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REVERSIBILITY OF CIRCLE HOMEOMORPHISMS AND FLOWS

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Abstract

In this paper, we study sufficient conditions for the reversibility of circle homeomorphisms. We show that every homeomorphism of the circle, which has only one periodic orbit, is continuously reversible. And we prove that a positively equicontinuous flow is continuously reversible.

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1. Introduction

In classical terminology, a reversible mechanical system is one whose Hamiltonian assumes the particularly simple form $K + V$. Here K is the kinetic energy of the system, and V is the potential energy. In 1976, Devaney [3] showed that it was to generalize the notion of reversibility and studied the qualitative properties of such generalized systems from a more geometric point of view. In 1986, Sevryuk [5] has written a monograph that provides a wide exposition of reversibility in smooth multidimensional analysis.

A process is reversible if its dynamics is unchanged by reversing the direction of its time. Mathematically this can be realized if we postulate that the function F describing the process is a bijection of a set X and, moreover, F and the inverse F^{-1} are conjugate by a function, $\gamma : X \to X$ which is the inverse of itself, i.e., involution:

$$F^{-1} = \gamma^{-1} \circ F \circ \gamma$$

or, since $\gamma^{-1} = \gamma$,

$$F^{-1} = \gamma \circ F \circ \gamma.$$

Then, consequently,

$$F^{-n} = \gamma^{-1} \circ F^{n} \circ \gamma \quad \text{for } n \in \mathbb{Z},$$

which yields the desired property of similarity of the dynamics of the given process and its reverse.

In 2002, Jarczyk [4] proved that any homeomorphism mapping a real interval onto itself and having no fixed points is conjugate to its inverse by a continuous involution or, equivalently, is a composition of two continuous decreasing involutions.

In this paper, we investigate sufficient conditions for the reversibility for homeomorphisms defined on an interval and the circle. We will show that every homeomorphism of the circle, which has only one periodic orbit, is continuously reversible. And we prove that a positively equicontinuous flow is continuously reversible.

2. Reversibility of Circle Homeomorphisms

Definition. Let X be a topological space. A homeomorphism $F : X \to X$ is said to be (*continuously*) *reversible* if there exists an (continuous) involution $\gamma : X \to X$ such that

$$F^{-1} = \gamma^{-1} \circ F \circ \gamma.$$

The following remark is founded in [4].

Remark. (1) A map of a topological space is reversible if and only if it is a composition of two involutions. A continuous map is continuously reversible if and only if it is a composition of two continuous involutions.

(2) If F is a reversible (continuously reversible) map, then also F^{-1} is reversible (continuously reversible). Moreover, if $F = \beta \circ \gamma$ is a representation of F as a composition of two involutions (continuous involutions), then $F^{-1} = \gamma \circ \beta$.

(3) The only increasing involution defined on a set of reals is the identity function.

Note that every continuously reversible map is a homeomorphism. In particular, being a homeomorphism of a real interval it is necessarily a monotonic function. And a decreasing homeomorphism of an interval is continuously reversible if and only if it is an involution. And in the case of increasing homeomorphisms, Jarczyk [4] had a result by Theorem 2.1 below. It is well-known how to obtain all continuous solutions $\gamma : X \to X$ of the conjugacy equation

$$\gamma(F(x)) = G(\gamma(x)),$$

where $F, G : X \to X$ are given continuous functions, F has no fixed points and is invertible. Similarly, also homeomorphic solutions can be found whenever F and G are, in addition, homeomorphisms. Below it is shown that in the case where $G = F^{-1}$ we may additionally require γ to be an involution.

Theorem 2.1 [4]**.** *Every homeomorphism of a real interval having no fixed points is continuously reversible. More exactly: if F is a homeomorphism of a real interval X and*

$$F(x) \neq x \quad \textit{for } x \in X,$$

then there is a decreasing continuous involution $\gamma : X \to X$ satisfying the equation

$$F^{-1} = \gamma^{-1} \circ F \circ \gamma.$$

Now we have following proposition from Theorem 2.1. Actually the proof is a slight modification of that of Theorem 2.1.

