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APPLICATIONS OF CONVEXITY ON QUANTITIES AND INEQUALITIES

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Abstract

The convex combinations supported by the geometrical images have been applied in determining the position of the center of physical and mathematical quantity. The main results of the paper include the presentation of the center (barycenter) of quantity in the integral form, and the application of the convexity on mathematical inequalities.

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1. Introduction - Geometry and Analytics of Convexity

In our three-dimensional space $\mathcal{S}$, a set $\mathcal{C} \subseteq \mathcal{S}$ is convex if for any two points $A, B \in \mathcal{C}$ the corresponding segment (line segment) $\overline{AB} \subseteq \mathcal{C}$. In the real vector space $\mathcal{X}$ a set $\mathcal{C} \subseteq \mathcal{X}$ is convex if for any two vectors $x, y \in \mathcal{C}$ the vector $\alpha x + \beta y \in \mathcal{C}$ for all real numbers $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$.

If $x_1, \ldots, x_n \in \mathcal{X}$ are vectors, and $\alpha_1, \ldots, \alpha_n \in [0, 1]$ are real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$, then the vector expression

$$\sum_{i=1}^{n} \alpha_i x_i \tag{1.1}$$

is called the *convex combination* of the vectors x_i with the coefficients α_i. Thus, the segment corresponding to the binomial convex combinations.

The convex hull $\mathrm{co}\mathcal{A}$ of the set $\mathcal{A} \subseteq \mathcal{S}$ is the smallest convex set which contains the set $\mathcal{A}$. The convex hull of the vector set $\mathcal{A} \subseteq \mathcal{X}$ consists of all binomial convex combinations of the vectors from $\mathcal{A}$.

Any convex combination with at least two positive coefficients can be expressed in the binomial form, that is,

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha \sum_{i=1}^{k} \frac{\alpha_i}{\alpha} x_i + \beta \sum_{i=k+1}^{n} \frac{\alpha_i}{\beta} x_i, \tag{1.2}$$

where $1 \le k \le n-1$, $\alpha = \sum_{i=1}^{k} \alpha_i > 0$ and $\beta = \sum_{i=k+1}^{k} \alpha_i > 0$. Using the above binomial form and the mathematical induction we find that a convex vector set $\mathcal{C}$ contains all convex combinations of the finite number of its vectors.

Boundary points of a convex set $\mathcal{C} \subset \mathcal{S}$ which most protrude outward are called *extreme points*. The extreme points of a polygon or polyhedron are just its vertices. The extreme points of the convex set in Figure 1 left, the three points and the arc, are bolded. A point $z \in \mathcal{C} \subset \mathcal{X}$ is an extreme if

$z \neq \alpha x + \beta y$ for all distinct points $x, y \in \mathcal{C}$ and all $\alpha, \beta \in \langle 0, 1\rangle$ such that $\alpha + \beta = 1$ (see [4, pp. 110-112]). The importance of extreme points is evident from the following fact: every point of a bounded closed convex set $\mathcal{C} \subset \mathcal{S}$ can be expressed as the convex combination of at most four extreme points from $\mathcal{C}$. Otherwise stated, every bounded closed convex set from $\mathcal{S}$ is the convex hull of its extreme points.

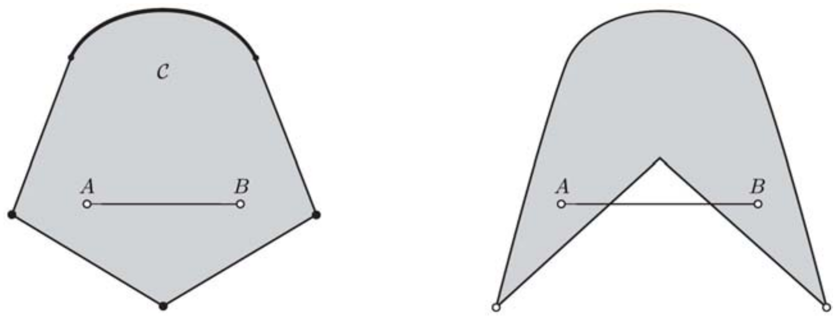

Figure 1. Convex and non-convex sets.

The convexity of a set from $\mathcal{S}$ can also be described by using the points that are outside of the set interior. A planar set $\mathcal{C} \subset \mathcal{S}$ is convex if the line $\mathcal{L}$ through each point outside of $\operatorname{int}\mathcal{C}$ exists so that $\mathcal{L} \cap \operatorname{int}\mathcal{C} = \emptyset$, and the entire $\operatorname{int}\mathcal{C}$ is contained in one of the half-planes determined by $\mathcal{L}$ (see [3]). In the description of convexity of a spatial set we use the plane instead of the line.

2. Geometry of the Space with Convex Combinations

In this section, we give the application overview of convex combinations in the analytical representation of the basic planar figures (triangle and polygon) and spatial bodies (tetrahedron and polyhedron).

