International Journal of Numerical Methods and Applications
Volume 9, Number 1, 2013, Pages 57-68
Published Online: April 2013
Available online at http://pphmj.com/journals/ijnma.htm
Published by Pushpa Publishing House, Allahabad, INDIA

AN EFFICIENT COMPUTATION OF STURM-LIOUVILLE EIGENVALUES BY MEANS OF POLYNOMIAL EXPANSION

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Abstract

The purpose of this study is to present a simple and efficient approach to calculate the eigenvalues of the Sturm-Liouville problem. By expanding the unknown function as power series, we directly get the corresponding polynomial characteristic equations for kinds of boundary conditions, and the lower- and higher-order eigenvalues can be determined simultaneously from the multi-roots. Several examples for numerical computation used frequently in Sturm-Liouville problem of estimating eigenvalues are given to show that our method has fast convergence and the obtained numerical results have high accuracy.

2010 Mathematics Subject Classification: 65L15, 65L60.

Keywords and phrases: approximate solution, Sturm-Liouville problem, eigenvalues, polynomial expansion.

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Communicated by K. K. Azad

Received October 30, 2012

1. Introduction

In this paper, the Liouville normal form of the Sturm-Liouville problem is considered:

$$-y''(x) + q(x)y(x) = \lambda y(x), \tag{1}$$

where $q(x)$ is defined on the interval of $a \leq x \leq b$ and the parameter λ is the characteristic value needed to be determined. Such Sturm-Liouville value problem arises in many physical, engineering and other scientific fields, which plays a very important role in both theory and applications [1, 2].

Generally speaking, it seems unlikely to obtain an analytical solution in explicit form or it is difficult to obtain exact eigenvalues of the above problems. Therefore, some numerical and approximate methods have been proposed for determining the solution of Sturm-Liouville eigenvalue problem [3]. Based on the asymptotic correction with Numerov's method and finite element method, Andrew and Paine improved the results for Sturm-Liouville problems with natural boundary conditions [4-6]. Ghelardoni [7] used some linear multistep methods, called Boundary Value Methods (BVM), to discretize the Sturm-Liouville problem and investigate the approximations of eigenvalues, where the correction technique of Andrew-Paine [5] is extended to BVM. In [8] and [9], Çelik and Gokmen used the Chebyshev collocation method to investigate for the approximate computation of higher Sturm-Liouville eigenvalues by transforming the problems and given boundary conditions to matrix equation. For computing the approximate eigenvalues of regular Sturm-Liouville problems with two points or periodic or semi-periodic boundary conditions, Yuan et al. [10] utilized the Chebyshev collocation method combined with an improved step in order to solve a generalized eigenvalue problem. Unlike the classical Chebyshev collocation method, Chen and Ma [11] introduced the Legendre-Galerkin-Chebyshev collocation method, which preserves the symmetry of the problem, to compute the approximate eigenvalues of the Sturm-Liouville problem with kinds of different boundary conditions. Recently, Zhang [12] discretized the Sturm-Liouville problems (SLPs) into standard matrix eigenvalue problems in order to achieve high accuracy and high efficiency by using the mapped barycentric Chebyshev differentiation matrix method.

In this paper, we will introduce a simple approach for determining the eigenvalues of Sturm-Liouville problem with kinds of boundary conditions. By expanding the mode shapes as power series, we transform the governing differential equation to a system of algebraic equations in unknown coefficients. The characteristic values λ can be easily determined from the existence condition of a nontrivial solution in the resulting system. We will apply the method to evaluate the eigenvalues for the examples used frequently in Sturm-Liouville problem to demonstrate the accuracy of approximation.

2. Formulas and Method

For the Sturm-Liouville problem, the key thing is to calculate the characteristic values λ of resulting governing equation (1). Since the eigenvalues are closely related to the end supports, it is instructive to give explicit expressions for boundary conditions. Without loss of generality, the most homogeny conditions can be stated below [8]:

$$\sum_{j=0}^{1} a_{1j} y^{(j)}(a) + b_{1j} y^{(j)}(b) = 0, \tag{2}$$

$$\sum_{j=0}^{1} a_{2j} y^{(j)}(a) + b_{2j} y^{(j)}(b) = 0. \tag{3}$$

Actually, the problem becomes solving a set of a second-order ordinary differential governing equation (1) combining with the above boundary conditions. In this section, avoiding solving the differential equation (1) directly, we introduce a simple method to determine the eigenvalues of Sturm-Liouville equation. First of all, we expand $y(x)$ as power series. Or rather, if neglecting sufficient small error, the unknown $y(x)$ can be approximately expanded as:

