset. Let $r \in [0, 1]$ and $q \in X$. Then, the element q_r is irreducible if and only if q is irreducible in (X, μ). If Q is a down-set of (X, μ), then the set $Q_r = \{q_r \mid q \in Q\}$ is a down-set of $(X_{(r)}, \tilde{\mu}_r)$ for every $r \in [0, 1]$.

The following proposition gives an example of a down-set of $(\mathcal{FS}(X), \tilde{\mu})$.

Proposition 2.3. *Let A be a fuzzy subset of X and*

$$\xi(A) = \{e_r \mid e \in X \ \text{and} \ 0 \leq r \leq A(e)\}.$$

If A is decreasing, then $\xi(A)$ *is a down-set of* $(\mathcal{FS}(X), \tilde{\mu})$. □

Proof. Let $q_r \in \xi(A)$ and $p_s \leq q_r$. Then $p \leq q$ and $s \leq r$. Hence,

$$s \leq r \leq A(q) \leq A(p)$$

if and only if $p_s \in \xi(A)$. This proves the proposition.

Remark 2.1. If A is a decreasing fuzzy subset of X and Canon (A) is the canonical chain $A = A_0 \subseteq A_1 \subseteq \cdots$ of fuzzy subsets of X generated by A as defined in [6], then one can verify directly that $\xi(A_i) \subseteq \xi(A_{i+1})$ for all nonnegative integers i. This gives an increasing sequence of down-sets of $\mathcal{FS}(X)$ whenever Canon (A) is strictly increasing and A is a decreasing fuzzy subset of X as in Proposition 2.3.

3. Direct Sums of Posets

We now give the definitions of the projection of a fuzzy relation and the direct sum of a family of subposets.

Definition 3.1. Let (X, μ) be a poset and X' be a subset of X. By the projection of μ onto X', denoted by $Proj_{X'}(\mu)$, we mean the fuzzy relation μ' on X defined by

$$\mu'(x, y) = \begin{cases} \mu(x, y), & \text{if } x, y \in X', \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.2. Let (X, μ) be a poset and let $\{X_\alpha\}$ be a family of subsets of X such that (X_α, μ_α) is a subposet of (X, μ). Then (X, μ) is called a *direct sum* of the family $\{(X_\alpha, \mu_\alpha)\}$ if

(1) $\bigcup_\alpha X_\alpha = X$,

(2) $X_\alpha \cap X_\beta = \varnothing$ if $\alpha \neq \beta$,

(3) μ_α is the projection of μ onto X_α,

(4) If $\mu(x, y) > 0$, then their exists a unique α such that $\mu_\alpha(x, y) > 0$.

Such a direct sum is denoted by $\sum(X_\alpha, \mu_\alpha)$. We note the following proposition whose proof is immediate.

Proposition 3.1. *If* (X, μ) *is a direct sum of a family* $\{(X_\alpha, \mu_\alpha)\}$ *of subposets, then* $(\mathcal{FS}(X), \tilde{\mu})$ *is a direct sum of the family* $\{(\mathcal{FS}(X_\alpha), \tilde{\mu}_\alpha)\}$.

Proof. It is easy to verify that the canonical extension of $\sum(X_\alpha, \mu_\alpha)$ is

$$\sum(\mathcal{FS}(X_\alpha), \tilde{\mu}_\alpha).$$

Hence, the result follows. $\square$

Let (Y, θ) be an extension of (X, μ), where X is a subset of Y. Then, the restriction of the identity function of Y to X is called the '*inclusion*' of X into Y and denoted by $i : X \to Y$.