Let us start with a segment with end-points A and B. If P is any point from the segment $\overline{AB}$, then $\overrightarrow{BP} = \lambda\overrightarrow{BA}$, where unique number $\lambda \in [0, 1]$. Inserting radius-vectors considering some fixed point O, we have

$$\vec{r}_P = \lambda\vec{r}_A + (1-\lambda)\vec{r}_B. \tag{2.1}$$

Using the notion of convex combinations and applying the vector algebra we get the three following basic propositions:

Segment proposition. *Let* $\overline{AB} \subset \mathcal{S}$ *be a segment with end-points* A *and* B. *Then the vector equality as the unique convex combination*

$$\vec{r}_P = \alpha\vec{r}_A + \beta\vec{r}_B \tag{2.2}$$

holds for every point $P \in \overline{AB}$.

Triangle proposition. *Let* $\mathcal{C}(A,\ B,\ C) \subset \mathcal{S}$ *be a triangle with vertices* A, B *and* C. *Then the vector equality as the unique convex combination*

$$\vec{r}_P = \alpha\vec{r}_A + \beta\vec{r}_B + \gamma\vec{r}_C \tag{2.3}$$

holds for every point $P \in \mathcal{C}(A,\ B,\ C)$.

Tetrahedron proposition. *Let* $\mathcal{C}(A,\ B,\ C,\ D) \subset \mathcal{S}$ *be a tetrahedron with vertices* A, B, C *and* D. *Then the vector equality as the unique convex combination*

$$\vec{r}_P = \alpha\vec{r}_A + \beta\vec{r}_B + \gamma\vec{r}_C + \delta\vec{r}_D \tag{2.4}$$

holds for every point $P \in \mathcal{C}(A,\ B,\ C,\ D)$.

Applying the additional geometrical observations of convex sets in the plane and space we achieve the following double theorem:

Polygon and polyhedron theorem. *Let* $\mathcal{C}(P_1,\ ...,\ P_n) \subset \mathcal{S}$ *be a convex polygon or polyhedron with vertices* $P_1,\ ...,\ P_n$. *Let* $P \in \mathcal{S}$ *be a point. Then the vector equality as the convex combination*

$$\vec{r}_P = \sum_{i=1}^{n} \alpha_i \vec{r}_i \tag{2.5}$$

holds if and only if $P \in \mathcal{C}(P_1,\ ...,\ P_n)$.

Thus, every convex polygon or polyhedron $\mathcal{C}(P_1,\ ...,\ P_n) \subset \mathcal{S}$ can be represented using the convex combinations of the radius-vectors $\vec{r}_i$ of its vertices P_i, by the set

$$\mathcal{C}(P_1, \ldots, P_n) = \left\{ P \in \mathcal{S} \,|\, \vec{r}_P = \sum_{i=1}^{n} \alpha_i \vec{r}_i,\ \alpha_i \in [0, 1],\ \sum_{i=1}^{n} \alpha_i = 1 \right\}. \quad (2.6)$$

Remark 2.1. Convex combinations in (2.5) are not unique. For example, imagine that some point P from a quadrangle $\mathcal{C}(A, B, C, D)$ belongs to the triangles $\mathcal{C}(A, B, C)$ and $\mathcal{C}(A, B, D)$, and P does not belong to the edge $\overline{AB}$. By Triangle proposition, we have two different four-members convex combinations

$$\vec{r}_P = \alpha_1 \vec{r}_A + \beta_1 \vec{r}_B + \gamma_1 \vec{r}_C + 0\vec{r}_D = \alpha_2 \vec{r}_A + \beta_2 \vec{r}_B + 0\vec{r}_C + \delta_2 \vec{r}_D$$

because $\gamma_1 > 0$ and $\delta_2 > 0$. Similarly, we can take a pentahedron and apply tetrahedron proposition on suitable selected point from the observed pentahedron.

3. Physical and Probabilistic Interpretation of Convexity

In the practical sense, the presence of convex combinations is evident in the formulation of important issues. Work with the convex combinations enables simple and general way of expressing. Thus, the physical and probabilistic meaning will be expressed one and the same formula with the convex combinations. In all this we still have the geometrical fact that the convex combinations of finitely many given radius-vectors belong to the simplest convex sets, polygons in the plane, and polyhedrons in the space.

Consider a set of n particles (points) in the plane. The value of a certain physical quantity q (mass, density, potential, resistance) is measured at each of n particles. We want to specify the center P of the quantity q. It can also be assumed that a quantity q is the discrete random variable (temperature, humidity, contamination). In this case, we are looking for the expected position P of the variable q.