$$y(x) = \sum_{i=0}^{N} c_i x^i, \quad a \le x \le b, \tag{4}$$

where c_i are unknown coefficients and N is a certain positive integer, which is chosen large enough such that the rest have a negligible error. It should be pointed out that besides the above expansion equation (4) of $y(x)$, other expansion approximations of the unknown y such as Chebyshev polynomial, trigonometric functions, etc., are also applicable and even better because of their orthogonality. Bearing equation (4) in mind, application of the condition (2) leads to yield a linear equation of $N + 1$ unknown coefficients of $c_0, c_1, ..., c_N$,

$$\sum_{i=0}^{N} (f_{0i} - \lambda k_{0i}) c_i = 0, \tag{5}$$

where

$$f_{0i} = (a.a_{10} + i.a_{11}) a^{i-1} + (b.b_{10} + i.b_{11}) b^{i-1}, \;\; k_{0i} = 0, \quad i = 0, 1, ..., N. \tag{6}$$

On the other hand, using the condition (3), one has

$$\sum_{i=0}^{N} (f_{1i} - \lambda k_{1i}) c_i = 0 \tag{7}$$

with

$$f_{1i} = (a.a_{20} + i.a_{21}) a^{i-1} + (b.b_{20} + i.b_{21}) b^{i-1}, \;\; k_{1i} = 0, \quad i = 0, 1, ..., N. \tag{8}$$

It is easily found that in order to solve the linear equations (5) and (7), another $N - 1$ independent equations are needed for uniquely determining c_i. This can be achieved by using the following method. Inserting (4) into the differential equation (1) for each case leads to

$$\sum_{i=0}^{N} c_i [-i(i-1) + q(x) x^2] x^{i-2} - \lambda \sum_{i=0}^{N} c_i x^i = 0. \tag{9}$$

We multiply both sides of (9) by x^n $(n = 0, 1, 2, ..., N - 2)$ and then integrate with respect to x between a and b, yielding another $N - 1$ linear algebraic equations of unknown coefficients c_i:

$$\sum_{i=0}^{N} (f_{ji} - \lambda k_{ji}) c_i = 0, \quad j = 2, 3, ..., N, \tag{10}$$

with

$$f_{ji} = \int_a^b [-i(i-1) + q(x)x^2] x^{j+i-4} dx, \quad k_{ji} = \frac{b^{j+i-1} - a^{j+i-1}}{j+i-1}. \tag{11}$$

Therefore, equations (5), (7) and (10) form a system of $N + 1$ linear algebraic equations for $N + 1$ unknown coefficients c_i $(i = 0, 1, ..., N)$, which can be rewritten in a compact form as

$$(F_{nn} - \lambda K_{nn}) C_n = 0, \tag{12}$$

with

$$F_{nn} = (f_{ij})_{(N+1)\times(N+1)}, \quad K_{nn} = (k_{ij})_{(N+1)\times(N+1)}, \quad C_n = (c_i)_{(N+1)\times 1}.$$

To obtain a nontrivial solution of the resulting system, the determinant of the coefficient matrix of the system has to vanish, that is $\det(F_{nn} - \lambda K_{nn}) = 0$. Then we immediately get a characteristic equation in eigenvalues λ. By inspecting the above procedure, one readily finds that the obtained corresponding characteristic equation has multi-roots since that it is a polynomial in eigenvalue λ. Therefore, a numerical procedure to look for its roots is very simple and easy with the aid of commercial software. As a result, both the lower and higher characteristic values of Sturm-Liouville equation (1) can be determined simultaneously from the multi-roots.

It should be noted that for other general boundary conditions, we can deal with the problem in the same way. For example, if we choose the common conditions

$$y(0) = y(\pi) = 0, \tag{13}$$

where $a = 0$ and $b = \pi$. Substituting (4) into the above conditions, the first two linear equations can be derived as

$$c_0 = 0, \tag{14}$$

$$\sum_{i=0}^{N} c_i \pi^i = 0. \tag{15}$$

Then one can replace equations (5) and (7) for the above linear algebraic equations, while keeping the last $N - 1$ equations as the same form (10). For any other boundary conditions such as

$$y'(0) = y'(\pi) = 0, \tag{16}$$

and

$$y\left(-\frac{\pi}{2}\right) = y\left(\frac{\pi}{2}\right), \quad y'\left(-\frac{\pi}{2}\right) = y'\left(\frac{\pi}{2}\right), \tag{17}$$

the corresponding characteristic equations can be similarly obtained, which are omitted here.

3. Numerical Examples

In order to examine the effectiveness of the method proposed in this paper, several numerical examples used frequently in Sturm-Liouville problem are given to demonstrate the accuracy of approximation as compared to the exact solution and other numerical results.