In view of Proposition 3.1, we get a hierarchy of canonical extensions of the direct sum of a given family $\{(X_\alpha, \mu_\alpha)\}$ of posets. We shall denote the canonical extensions by $\pi(n), n \geq 0$, where $\pi(0)$ is denoted by $\sum(X_\alpha, \mu_\alpha)$,

$\pi(1) =$ the canonical extension of $\pi(0)$ denoted by $\sum(\mathcal{FS}(X_\alpha), (\tilde{\mu}_\alpha)^{[1]})$

$\pi(2) =$ the canonical extension of $\pi(1)$ denoted by $\sum(\mathcal{F}^2\mathcal{S}(X_\alpha), (\tilde{\mu}_\alpha)^{[2]})$

$$\vdots$$

$\pi(n) =$ the canonical extension of $\pi(n-1)$ denoted by $\sum(\mathcal{F}^n\mathcal{S}(X_\alpha), (\tilde{\mu}_\alpha)^{[n]})$.

Thus, we get the following commutative diagram where (X_α, μ_α) is now simply abbreviated by X_α.

$$\begin{array}{ccccccccc}
X_{\alpha_0} & \xrightarrow{\epsilon} & \mathcal{F}\mathcal{S}\left(X_{\alpha_0}\right) & \xrightarrow{\epsilon} & \mathcal{F}^2\mathcal{S}\left(X_{\alpha_0}\right) & \cdots & \xrightarrow{\epsilon} & \mathcal{F}^n\mathcal{S}\left(X_{\alpha_0}\right) \\
i\downarrow & & i\downarrow & & i\downarrow & & & i\downarrow \\
\pi\left(0\right) & \xrightarrow{\epsilon} & \pi\left(1\right) & \xrightarrow{\epsilon} & \pi\left(2\right) & \cdots & \xrightarrow{\epsilon} & \pi\left(n\right)
\end{array}$$

Figure 1. Hierarchy of canonical extensions.

Figure 1 shows relationships between the canonical extensions denoted by ε and the inclusions denoted by *i*.

We now give some examples of direct sums.

Example 3.1. Let X_0 and X_1 denote, respectively, the sets of even and odd natural numbers, $V = \mathbb{N}_0 \times \mathbb{N}_0$, $V_0 = X_0 \times X_0$, $V_1 = X_1 \times X_1$, and $V_2 = V - (V_0 \cup V_1)$. Define $\theta : V \times V \to [0, 1]$ by

$$\theta[(a, b), (c, d)] = \begin{cases} 1, & \text{if } (a, b) = (c, d), \\ \dfrac{1}{2^i}, & \text{if } (a, b) \neq (c, d), (a, b), (c, d) \in V_i \\ & \quad \text{(for the same } i\text{) and } a \mid c,\ b \mid d, \\ 0, & \text{otherwise.} \end{cases}$$

Then, (V, θ) is a poset. Let θ_i be the projection of θ onto V_i. Then (V, θ) is a direct sum of the subposets (V_i, θ_i).

Example 3.2. We now generalize Example 3.1. Let (X, μ) be a poset, where X is a nondegenerate set (i.e., a set consisting of more than one element) and $V = X \times X$. Let $X_0, X_1, \ldots, X_{n-1}$ be a partition of X, $V_i = X_i \times X_i$ for $i = 0, 1, \ldots, n-1$ and $V_n = V - \cup_{i=1}^{n-1} V_i$.

Define $\theta : V \times V \to [0, 1]$ by

$$\theta[(a, b), (c, d)] = \begin{cases} 1, & \text{if } (a, b) = (c, d), \\ \dfrac{1}{2^i}, & \text{if } (a, b) \neq (c, d) \text{ but } (a, b), (c, d) \in V_i \\ & \quad \text{(for the same } i\text{) and } \mu(a, c) > 0,\ \mu(b, d) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, (V, θ) is a direct sum of $\{(V_i, \theta_i)\}$, where $\theta_i : V_i \times V_i \to [0, 1]$ is the projection of θ onto V_i.