Let the particles be located at the points $P_1, \ldots, P_n$ with non-negative quantity values $q_1, \ldots, q_n$ and positive quantity total value

$$q_{\text{tot}} = \sum_{i=1}^{n} q_i.$$

So, we can take the relative quantity values

$$\alpha_i = \frac{q_i}{q_{\text{tot}}}$$

for $i = 1, \ldots, n.$ The quantity arithmetic mean (variable expected value) $\overline{q}$ is the "scalar convex combination" of the given quantity values q_i, as evidenced by

$$\overline{q} = \frac{q_{\text{tot}}}{n} = \sum_{i=1}^{n} \frac{1}{n} q_i. \tag{3.1}$$

It is reasonable to assume that the radius-vector $\vec{r}_P$ of the quantity center (variable expected position) P is the convex combination of the given position vectors $\vec{r}_i$ with the coefficients α_i. Accordingly, we have

$$\vec{r}_P = \sum_{i=1}^{n} \alpha_i \vec{r}_i = \sum_{i=1}^{n} \frac{q_i}{q_{\text{tot}}} \vec{r}_i. \tag{3.2}$$

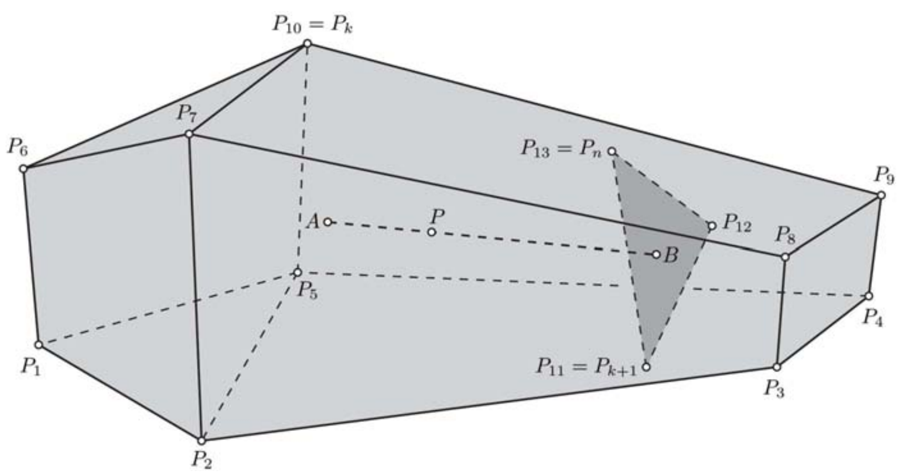

Figure 2. Quantity center and expected position as the convex combinations.

Complementing the geometric image it is necessary to determine the convex hull of the given point set, that is, the smallest convex polyhedron that contains all *n* points. Suppose that $P_1, ..., P_k$ as the external points are the vertices of such convex polyhedron $\mathcal{C}(P_1, ..., P_k)$, and suppose another that $P_{k+1}, ..., P_k$ as the internal points are also the vertices of the convex polyhedron or polygon $\mathcal{C}(P_{k+1}, ..., P_n)$, as can be seen in Figure 2.

Applying the formula in (3.1) for the quantity arithmetic mean (expected value) $\overline{q}$, and presenting it as the binomial "scalar convex combination", it follows:

$$\overline{q} = \sum_{i=1}^{n} \frac{1}{n} q_i = \frac{k}{n} \overline{q}_k + \frac{n-k}{n} \overline{q}_{n-k}, \tag{3.3}$$

where

$$\overline{q}_k = \sum_{i=1}^{k} \frac{1}{k} q_i, \qquad \overline{q}_{n-k} = \sum_{i=k+1}^{n} \frac{1}{n-k} q_i.$$

Thus, the quantity arithmetic mean (expected value) $\overline{q}$ is the binomial "scalar convex combination" of the *k*-member "scalar convex combination" $\overline{q}_k$ and the $(n-k)$-member "scalar convex combination" $\overline{q}_{n-k}$.

Using the formula in (3.2) for the radius-vector $\vec{r}_P$ of the quantity center (expected position) *P*, and presenting it as the binomial convex combination, it follows:

$$\vec{r}_P = \sum_{i=1}^{n} \frac{q_i}{q_{\text{tot}}} \vec{r}_i = \alpha \vec{r}_A + \beta \vec{r}_B, \tag{3.4}$$

where

$$\alpha = \sum_{i=1}^{k} \frac{q_i}{q_{\text{tot}}}, \quad \beta = \sum_{i=k+1}^{n} \frac{q_i}{q_{\text{tot}}}, \quad \vec{r}_A = \sum_{i=1}^{k} \frac{q_i}{\alpha q_{\text{tot}}} \vec{r}_i, \quad \vec{r}_B = \sum_{i=k+1}^{n} \frac{q_i}{\beta q_{\text{tot}}} \vec{r}_i.$$

We find that the radius-vector $\vec{r}_P$ is the binomial convex combination of the k-member convex combination $\vec{r}_A$ and the $(n-k)$-member convex combination $\vec{r}_B$.

Example 3.1 (Particles on the convex curve)**.** Let particles $P_1, ..., P_n$ be distributed along the graph of a strictly convex curve. Because of the strict convexity of the curve the convex hull of given points is exactly the convex polygon $\mathcal{C}(P_1, ..., P_n)$ with vertices $P_1, ..., P_n,$ as shown in Figure 3 left. Then the quantity center (expected position) P is placed in $\mathcal{C}(P_1, ..., P_n)$.