Proposition 2.2. *Every homeomorphism of a closed bounded interval having exactly two fixed points is continuously reversible.*

Remark that Proposition 2.2 provides the construction of all decreasing continuous involutions satisfying

$$F^{-1} = \gamma^{-1} \circ F \circ \gamma.$$

Indeed, let $\gamma : [a, b] \to [a, b]$ be such a function and assume, for instance, that $F(x) > x$ for $x \in (a, b)$. Then there is a unique $x_0 \in (a, b)$ such that $\gamma(x_0) = x_0$ and a unique $y_0 \in (x_0, F(x_0))$ with $\gamma(y_0) = F^{-1}(y_0)$. Put $\widetilde{\gamma} = \gamma|_{[x_0, y_0]}$. Since the function constructed in the proof of Proposition 2.2 is the unique extension of $\widetilde{\gamma}$ on $[a, b]$ to a decreasing continuous involution satisfying $F^{-1} = \gamma^{-1} \circ F \circ \gamma$, it coincides with γ.

For homeomorphisms on the closed interval with three fixed points the situation is more complicated. Let $a < c < b$ and $F : [a, b] \to [a, b]$ be a homeomorphism whose fixed point set is $\{a, b, c\}$. Let $\gamma : [a, b] \to [a, b]$ be a continuous involution satisfying $F^{-1} = \gamma^{-1} \circ F \circ \gamma$. Then γ is decreasing and

$$\gamma(a) = b, \quad \gamma(b) = a \quad \text{and} \quad \gamma(c) = c.$$

Now for $a < x < c$, assume $\gamma(x) < F(\gamma(x))$. Then we have $\gamma(F(\gamma(x))) < x$ because γ is an involution. Therefore, we get $F^{-1}(x) < x$. As a consequence if $x < F(x)$ for $a < x < c$ and $x > F(x)$ for $c < x < b$, then F cannot be continuously reversible.

Let $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ be the circle and let $\pi : \mathbb{R} \to S^1$ be the covering map defined by $\pi(x) = e^{2\pi i x}$.

For two points $p, q \in S^1$, by a closed arc $[p, q]$ means the set $\{\pi(t), t \in [t_1, t_2]\}$, where $t_1, t_2 \in \mathbb{R}$ are such that $\pi(t_1) = p$, $\pi(t_2) = q$ and $0 < t_2 - t_1 < 1$. An open arc (p, q) means the set $[p, q] - \{p, q\}$.

For homeomorphisms of the circle we have the following result.

Theorem 2.3. *Every homeomorphism of the circle, which has only one periodic orbit, is continuously reversible: More precisely, if f is a homeomorphism of the circle such that there exists an orbit* $\{p_1, p_2, \cdots, p_n\}$ *of a periodic point* p_1 *of period* n *and* $f^n(x) \neq x$ *for* $x \in S^1/\{p_1, p_2, \cdots, p_n\}$, *then f is continuously reversible.*

Proof. First assume $n = 1$. Then p_1 is the only fixed point of f. Let F be a lifting of f. Then we can assume that $F(a) = a$ with $\pi(a) = p_1$ for some $a \in \mathbb{R}$. The map F is a homeomorphism of $[a, a+1]$, whose only fixed points are a and $a + 1$.

By Proposition 2.2, there exists a decreasing continuous involution $\gamma : [a, a+1] \to [a, a+1]$ such that $F^{-1}(t) = \gamma^{-1} F \gamma(t)$ for $a \leq t \leq a+1$. Since $\gamma(a) = a + 1$ and $\gamma(a+1) = a$ we get

$$\pi\gamma(a) = \pi(a+1) \quad \text{and} \quad \pi\gamma(a+1) = \pi(a),$$

hence $\pi\gamma(a) = \pi\gamma(a+1)$.