Example 3.3. Let $\{(X_\alpha, \mu_\alpha)\}$ be a family of posets satisfying $X_\alpha \cap X_\beta = \varnothing$ if $\alpha \neq \beta$. By letting $X = \bigcup_\alpha X_\alpha$ and the fuzzy partial ordering μ be defined by

$$\mu(a, b) = \begin{cases} \mu_\alpha(a, b), & \text{if } a, b \in X_\alpha \text{ for the same } \alpha, \\ 0, & \text{otherwise} \end{cases}$$

we create a direct sum of the family of fuzzy posets $\{(X_\alpha, \mu_\alpha)\}$.

The following proposition shows that the direct sum of a family of subposets is unique.

Proposition 3.2. *Any two direct sums (X, μ) and (Q, θ) of a family $\{(X_\alpha, \mu_\alpha)\}$ of posets are isomorphic. Thus, the direct sum of a family of posets is unique up to isomorphism.*

Proof. Since (X, μ) and (Q, θ) are direct sums of $\{(X_\alpha, \mu_\alpha)\}$, we have

$$X = \bigcup_\alpha X_\alpha = Q.$$

Now, let $\mu(x, y) > 0$ for some $x, y \in X$. Then, there exists a unique α such that $\mu_\alpha(x, y) > 0$. Since

$$Proj_{X_\alpha}(\mu) = \mu_\alpha = Proj_{X_\alpha}(\theta),$$

we have by Definition 3.2(3) of a direct sum that $\theta(x, y) > 0$. Hence, (X, μ) is isomorphic to (Q, θ). This proves the assertion. □

Proposition 3.3. *If the posets (X, μ) and (Q, θ) are isomorphic and (X, μ) is the direct sum of the family $\{(X_\alpha, \mu_\alpha)\}$ of posets, then (Q, θ) is also a direct sum of some family of posets.*

Proof. Let $f : (X, \mu) \to (Q, \theta)$ be an isomorphism, where (X, μ) is a direct sum of the family $\{(X_\alpha, \mu_\alpha)\}$ of posets. Since f is an isomorphism, we have

$$Q = f(X) = f(\cup_\alpha X_\alpha) = \cup_\alpha f(X_\alpha).$$

Besides

$$f(X_\alpha) \cap f(X_\beta) = f(X_\alpha \cap X_\beta) = f(\varnothing) = \varnothing, \text{ if } \alpha \neq \beta.$$

Let θ_α be the projection of θ to $f(X_\alpha)$. Suppose $a, b \in Q$ satisfying $\theta(a, b) > 0$. Then, there exist $x = f^{-1}(a)$, $y = f^{-1}(b) \in X$ such that $\mu(x, y) > 0$. Hence, there exists a unique α such that $x, y \in X_\alpha$ and $\mu_\alpha(x, y) > 0$. Consequently, $a = f(x), b = f(y) \in f(X_\alpha)$ and $\theta_\alpha(a, b) > 0$. Thus, we have shown that (Q, θ) is a direct sum of $\{(f(X_\alpha), \theta_\alpha)\}$. This proves the assertion. $\square$

Proposition 3.4. *Every poset* (X, μ) *is embeddable into a direct sum of some family* $\{(X_\alpha, \mu_\alpha)\}$ *of posets*.

Proof. Let (X, μ) be a given poset. First consider the trivial case. If X consists of a single element, then the direct sum has only one summand, namely (X, μ) itself. Now consider the case when $|X| \geq 2$. Fix an element $a \in X$ and let $X_0 = \{a\}$, $X_1 = X - \{a\}$, $V = X \times X$, $V_0 = X_0 \times X_0$, $V_1 = X_1 \times X_1$, and $V_2 = V - (V_0 \cup V_1)$. The fuzzy relation θ defined on $V \times V$ as given by Example 3.2, that is,

$$\theta[(a, b), (c, d)] = \begin{cases} 1, & \text{if } (a, b) = (c, d), \\ \dfrac{1}{2^i}, & \text{if } (a, b) \neq (c, d) \text{ but } (a, b), (c, d) \in V_i \\ & \quad (\text{for the same } i) \text{ and } \mu(a, c) > 0, \mu(b, d) > 0, \\ 0, & \text{otherwise} \end{cases}$$

makes (V, θ) a poset. Besides (V, θ) is a direct sum of the family $\{(V_i, \theta_i)\}$, where for $i = 0, 1, 2$, θ_i is the restriction of θ to $V_i \times V_i$. Now the mapping

$$f : (X, \mu) \to (V, \theta)$$

defined by $f(b) = (a, b)$ injects (X, μ) into (V, θ), where the image of f is $V_0 \cup V_2$. This proves the assertion.