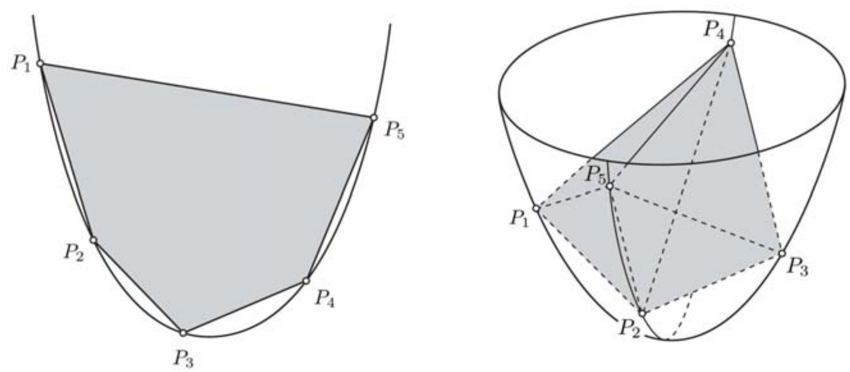

Figure 3. Particles on the convex curve and surface.

Example 3.2 (Particles on the convex surface)**.** Let particles $P_1, ..., P_n$ be distributed on the graph of a strictly convex surface. Due to the strict convexity of the surface the convex hull of the given points is just the convex polyhedron $\mathcal{C}(P_1, ..., P_n)$ with vertices $P_1, ..., P_n,$ as shown in Figure 3 right. Then the quantity center (expected position) P is placed in $\mathcal{C}(P_1, ..., P_n)$.

4. Transition to Integrals

This section, together with the next section, presents the main results of the paper.

Using the integral method, a sequence of convex combinations passes into integral, so a center of quantity becomes a barycenter of quantity. Some results on barycentres and integral arithmetic means for the subsets of the line are obtained in [5, pp. 1-5].

Continue with observations in the plane where the particles are not separate, but form a continuous set. Let $\mathcal{A} \subset \mathbb{R}^2$ be a set with positive area $\mu(\mathcal{A})$, and $q : \mathcal{A} \to \mathbb{R}$ be a non-negative quantity as the Riemann integrable function of two variables with positive $\iint_{\mathcal{A}} q(x, y)dxdy$. Assisting with Figure 4 left at first we imitate the previous discrete case. Given a positive integer *n*, it is necessary to make the partition of the set $\mathcal{A}$ into the union

$$\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_{ni}$$

of pairwise disjoint sets $\mathcal{A}_{ni}$ with positive areas $\mu(\mathcal{A}_{ni})$, where every $\mathcal{A}_{ni}$ contracts to the point or vanishes in infinity as *n* goes to infinity. For every $i = 1, \ldots, n$ we take the one point $P_{ni} \in \mathcal{A}_{ni}$, and its quantity value $q_{ni} = q(P_{ni})$. Keeping in mind the formulae in (3.1)-(3.2) we make adaptations and approximations:

$$q_{ni} \approx \frac{n}{\mu(\mathcal{A})} \mu(\mathcal{A}_{ni}) q_{ni}, \quad q_{\text{tot}} \approx \frac{n}{\mu(\mathcal{A})} \sum_{i=1}^{n} \mu(\mathcal{A}_{ni}) q_{ni},$$

$$\bar{q} \approx \frac{1}{\mu(\mathcal{A})} \sum_{i=1}^{n} \mu(\mathcal{A}_{ni}) q_{ni}, \tag{4.1}$$

$$\vec{r}_P \approx \frac{1}{\sum_{i=1}^{n} \mu(\mathcal{A}_{ni}) q_{ni}} \sum_{i=1}^{n} \mu(\mathcal{A}_{ni}) q_{ni} \vec{r}_{ni}. \tag{4.2}$$

Put $\vec{r}_{(x, y)} = x\vec{i} + y\vec{j}$. Letting *n* to infinity, the above discrete approximations get the Riemann integral forms:

$$\bar{q} = \frac{1}{\mu(\mathcal{A})} \iint_{\mathcal{A}} q(x, y)dxdy \tag{4.3}$$

as the integral arithmetic mean of the quantity function $q(x,\ y)$ on the set $\mathcal{A}$, and

$$\vec{r}_P = \frac{1}{\iint_{\mathcal{A}} q(x,\ y)dxdy} \iint_{\mathcal{A}} q(x,\ y)\vec{r}_{(x,\ y)}dxdy$$

$$= \frac{1}{\iint_{\mathcal{A}} q(x,\ y)dxdy} \left(\vec{i} \iint_{\mathcal{A}} xq(x,\ y)dxdy + \vec{j} \iint_{\mathcal{A}} yq(x,\ y)dxdy \right) \quad (4.4)$$

as the radius-vector of the barycenter P of the quantity $q(x,\ y)$ on the set $\mathcal{A}$.