One can have a continuous circle map $\alpha : S^1 \to S^1$ satisfying

$$\alpha(x) = \pi\gamma\pi^{-1}(x) \quad \text{for } \pi^{-1}(x) \in [a, a+1].$$

By the following commuting diagram:

$$\begin{tikzcd}
{[a,\ a+1]} \arrow[r, "\gamma"] \arrow[d, "\pi"'] & {[a,\ a+1]} \arrow[r, "F"] \arrow[d, "\pi"'] & {[a,\ a+1]} \arrow[r, "\gamma"] \arrow[d, "\pi"'] & {[a,\ a+1]} \arrow[d, "\pi"'] \\
S^1 \arrow[r, "\alpha"'] & S^1 \arrow[r, "f"'] & S^1 \arrow[r, "\alpha"'] & S^1
\end{tikzcd}$$

we have

$$\alpha f \alpha(x) = \pi\gamma F \gamma \pi^{-1}(x) = \pi\gamma^{-1} F \gamma \pi^{-1}(x) = \pi F^{-1} \pi^{-1}(x) = f^{-1}(x).$$

Clearly α is an involution, and hence f is continuously reversible.

Next assume $n \geq 2$ and $p_i = \pi(t_i)$, where $t_1, t_2, \cdots, t_n \in \mathbb{R}$ are such that $t_1 < t_2 < \cdots < t_n < t_{n+1}$. Then f^n is a homeomorphism of the arc $[p_1, p_2]$ having no fixed points in (p_1, p_2). By Proposition 2.2, there exists a continuous involution $\alpha : [p_1, p_2] \to [p_1, p_2]$ such that

$$f^{-n}(x) = \alpha f^n \alpha(x) \quad \text{for } x \in [p_1, p_2] \tag{1}$$

with $\alpha(p_1) = p_2$ and $\alpha(p_2) = p_1$.

We know that

$$S^1 = \bigcup_{i=1}^{n} f^{-i}([p_1, p_2])$$

and

$$f^{-i}([p_1, p_2]) \cap f^{-j}([p_1, p_2]) = \varnothing \quad \text{or} \quad \{p_k\}$$

for some k if $i \neq j$, $1 \leq i, j \leq n$.

For $i = 1, 2, \cdots, n-1$ if $x \in f^{-i}([p_1, p_2])$, define

$$\alpha(x) = f^i \alpha f^i(x). \tag{2}$$

Then α is continuous on $f^{-i}([p_1, p_2])$ for each $i = 1, 2, \cdots, n$ and so if α is well-defined, then α is continuous on S^1.

To prove that α is well-defined, let

$$p_k \in f^{-i}([p_1, p_2]) \cap f^{-j}([p_1, p_2])$$

for $1 \leq i, j \leq n, i \neq j$. Then we want to prove $f^i \alpha f^i(p_k) = f^j \alpha f^j(p_k)$. Since we know

$$f^i(p_k) = p_1 \text{ and } f^j(p_k) = p_2$$

or

$$f^i(p_k) = p_2 \text{ and } f^j(p_k) = p_1$$

we may assume

$$f^i(p_k) = p_1 \text{ and } f^j(p_k) = p_2.$$

Then we have

$$f^i \alpha f^i(p_k) = f^i \alpha(p_1) = f^i(p_2) = f^{i+j}(p_k),$$

and

$$f^j \alpha f^j(p_k) = f^j \alpha(p_2) = f^j(p_1) = f^{i+j}(p_k).$$

Therefore,

$$\alpha(p_k) = f^i \alpha f^i(p_k) = f^j \alpha f^j(p_k)$$

is well-defined.

Note that (1) implies that

$$\alpha(x) = f^n \alpha f^n(x) \quad \text{for } x \in [p_1, p_2],$$

and hence by (2) we have

$$\alpha(x) = f^{kn+i} \alpha f^{kn+i}(x) \tag{3}$$

for $x \in f^{-i}[p_1, p_2] = f^{-kn-i}[p_1, p_2]$ and for each $k = 1, 2, \cdots$.