4. Lattices and Cover Functions

We start here with the definitions of lattice, upper and lower covers and cover function for a distributive lattice.

Definition 4.1. Let (X, μ) be a poset and $a, b \in X$. If there exists an element $c \in X$ such that (1) $\mu(a, c) > 0$, (2) $\mu(b, c) > 0$, and (3) whenever $d \in X$ satisfies $\mu(a, d) > 0$ and $\mu(b, d) > 0$, then $\mu(c, d) > 0$, we call c the *lub* of $\{a, b\}$ and denote it by $a \vee b$. The *glb* of $\{a, b\}$ is defined in a similar manner and it is denoted by $a \wedge b$.

Definition 4.2. A lattice is a poset (X, μ), where every pair of elements a and b in X has a *lub* and *glb*. The lattice is said to be *complete* if every nonempty subset of (X, μ) has a *lub* and *glb*. The lattice (X, μ) is said to be *distributive* if the *lub* and *glb* operations are distributive over one another.

Example 4.1. Consider the fuzzy partial order μ_d given by equation (1). Define $\vee$ and $\wedge$ in $(\mathbb{N}, \mu_d)$ by

$$m \vee n = \operatorname{lcm}(m, n) \text{ and } m \wedge n = \gcd(m, n).$$

Then $(\mathbb{N}, \mu_d)$ becomes a lattice subject to these operations. This lattice is called the *divisor lattice*.

Proposition 4.1. *Let* (X, μ) *be a fuzzy lattice. Define* $\vee$ *and* $\wedge$ *for* $(\mathcal{FS}(X), \tilde{\mu})$ *by*

$$e_r \vee d_s \equiv (e \vee d)_{\max(r, s)},$$

$$e_r \wedge d_s \equiv (e \wedge d)_{\min(r, s)}.$$

Then $(\mathcal{FS}(X), \tilde{\mu})$ *is a fuzzy lattice. If* (X, μ) *is a complete fuzzy lattice, so is* $(\mathcal{FS}(X), \tilde{\mu})$. *Also if* (X, μ) *is a distributive fuzzy lattice, then so is* $(\mathcal{FS}(X), \tilde{\mu})$.

Proof. The *lub* and *glb*, as defined for $(\mathcal{FS}(X), \tilde{\mu})$, satisfy the requirements for *lub* and *glb*. Moreover 0_0 and 1_1, which we shall denote by 0 and 1, respectively, satisfy the requirements for absolute minimum and absolute maximum. Now if $s = \{(e_\alpha)_{r_\alpha}\}$ is any family of elements of $\mathcal{FS}(X)$, then we observe that $lub\, s = (lub\, s_1)_{lub\, s_2}$, where s_1 is the family $\{e_\alpha\} \subseteq X$, and s_2 is the family $\{r_\alpha\} \subseteq [0, 1]$. The *glb* s is obtained in a similar manner. Hence, $(\mathcal{FS}(X), \tilde{\mu})$ is complete.