Example 1. We first consider the simple Sturm-Liouville problem

$$-y''(x) = \lambda y(x), \tag{18}$$

subjected to

$$y(0) = y'(\pi) = 0,$$

where $q(x) = 0$. Obviously, the exact characteristic values of (18) is $\lambda = k^2$, $k \in Z$ [12]. For checking the convergence of the suggested method, we have calculated the lower and higher characteristic values of Sturm-Liouville equation (1) by taking different N values in (4). Evaluated absolute errors $|\lambda_{\text{num}} - \lambda_{\text{exact}}|$ of first six eigenvalues between the numerical results and the exact ones are tabulated in Table 1. By comparing our numerical results with the exact ones, it is clear from Table 1 that the numerical results have a rapid convergence. With N increasing from 6 to 14, the errors between the numerical and exact results dramatically decrease and the results when taking $N = 14$ are nearly identical to the exact ones up to 6 decimal digits for the first four eigenvalues, which indicates that the present approach is very efficient. But the accuracy of the results drops with the modes increasing. However, if increasing N, the accuracy becomes very satisfactory. As a result, a higher accuracy can be achieved through increasing N.

Table 1. Absolute errors $|\lambda_{\text{num}} - \lambda_{\text{exact}}|$ for Example 1

k	$N = 6$	$N = 8$	$N = 10$	$N = 12$	$N = 14$
1	2.64E-7	3.39E-11	2.89E-15	1.11E-15	1.55E-15
2	2.90E-2	1.34E-4	1.85E-7	1.04E-10	2.04E-12
3	2.43E-1	3.78E-3	1.94E-5	2.38E-8	8.38E-9
4	22.52	1.01	3.45E-2	4.34E-4	2.66E-6
5	49.87	2.94	1.73E-1	3.11E-3	2.77E-3
6	-	96.18	6.84	5.94E-1	1.19E-1

Example 2. As a second example, we solve the following Sturm-Liouville problem:

$$-y''(x) + e^{x} y(x) = \lambda y(x). \tag{19}$$

Here two boundary conditions are evaluated. One is that $y(0) = y(\pi) = 0$, which has been studied in [4] and [7] based on Numerov's method and boundary value methods, respectively; and the other is that $y(0) = y'(\pi) = 0$ and the corresponding numerical solutions have been investigated by using

Chebyshev collocation method [8] and Numerov's method (NM) [6]. According to the present method, the numerical results of first six eigenvalues are calculated for $N = 14$. The numerical results compared to the exact solutions and those obtained by some other methods, such as the corrected Numerov method (CNM), the uniderivative Simpson method (USM) [10], are given for two cases in Tables 2 and 3, respectively. As comparing to the existing results, we can find that the accuracy is very satisfactory by using a relatively small order $(N = 14)$ for both cases.

Table 2. Numerical results for Example 2 with $y(0) = y(\pi) = 0$

k	Eigenvalues λ		Absolute errors $\lvert \lambda_{num} - \lambda_{exact} \rvert$			
	Exact [10]	Present $(N = 14)$	[6] $(N = 40)$	USM [10] $(N = 39)$	CNM [10] $(N = 39)$	Present $(N = 14)$
1	4.8966694	4.89666937998	2.25E-6	1.39E-5	2.6E-6	2.0023E-8
2	10.045190	10.04518989709	2.87E-5	7.53E-5	3.26E-5	1.0291E-7
3	16.019267	16.01927824811	4.54E-5	2.940E-4	1.111E-4	1.1248E-5
4	23.266271	23.26666096023	9.35E-5	4.927E-4	2.318E-4	3.8996E-4
5	32.263707	32.25761510080	4.76E-4	1.1286E-3	3.878E-4	6.0918E-3
6	43.220020	43.42520272001	9.76E-4	1.6781E-3	5.823E-4	2.0518E-1

Table 3. Numerical results for Example 2 with $y(0) = y'(\pi) = 0$

k	Eigenvalues λ		Absolute errors $\lvert \lambda_{num} - \lambda_{exact} \rvert$		
	Exact [6]	Present $(N = 14)$	NM [6] $(N = 40)$	CNM [6] $(N = 40)$	Present $(N = 14)$
1	4.89571	4.895713259646	2.52E-6	2.52E-6	3.2596E-6
2	9.99955	9.999549844573	3.04E-5	2.87E-5	1.5542E-7
3	15.4685	15.468519155137	8.41E-5	4.54E-5	1.9155E-5
4	21.0371	21.037295988923	3.86E-4	9.35E-5	1.9598E-4
5	28.1893	28.188932386472	1.80E-3	4.76E-4	3.6761E-4
6	37.7907	37.796206842212	5.40E-3	9.76E-4	5.5068E-3