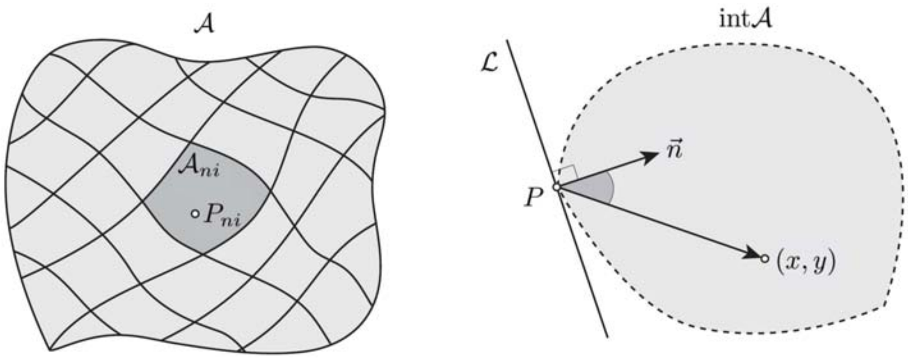

Figure 4. Partition of the set and geometrical interpretation of the convex set.

If $\mathcal{A}$ is connected and $q(x,\ y)$ is continuous, then the integral arithmetic mean $\bar{q}$ belongs to the set $q(\mathcal{A})$. The coordinates of the barycenter P are

$$x_P = \frac{\iint_{\mathcal{A}} xq(x,\ y)dxdy}{\iint_{\mathcal{A}} q(x,\ y)dxdy}, \qquad y_P = \frac{\iint_{\mathcal{A}} yq(x,\ y)dxdy}{\iint_{\mathcal{A}} q(x,\ y)dxdy}.$$

If we put $q(x,\ y) = 1$, then the expression in (4.4) presents the radius-vector of the barycenter P of the set $\mathcal{A}$, that is,

$$\vec{r}_P = \frac{1}{\mu(\mathcal{A})} \iint_{\mathcal{A}} \vec{r}_{(x,\ y)}dxdy. \quad (4.5)$$

From the convex combination formula in (4.2), it can be concluded that the barycenter P of the quantity q on the set $\mathcal{A}$ belongs to the convex hull $\mathrm{co}\mathcal{A}$ of the set $\mathcal{A}$. The formula in (4.4) arises after taking the limit, so the barycenter P belongs to the closed convex hull of the set $\mathcal{A}$. We would like to verify the intuitive assumption that the barycenter belongs to the interior of the convex hull. The geometrical description of convexity will assist.

Theorem 4.1. *Let* $\mathcal{A} \subset \mathbb{R}^2$ *be a set with* $\mu(\mathcal{A}) > 0$, *and* $q : \mathcal{A} \to \mathbb{R}$ *be a Riemann integrable quantity function such that* $q(x, y) > 0$ *for every* $(x, y) \in \operatorname{int} \mathcal{A}$. *If P is the barycenter of q on* $\mathcal{A}$, *then*

$$P \in \operatorname{int}(\mathrm{co}\mathcal{A}). \tag{4.6}$$

Proof. Assume $\mathcal{A}$ is convex, and prove that $P \in \operatorname{int} \mathcal{A}$.

The geometrical characterization of convexity of the set $\mathcal{A}$ with $\operatorname{int} \mathcal{A} = \emptyset$ states that the line $\mathcal{L}$ through each point outside of $\operatorname{int} \mathcal{A}$ exists so that $\mathcal{L} \cap \operatorname{int} \mathcal{A} = \emptyset$, and the entire $\operatorname{int} \mathcal{A}$ is contained in one of the half-planes determined by $\mathcal{L}$. Suppose $P \notin \operatorname{int} \mathcal{A}$, and the characteristic line $\mathcal{L}$ passes through the point P. Let $\vec{n}$ be the vector normal to the line $\mathcal{L}$, oriented towards set $\mathcal{A}$ as shown in Figure 4 right.

Using the inner product with vector $\vec{n}$, we find that the following holds for every $(x, y) \in \operatorname{int} \mathcal{A}$:

$$\langle \vec{n}, \vec{r}_{(x,y)} - \vec{r}_P \rangle > 0,$$

$$\left\langle \vec{n}, \vec{r}_{(x,y)} - \frac{1}{\iint_{\mathcal{A}} q(x, y)\,dxdy} \iint_{\mathcal{A}} q(x, y)\vec{r}_{(x,y)}\,dxdy \right\rangle > 0,$$

$$\langle \vec{n}, q(x, y)\vec{r}_{(x,y)} \rangle - \frac{q(x, y)}{\iint_{\mathcal{A}} q(x, y)\,dxdy} \left\langle \vec{n}, \iint_{\mathcal{A}} q(x, y)\vec{r}_{(x,y)}\,dxdy \right\rangle > 0.$$