Now let (X, μ) be a distributive fuzzy lattice and let $b_t, e_r, d_s \in \mathcal{FS}(X)$. Then

$$
\begin{aligned}
e_r \wedge (d_s \vee b_t) &= e_r \wedge [(d \vee b)]_{\max(s, t)} \\
&= e_r \wedge (d \vee b)_{t'} \ (t' = \max(s, t)) \\
&= [e \wedge (d \vee b)]_{\min(r, t')} \\
&= [(e \wedge d) \vee (e \wedge b)]_{\min(r, t')}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
(e_r \wedge d_s) \vee (e_r \wedge b_t) &= (e \wedge d)_{\min(r, s)} \vee (e \wedge b)_{\min(r, t)} \\
&= [(e \wedge d) \vee (e \wedge b)]_{\max\{\min(r, s), \min(r, t)\}}.
\end{aligned}
$$

Now, we must show $\max\{\min(r, s), \min(r, t)\} = \min(r, t')$. We consider the following cases: (i) $\min(r, s) = r = \min(r, t)$, (ii) $\min(r, s) = s$ and $\min(r, t) = t$, (iii) $\min(r, s) = s$ and $\min(r, t) = r$, and (iv) $\min(r, s) = r$ and $\min(r, t) = t$.

Case (i) implies $r \leq t'$. Hence the result.

Case (ii) implies $r \geq t'$. Hence we have

$$\max\{\min(r, s), \min(r, t)\} = \max(s, t) = t' = \min(r, t').$$

Case (iii) implies $t' = t \geq r \geq s$. Hence

$$\max\{\min(r, s), \min(r, t)\} = \max(s, r) = r = \min(r, t').$$

Case (iv) implies $s \geq r \geq t$. Therefore, $t' = s$. The result is obvious.

Now to show $e_r \vee (d_s \wedge b_t) = (e_r \vee d_s) \wedge (e_r \vee b_t)$ we observe that

$$\begin{aligned} e_r \vee (d_s \wedge b_t) &= e_r \vee [(d \wedge b)]_{\min(s, t)} \\ &= [e \vee (d \wedge b)]_{\max(r, s')} \ (s' = \min(s, t)) \\ &= [(e \vee d) \wedge (e \vee b)]_{\max(r, s')}. \end{aligned}$$

On the other hand

$$\begin{aligned} (e_r \vee d_s) \wedge (e_r \vee b_t) &= (e \vee d)_{\max(r, s)} \wedge (e \vee b)_{\max(r, t)} \\ &= [(e \vee d) \wedge (e \vee b)]_{\min\{\max(r, s), \max(r, t)\}}. \end{aligned}$$

By considering the four cases as we did earlier one can show that

$$\min\{\max(r, s), \max(r, t)\} = \max(r, s').$$

This completes the proof. $\square$

$(\mathcal{FS}(X), \tilde{\mu})$ shows that every fuzzy lattice (X, μ) (finite or infinite) has an infinite lattice extension (see Definition 4.6).

Definition 4.3. Let (X, μ) be a fuzzy lattice. Then a fuzzy sublattice of (X, μ) is a fuzzy subposet (X', μ') of (X, μ) which forms a lattice under the restriction μ' of μ to X'.

We now give a sufficient condition for a fuzzy subposet of a fuzzy lattice (X, μ) to be a fuzzy sublattice.

Proposition 4.2. *Let* (X, μ) *be a fuzzy lattice. If every fuzzy subposet* (X', μ') *of* (X, μ) *is a down-set of* (X, μ) *and has a maximum M, then every fuzzy subposet of* (X, μ) *is a fuzzy sublattice of* (X, μ).

Proof. Let $a, b \in (X', \mu')$. Since (X', μ') is a down-set and $a \wedge b \leq a$, $a \wedge b \in (X', \mu')$. Similarly, $a \leq M$ and $b \leq M$. Therefore $a \vee b \in (X', \mu')$. This proves the proposition. □

One wonders whether a direct sum of a family of sublattices is always a lattice. The following example shows that this is not always the case.

Example 4.2. For each prime natural number $p > 2$ let $S_p = \{p, 2p\}$. Let

$$S = \bigcup_p S_p$$

and define λ_S by

$$\lambda_S(m, n) = \begin{cases} 1, & \text{if } m \mid n, \\ 0, & \text{otherwise.} \end{cases}$$

We also let λ_p be the restriction of λ_S to S_p for each prime *p*. Then (S, λ_S) is a fuzzy poset which is a direct sum of the family of fuzzy sublattices $\{(S_p, \lambda_p)\}$. However, (S, λ_S) is not a fuzzy lattice because it is not closed under the *lub* operation $\vee$ since $6, 10 \in S$ but $6 \vee 10 = \text{lcm}(6, 10) = 30 \notin S$.