Example 3. Next, we discuss the following periodic Sturm-Liouville problem:

$$-y''(x) + 10\cos(2x)\,y(x) = \lambda y(x), \tag{20}$$

under the constraints

$$y\left(-\frac{\pi}{2}\right) = y\left(\frac{\pi}{2}\right), \quad y'\left(-\frac{\pi}{2}\right) = y'\left(\frac{\pi}{2}\right), \text{ for case 1,}$$

$$y\left(-\frac{\pi}{2}\right) = -y\left(\frac{\pi}{2}\right), \quad y'\left(-\frac{\pi}{2}\right) = -y'\left(\frac{\pi}{2}\right), \text{ for case 2.}$$

In the following, we employ the approach described in this paper to determine the first six eigenvalues' approximations when $N = 14$. The absolute errors between the exact solution and its approximations are tabulated in Tables 4 and 5 for two cases, respectively. Other numerical results based on the Legendre-Galerkin-Chebyshev collocation method (LFCC) [11] and the corrected finite difference (CFD) [5] are also presented in those tables. From Tables 4 and 5, it can be seen that our results are in excellent agreement with the existing results for both cases.

Table 4. Numerical results of eigenvalues and absolute errors for case 1

k	Eigenvalues λ		Absolute errors $\lvert \lambda_{\text{num}} - \lambda_{\text{exact}} \rvert$		
	Exact [12]	Present ($N = 14$)	CFD ($N = 40$)	LGCC ($N = 39$)	Present ($N = 14$)
1	2.09946044548547	2.09945653298	1.75E-2	4.55E-8	3.9125E-6
2	7.44910973952939	7.44911398503	2.35E-2	3.95E-8	4.2455E-6
3	16.6482199371686	16.64825148895	1.69E-2	6.28E-8	3.1551E-5
4	17.0965816843648	17.09668059919	4.8E-3	3.16E-7	9.8914E-5
5	36.3588668480280	36.36896284201	1.34E-2	1.52E-7	1.0095E-2

Table 5. Numerical results of eigenvalues and absolute errors for case 2

k	Exact λ [11]	Present λ	$\lvert \lambda_{num} - \lambda_{exact} \rvert$
1	–5.79008059863	–5.79007822454	2.3740E-6
2	1.85818754154	1.85819009746	2.5559E-6
3	9.236327713693	9.23645545404	1.2774E-4
4	11.548832036343	11.54884282673	1.0790E-5
5	25.510816046303	25.51361310254	2.7971E-3
6	25.549971749981	25.55001133158	3.9581E-5

Example 4. Finally, we consider the following Sturm-Liouville problem:

$$-y''(x) + (x + 0.1)^{-2} y(x) = \lambda y(x), \tag{21}$$

where the boundary condition is $y(0) = y(\pi) = 0$. When N in (4) takes 14, the numerical results together with the exact results are tabulated in Table 6. Other absolute errors derived previously by the Numerov's method $(N = 19)$ [4] and boundary value methods with $N = 40$ [7] are also presented in the table. Table 6 demonstrates that the present numerical results are in very good agreement with the exact solutions and existing numerical results.

Table 6. Numerical results of eigenvalues and absolute errors

k	Eigenvalues λ		Absolute errors $\lvert \lambda_{num} - \lambda_{exact} \rvert$		
	Exact [4]	Present $(N = 14)$	[4] $(N = 19)$	[7] $(N = 40)$	Present $(N = 14)$
1	1.5198658	1.519865189	4.3250E-4	1.4300E-5	6.1100E-7
2	4.9433098	4.943306263	2.7664E-3	1.0390E-4	3.5370E-6
3	10.284663	10.28464398	8.6229E-3	6.0420E-4	1.9020E-5
4	17.559958	17.55965280	1.9436E-2	2.3847E-3	3.0520E-4
5	26.782863	26.79196486	3.6391E-2	8.0143E-3	9.1018E-3
6	37.964426	37.68308759	6.0481E-1	2.1852E-2	2.8133E-1

4. Conclusions

An efficient approach has been presented to deal with the Sturm-Liouville problem. Instead of directly solving the differential equation, we transform the governing equation to a system of linear algebraic equations by making use of power series; then a characteristic equation will be obtained, which can be effectively computed through using symbolic computing codes on any personal computer. The effectiveness of the method has been confirmed by comparing our numerical results with the exact ones and other numerical results available for numerical examples used frequently in Sturm-Liouville problem. It has been seen that our numerical results show excellent consistency with the existing results.

Acknowledgements

The authors would like to give their thanks to the financial support of the TianYuan Special Funds of the National Natural Science Foundation of China (Grant No. 11126340) and Guangdong Provincial Natural Science Foundation of China (Grant No. 10252800001000000).

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