Integrating the above inner products over $\operatorname{int}\mathcal{A}$, it arises

$$\left\langle \vec{n}, \iint_{\mathcal{A}} q(x, y)\vec{r}_{(x,y)}dxdy \right\rangle - \left\langle \vec{n}, \iint_{\mathcal{A}} q(x, y)\vec{r}_{(x,y)}dxdy \right\rangle > 0$$

because the integrals over $\operatorname{int}\mathcal{A}$ and $\mathcal{A}$ are the same. The resulting contradiction says that must be $P \in \operatorname{int}\mathcal{A}$. □

Theorem 4.2. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathbb{R}^2$ be sets with $\mu(\mathcal{A}_i) > 0$, and $q_i : \mathcal{A}_1 \to \mathbb{R}$ be Riemann integrable quantity functions. If P_i are the barycenters of q_i on $\mathcal{A}_i$, and P is the center of all integral arithmetic means $\overline{q}_i$, then*

$$\vec{r}_P = \frac{1}{\sum_{i=1}^{n} \frac{1}{\mu(\mathcal{A}_i)} \iint_{\mathcal{A}_i} q_i(x, y)dxdy} \sum_{i=1}^{n} \frac{1}{\mu(\mathcal{A}_i)} \iint_{\mathcal{A}_i} q_i(x, y)dxdy\vec{r}_i. \tag{4.7}$$

Proof. If we insert $\overline{q}_i = \frac{1}{\mu(\mathcal{A}_i)} \iint_{\mathcal{A}_i} q_i(x, y)dxdy$ in the discrete expression

$$\vec{r}_P = \frac{1}{\overline{q}_{tot}} \sum_{i=1}^{n} \overline{q}_i \vec{r}_i,$$

then we get the integral expression in (4.7). □

Theorem 4.3. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathbb{R}^2$ be pairwise disjoint sets with $\mu(\mathcal{A}_i) > 0$, and $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$. Let $q : \mathcal{A} \to \mathbb{R}$ be a Riemann integrable quantity function. If P_i are the barycenters of q on $\mathcal{A}_i$, and P is the barycenter of q on $\mathcal{A}$, then*

$$\vec{r}_P = \frac{1}{\iint_{\mathcal{A}} q(x, y)dxdy} \sum_{i=1}^{n} \iint_{\mathcal{A}_i} q(x, y)dxdy\vec{r}_i. \tag{4.8}$$

Proof. Using the equality

$$\int\int_{\mathcal{A}} q(x, y)\vec{r}_{(x, y)}dxdy = \sum_{i=1}^{n}\int\int_{\mathcal{A}_i} q(x, y)\vec{r}_{(x, y)}dxdy$$

$$= \sum_{i=1}^{n}\frac{\int\int_{\mathcal{A}_i} q(x, y)dxdy}{\int\int_{\mathcal{A}_i} q(x, y)dxdy}\int\int_{\mathcal{A}_i} q(x, y)\vec{r}_{(x, y)}dxdy$$

$$= \sum_{i=1}^{n}\int\int_{\mathcal{A}_i} q(x, y)dxdy\vec{r}_i$$

in the formula in (4.4), it follows the barycenter expression in (4.8). □

Taking $q(x, y) = 1$ in the quantity barycenters formula in (4.8), we have the known rule for the set barycenters:

Corollary 4.4. *Let* $\mathcal{A}_1, ..., \mathcal{A}_n \subset \mathbb{R}^2$ *be pairwise disjoint sets with* $\mu(\mathcal{A}_i) > 0,$ *and* $\mathcal{A} = \bigcup_{i=1}^{n}\mathcal{A}_i$. *If* P_i *are the barycenters of* $\mathcal{A}_i$, *and P is the barycenter of* $\mathcal{A}$, *then*

$$\vec{r}_P = \frac{1}{\mu(\mathcal{A})}\sum_{i=1}^{n}\mu(\mathcal{A}_i)\vec{r}_i. \tag{4.9}$$

Everything we have done in the plane can be easily extended to the space. Let $\mathcal{A} \subset \mathbb{R}^3$ be a set with $\mu(\mathcal{A}) > 0,$ and $q : \mathcal{A} \to \mathbb{R}$ be a non-negative quantity as the Riemann integrable function of three variables with positive $\int\int\int_{\mathcal{A}} q(x, y, z)dxdydz.$

Let $\vec{r}_{(x, y, z)} = x\vec{i} + y\vec{j} + z\vec{k}$. Then the integral arithmetic mean of the quantity function $q(x, y, z)$ on the set $\mathcal{A}$ is expressed with

$$\bar{q} = \frac{1}{\mu(\mathcal{A})}\int\int\int_{\mathcal{A}} q(x, y, z)dxdydz, \tag{4.10}$$

and the radius-vector of the barycenter P is expressed with

$$\vec{r}_P = \frac{1}{\iiint_{\mathcal{A}} q(x, y, z)dxdydz} \iiint_{\mathcal{A}} q(x, y, z)\vec{r}_{(x, y, z)}dxdydz. \tag{4.11}$$

Spatial analogies of Theorems 4.1-4.3 and Corollary 4.4 are also valid. In the proof of the spatial variant of Theorem 4.1 the plane $\mathcal{P}$ which does not intersect the set $\operatorname{int}\mathcal{A}$ is used instead of the line $\mathcal{L}$.