Definition 4.4. A fuzzy lattice (J, μ) which is the direct sum of a family of fuzzy sublattices $\{(J_\alpha, \mu_\alpha)\}$ is called the *lattice direct* sum of $\{(J_\alpha, \mu_\alpha)\}$.

Remark 4.1. In Example 3.1 each (V_i, θ_i) is a lattice if we define $\vee$ and $\wedge$ as follows:

(1) $(a, b) \vee (c, d) = [\text{lcm}(a, c), \text{lcm}(b, d)]$,

(2) $(a, b) \wedge (c, d) = [\text{gcd}(a, c), \text{gcd}(b, d)]$.

Similar definitions of $\vee$ and $\wedge$ show that (V, θ) is also a lattice. Thus, we have an example of a lattice direct sum.

Definition 4.5. Let (X, μ) and (Y, θ) be fuzzy lattices. An isomorphism from (X, μ) onto (Y, θ) is a poset isomorphism *f* of (X, μ) onto (Y, θ) which satisfies for all $x, x' \in X$,

(1) $f(x \vee x') = f(x) \vee f(x')$,

(2) $f(x \wedge x') = f(x) \wedge f(x')$.

Definition 4.6. An extension of a fuzzy lattice (X, μ) is a fuzzy lattice (E, ω) which contains a fuzzy lattice isomorphic copy of (X, μ).

An example of such an extension of (X, μ) is $(\mathcal{FS}(X), \tilde{\mu})$, as earlier mentioned.

Remark 4.2. Not every extension of (X, μ) is complete. For instance, let

$$G = \{x_r \mid x \in X \text{ and } r \in (0, 1]\}$$

and μ' be the restriction of $\tilde{\mu}$ to *G*. Then (G, μ') is an extension of (X, μ) which is not complete.

Proposition 4.3. *If* (X, μ) *is isomorphic to* (Y, ω), *then* $(\mathcal{FS}(X), \tilde{\mu})$ *is isomorphic to* $(\mathcal{FS}(Y), \tilde{\omega})$.

Proof. Let $\sigma : (X, \mu) \to (Y, \omega)$ be an isomorphism. Then, define

$$\tilde{\sigma} : (\mathcal{FS}(X), \tilde{\mu}) \to (\mathcal{FS}(Y), \tilde{\omega})$$

as $\tilde{\sigma}(x_r) = (\sigma(x))_r$. Now we must show $\tilde{\sigma}$ is injective, surjective and the lattice operations $\vee$ and $\wedge$ are preserved. However, to show $\tilde{\sigma}$ is an isomorphism and the operations are preserved we need the notion of equality of fuzzy singletons. Two fuzzy singletons x_r and y_s are said to be *equal* if and only if $x = y$ and $r = s$. To prove injectivity, suppose $\tilde{\sigma}(x_r) = \tilde{\sigma}(y_s)$.

Then $(\sigma(x))_r = (\sigma(y))_s$ and by equality of fuzzy singletons $\sigma(x) = \sigma(y)$ and $r = s$. Since σ is an isomorphism $x = y$. Therefore, applying equality of fuzzy singletons again gives $x_r = y_s$. To prove surjectivity, let $z_t \in (\mathcal{FS}(Y), \tilde{\omega})$. Then $z \in Y$ implies there exists $z' \in X$ such that $\sigma(z') = z$. Hence $\sigma(z'_t) = (\sigma(z'))_t = z_t$. To show $\tilde{\sigma}$ preserves $\wedge$ and $\vee$ we have to show $\tilde{\sigma}(x_r \vee y_s) = \tilde{\sigma}(x_r) \vee \tilde{\sigma}(y_s)$ and $\tilde{\sigma}(x_r \wedge y_s) = \tilde{\sigma}(x_r) \wedge \tilde{\sigma}(y_s)$. Consider $\tilde{\sigma}(x_r \vee y_s)$, then by the definitions of $\tilde{\sigma}$ and *lub* (see Proposition 4.1) and σ being an isomorphism we obtain