By including the Lebesgue integral, we can get more general expressions for the integral arithmetic mean and the barycenter of the quantity $q(x, y, z)$. Let μ be a measure on $\mathbb{R}^3$. Let $\mathcal{A} \subset \mathbb{R}^3$ be a μ-measurable set with $\mu(\mathcal{A}) > 0$, and $q : \mathcal{A} \to \mathbb{R}$ be a μ-integrable non-negative quantity function on the set $\mathcal{A}$ with $\int_{\mathcal{A}} qd\mu > 0$.

The integral arithmetic mean of the quantity function $q(x, y, z)$ on the set $\mathcal{A}$ can be presented with

$$\overline{q} = \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} q(x, y, z)d\mu(x, y, z), \tag{4.12}$$

and the radius-vector of the barycenter P can be presented with

$$\vec{r}_P = \frac{1}{\int_{\mathcal{A}} q(x, y, z)d\mu(x, y, z)} \int_{\mathcal{A}} q(x, y, z)\vec{r}_{(x, y, z)}d\mu(x, y, z) \tag{4.13}$$

provided that the functions $xq(x, y, z)$, $yq(z, y, z)$ and $zq(x, y, z)$ are μ-integrable on the set $\mathcal{A}$.

In the case of presenting Theorem 4.1 by the Lebesgue integral with respect to some measure μ on $\mathbb{R}^2$, it is needed to write $P \in \operatorname{co}\mathcal{A}$ instead of $P \in \operatorname{int}(\operatorname{co}\mathcal{A})$. Everything else that we presented with the Riemann integral is also true with the Lebesgue integral.

Using the measure and integral theory, the barycenter of the non-negative quantity q can be reduced to the barycenter of the set. Let the measure ν of the μ-measurable set $\mathcal{A}$ be defined by

$$\nu(\mathcal{A}) = \int_{\mathcal{A}} q d\mu$$

(see [6, p. 23]). If $\nu(\mathcal{A}) > 0$, then the μ-barycenter of the non-negative quantity q on the set $\mathcal{A}$ corresponds to the ν-barycenter of the set $\mathcal{A}$, that is,

$$\frac{1}{\int_{\mathcal{A}} q d\mu} \int_{\mathcal{A}} q \vec{r} d\mu = \frac{1}{\nu(\mathcal{A})} \int_{\mathcal{A}} \vec{r} d\nu.$$

5. Applications of Convexity on Inequalities

We recall the discrete and integral form of Jensen's inequality from [2]. The discrete form will be extended by using Polygon theorem, and the integral form will be extended by using Theorem 4.1.

Theorem E. (The discrete form of Jensen's inequality)**.** *Let* $\mathcal{I} \subseteq \mathbb{R}$ *be an interval, and* $f : \mathcal{I} \to \mathbb{R}$ *be a function. Let* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *be an n-tuple with points* $x_i \in \mathcal{I}$, *and* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *be an n-tuple with numbers* $\alpha_i \in [0, 1]$ *such that* $\sum_{i=1}^{n} \alpha_i = 1$.

A function f is convex if and only if the inequality

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i) \tag{5.1}$$

holds for all above n-tuples $\boldsymbol{\alpha}$ *and* $\boldsymbol{x}$.

Consequently, if $\sum_{i=1}^{n} \alpha_i = \alpha > 0$, *not necessarily equals* 1, *then f is convex if and only if*

$$f\left(\frac{1}{\alpha} \sum_{i=1}^{n} \alpha_i x_i\right) \leq \frac{1}{\alpha} \sum_{i=1}^{n} \alpha_i f(x_i). \tag{5.2}$$

A function f is concave if and only if the reverse inequality is valid in (5.1) *and* (5.2).

Let $\mathcal{C}(P_1, \ldots, P_n) \subset \mathbb{R}^2$ be a convex polygon with vertices $P_i(x_i, y_i)$ bounded below with the convex polygonal line $y = f_1(x)$, and bounded above with the concave polygonal line $y = f_2(x)$. Let q_i be non-negative quantity values at the points P_i such that $q_{\text{tot}} = \sum_{i=1}^{n} q_i > 0$. Then the inequality

$$f_1(x_P) \le y_P \le f_2(x_P) \tag{5.3}$$

holds for the center coordinates

$$x_P = \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i x_i, \qquad y_P = \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i y_i$$

by Polygon theorem. The inequality in (5.3) can be extended to the series of inequalities by using the discrete form of Jensen's inequality.