$$\begin{aligned}
\tilde{\sigma}(x_r \vee y_s) &= (\sigma(x \vee y))_{\max(r, s)} \\
&= (\sigma(x) \vee \sigma(y))_{\max(r, s)} \\
&= (\sigma(x))_r \vee (\sigma(y))_s \\
&= \tilde{\sigma}(x_r) \vee \tilde{\sigma}(y_s).
\end{aligned}$$

Similarly, $\tilde{\sigma}(x_r \wedge y_s) = \tilde{\sigma}(x_r) \wedge \tilde{\sigma}(y_s)$ follows. This proves the proposition. □

Example 4.3. $(\mathcal{FS}(B_n), \tilde{\mu})$ is isomorphic to $(\mathcal{FS}((0, 1)^n), \tilde{\omega})$. This follows from the know fact that B_n is isomorphic to $(0, 1)^n$, where B_n denotes the lattice made up of subsets of $[n] = \{1, 2, ..., n\}$, order by inclusion and $(0, 1)^n$ is the lattice of all n-tuples of 0'*s* and 1'*s* with $x \leq y$ meaning $x_i \leq y_i$ for each of the n components of x and y, see Aigner [1]. The fuzzy relations μ and ω are chosen, respectively, to form fuzzy lattices that are associated with these crisp orderings.

Definition 4.7. Let $x, y \in (X, \mu)$. Then, y is called an *upper cover* for x if $\mu(x, y) > 0$ and there is no $z \in (X, \mu)$ such that $\mu(x, z) > 0$ and $\mu(z, y) > 0$. The element x is called a *lower cover* for y if y is an upper cover for x.

Definition 4.8. A function $f : \mathbb{N}_0 \to \mathbb{N}_0$ is called a *cover function* for a distributive lattice (X, μ) if every element of (X, μ) having n lower covers has $f(n)$ upper covers.

Proposition 4.4. *Let (X, μ) be the direct sum of a family $\{(X_\alpha, \mu_\alpha)\}$ of distributive lattices. Then, any cover function for (X, μ) induces a cover function f_α of (X_α, μ_α) for each α.*

Proof. Let (X, μ) be a direct sum of the family $\{(X_\alpha, \mu_\alpha)\}$ of distributive lattices and let f be a cover function for (X, μ). For each $x \in X_\alpha$, if n_α is the number of lower covers of x in (X_α, μ_α) and n is the number of lower covers of x in (X, μ), then $n_\alpha = n$. Let N_α denote the set

$$N_\alpha = \{n_\alpha \in \mathbb{N}_0 \mid n_\alpha = \text{number of lower covers of some } x \in X_\alpha\}.$$

Then, $f_\alpha = f$, where f is restricted to the set N_α is a cover function for (X_α, μ_α). This proves the assertion.

Proposition 4.5. *Let $\{(X_\alpha, \mu_\alpha)\}$ be a family of distributive lattices and (X, μ) be the direct sum of $\{(X_\alpha, \mu_\alpha)\}$. If for all α, (X_α, μ_α) has a cover function f_α, then the family $\{f_\alpha\}$ of cover functions generates a lower-bound for a cover function for (X, μ).*

Proof. Let (X, μ) be a direct sum of the family $\{(X_\alpha, \mu_\alpha)\}$ of distributive lattices where, for all α, (X_α, μ_α) has a cover function f_α. For each integer $n \in \mathbb{N}_0$ let n_α denote $f_\alpha(n)$. Let $\bar{n}$ be the smallest element of the set $\{n_\alpha\}$. Then, the function $f : \mathbb{N}_0 \to \mathbb{N}_0$ given by $f(n) = \bar{n}$ is a lower-bound for a cover function for (X, μ). □