Corollary 5.1. *Let $\mathcal{C}(P_1, \ldots, P_n) \subset \mathbb{R}^2$ be a convex polygon with vertices $P_i(x_i, y_i)$ bounded below and above with the polygonal lines $y = f_1(x)$ and $y = f_2(x)$, respectively. Let q_i be non-negative quantity values at the points P_i such that $q_{\text{tot}} = \sum_{i=1}^{n} q_i > 0$. Then*

$$\begin{aligned} f_1\left(\frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i x_i\right) &\le \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i f_1(x_i) \\ &\le \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i y_i \\ &\le \frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i f_2(x_i) \le f_2\left(\frac{1}{q_{\text{tot}}} \sum_{i=1}^{n} q_i x_i\right). \end{aligned} \tag{5.4}$$

Proof. The left side of the inequality in (5.4) follows from the discrete form of Jensen's inequality for the convex function f_1, and the right side follows from Jensen's inequality for the concave function f_2. The middle part of the inequality in (5.4) follows from the inequalities $f_1(x_i) \leq y_i \leq f_2(x_i)$ for $i = 1, \ldots, n$. $\square$

If $\operatorname{co}\{P_1, \ldots, P_n\} = \mathcal{C}(P_{i_1}, \ldots, P_{i_k})$, not necessarily all points P_i vertices, then the inequality in (5.4) remains valid. In this case the polygonal lines $y = f_1(x)$ and $y = f_2(x)$ depend on the vertices $P_{i_1}, \ldots, P_{i_k}$ only.

If (Ω, μ) is a measure space, then it is assumed that every weighted function $q : \Omega \to \mathbb{R}$ is non-negative almost everywhere on Ω, that is, $q(\omega) \geq 0$ for almost all $\omega \in \Omega$.

Theorem F (The integral form of Jensen's inequality)**.** *Let* $\mathcal{I} \subseteq \mathbb{R}$ *be an interval, and* $f : \mathcal{I} \to \mathbb{R}$ *be a function. Let* (Ω, μ) *be a probability measure space,* $g : \Omega \to \mathcal{I}$ *be a measurable function, and* $q \in L^1(\Omega, \mu)$ *be a weighted function with* $\int_\Omega q d\mu = 1$ *so that* $q \cdot g,\ q \cdot (f \circ g) \in L^1(\Omega, \mu)$.

If a function f is convex, then the inequality

$$f\left(\int_\Omega q \cdot g d\mu\right) \leq \int_\Omega q \cdot (f \circ g) d\mu \tag{5.5}$$

holds for all above q, g and μ.

Consequently, if $\int_\Omega q d\mu = \alpha > 0$, *not necessarily equals* 1, *then*

$$f\left(\frac{1}{\alpha}\int_\Omega q \cdot g d\mu\right) \leq \frac{1}{\alpha}\int_\Omega q \cdot (f \circ g) d\mu. \tag{5.6}$$

If a function f is concave, then the reverse inequality is valid in (5.5) *and* (5.6).

Let $\mathcal{A} \subset \mathbb{R}^2$ be a convex set bounded below with the convex function $y = f_1(x)$, and bounded above with the concave function $y = f_2(x)$. Let $q : \mathcal{A} \to \mathbb{R}$ be a Riemann integrable non-negative quantity function such that $\alpha = \int\int_{\mathcal{A}} q(x, y)dxdy > 0.$ Then the inequality

$$f_1(x_P) \leq y_p \leq f_2(x_P) \tag{5.7}$$

holds for the barycenter coordinates

$$x_P = \frac{1}{\alpha}\int\int_{\mathcal{A}} q(x, y)xdxdy, \qquad y_P = \frac{1}{\alpha}\int\int_{\mathcal{A}} q(x, y)ydxdy$$

by Theorem 4.1. The inequality in (5.7) can be extended to the series of inequalities by using the integral form of Jensen's inequality.

Corollary 5.2. *Let* $\mathcal{A} \subset \mathbb{R}^2$ *be a convex set bounded below and above with the functions* $y = f_1(x)$ *and* $y = f_2(x)$, *respectively*. *Let* $q : \mathcal{A} \to \mathbb{R}$ *be a Riemann integrable quantity function such that* $\alpha = \int\int_{\mathcal{A}} q(x, y)dxdy$ $> 0.$ *Then*

$$\begin{aligned} f_1\left(\frac{1}{\alpha}\int\int_{\mathcal{A}} q(x, y)xdxdy\right) &\leq \frac{1}{\alpha}\int\int_{\mathcal{A}} q(x, y)\, f_1(x)dxdy \\ &\leq \frac{1}{\alpha}\int\int_{\mathcal{A}} q(x, y)ydxdy \\ &\leq \frac{1}{\alpha}\int\int_{\mathcal{A}} q(x, y)\, f_2(x)dxdy \\ &\leq f_2\left(\frac{1}{\alpha}\int\int_{\mathcal{A}} q(x, y)xdxdy\right). \end{aligned} \tag{5.8}$$

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