Theorem 4.6. *Let $\{(X_\alpha, \mu_\alpha)\}$ be a family of pairwise disjoint isomorphic lattices. Suppose for some α_0, $(X_{\alpha_0}, \mu_{\alpha_0})$ has a cover function f, then f is a cover function for (X_α, μ_α) for all α. Moreover, for all integers $n \geq 0$, f is a cover function for $(\mathcal{F}^n\mathcal{S}(X_\alpha), (\tilde{\mu}_\alpha)^{[n]})$ and $\pi(n)$. That is, f is a cover* *function for every poset appearing in the commutative diagram* (*Figure* 1) *for each* α.

Proof. Let f be a cover function for $(X_{\alpha_0}, \mu_{\alpha_0})$ and let (X_β, μ_β) be any other member of the family $\{(X_\alpha, \mu_\alpha)\}$. Let σ be an isomorphism from $(X_{\alpha_0}, \mu_{\alpha_0})$ onto (X_β, μ_β). Then, if $a, b \in X_{\alpha_0}$ and a is a lower cover for b, then $\sigma(a)$ is a lower cover for $\sigma(b)$. This shows that f is a cover function for (X_β, μ_β). Since X_β is arbitrary, f is a cover function for (X_α, μ_α) for all α.

Now for any elements $x, y \in (X_\alpha, \mu_\alpha)$ we observe that x is a lower cover for y if and only if x_r is a lower cover for y_r (same r) in $(\mathcal{FS}(X_\alpha), \mu_\alpha)$. Consequently, if f is a cover function for (X_α, μ_α), then f is a cover function for $(\mathcal{FS}(X_\alpha), \tilde{\mu}_\alpha)$. Inductively, f is a cover function for

$$(\mathcal{F}^n\mathcal{S}(X_\alpha), (\tilde{\mu}_\alpha)^{[n]})$$

for all integers $n \geq 0$. Proposition 4.3 now guarantees that

$$(\mathcal{F}^n\mathcal{S}(X_\alpha), (\tilde{\mu}_\alpha)^{[n]})$$

has the cover function f for every α.

In the case of the direct sums $\pi(n)$ we observe that x is a lower cover for y in $\pi(0)$ if and only if x is a lower cover for y in (X_α, μ_α) for a unique α. Hence, the direct sum $\pi(0)$ shares the cover function f with (X_α, μ_α) for every α. The fact that a cover function for (X, μ) is a cover function for the canonical extension $(\mathcal{FS}(X), \tilde{\mu})$ for every poset (X, μ), together with induction, guarantees that f is a cover function for $\pi(n)$ for all integers $n \geq 0$. $\square$

Corollary 4.7. *Let* (X, μ) *be a finitary distributive lattice. Then every cover function for* (X, μ) *is also a cover function for some nonfinitary distributive lattice.*

Proof. If (X, μ) is a finitary distributive lattice, then $(\mathcal{FS}(X), \tilde{\mu})$ is a nonfinitary distributive lattice. For $r \in [0, 1]$ and $q \in X$, a lower cover of q_r is an element p_r, where p is a lower cover of q in (X, μ). When p_r is a lower cover of q_r, q_r is an upper cover of p_r. Thus, every cover function for (X, μ) is a cover function for $(\mathcal{FS}(X), \tilde{\mu})$. This proves the corollary. $\square$

Remark 4.3. In the manner of Chon [2, p. 363], we can define levels of covers and cover functions. For instance, for $p \in [0, 1]$, we can call y a p-level upper cover for x if $\mu(x, y) > p$ and there is no $x' \in X$ satisfying $\mu(x, x') > p$ and $\mu(x', y) > p$. Thus, the cover functions studied by Stanley, Farley and discussed in this paper are the p-level cover functions for $p = 0$. The p-level cover functions for $p > 0$ are yet to be investigated.

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