Now we will show that α is an involution. Let $x \in f^{-i}([p_1, p_2])$ for $i = 1, 2, \cdots, n$. Then we have

$$\alpha(x) = f^i \alpha f^i(x) \in f^i([p_1, p_2]) = f^{i-n}([p_1, p_2]),$$

$$\alpha\alpha(x) = f^{n-i}\alpha f^{n-i} f^i \alpha f^i(x) = f^{n-i}\alpha f^n \alpha f^i(x) = f^{n-i} f^{-n} f^i(x) = x.$$

Hence α is an involution.

Also if $x \in f^{-i}([p_1, p_2])$ for $i = 1, 2, \cdots, n$, then

$$f\alpha(x) \in f^{i+1}([p_1, p_2]) = f^{i+1-2n}([p_1, p_2])$$

and hence by (3) we have

$$\alpha f \alpha(x) = f^{2n-i-1}\alpha f^{2n-i-1} f\alpha(x) = f^{2n-i-1}\alpha f^{2n-i-1} f f^i \alpha f^i(x)$$

$$= f^{2n-i-1}\alpha f^{2n-i-1} f^{i+1}\alpha f^i(x) = f^{2n-i-1} f^{-2n} f^i(x) = f^{-1}(x).$$

Thus f is reversible with respect to α. Therefore, f is continuously reversible.

A rotation map on S^1 is a map $f : S^1 \to S^1$ defined by $f(z) = \beta z$ for some $\beta \in S^1$. In this case $\beta = e^{2\pi i \rho}$, where ρ is the rotation number of a lifting F of f, and it is well-known that if $\rho(F) = \rho$ is an irrational number, then

$$\omega_f(x) = S^1 \quad \text{for all } x \in S^1,$$

where ω_f is the set of all ω-limit points of f.

Lemma 2.4. *Every rotation map is continuously reversible.*

Proof. The proof is clear.

Let f and g be continuous maps on the circle S^1. We say that two maps f and g are topologically conjugate if there exists a homeomorphism $h : S^1 \to S^1$ such that $g = h^{-1} f h$.

Lemma 2.5. *If f is continuously reversible and g is topologically conjugate to f, then g is also continuously reversible.*

Proof. The proof is clear.

Let f be a continuous map from the circle S^1 into itself. Then the family of iterates $\{f^n \mid n = 1, 2, \cdots\}$ of f is said to be *equicontinuous* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in S^1$, $|x - y| < \delta$ implies $|f^n(x) - f^n(y)| < \varepsilon$ for all $n \in \mathbb{N}$.

The following lemma is appeared in [2].

Lemma 2.6. *Let f be a continuous map of the circle. Then $\{f^n \mid n = 1, 2, \cdots\}$ is equicontinuous if and only if one of the following holds*:

(i) $\deg(f) = 0$ *and* $\bigcap_{n=1}^{\infty} f^n(S^1) = F_2$, *where* F_2 *is the fixed point set of* f^2;

(ii) $\deg(f) = 1$ *and f is topologically conjugate to a rotation map*;

(iii) $\deg(f) = -1$ *and* f^2 *is the identity*.

Theorem 2.7. *If f is a homeomorphism of the circle whose iterates are equicontinuous, then f is continuously reversible.*

Proof. If $\deg(f) = 1$, then by Lemma 2.4, Lemma 2.5 and Lemma 2.6(ii) f is continuously reversible. If $\deg(f) = -1$, then Lemma 2.6(iii) says that f is an *involution* so that f is continuously reversible.

3. Reversibility of a Positively Equicontinuous Flow

A family of continuous maps $f_t : S^1 \to S^1$, $t \in \mathbb{R}$ is called a *flow* on S^1 if

$$f_t \circ f_s = f_{t+s} \quad \text{for } t, s \in \mathbb{R}.$$

A flow $\{f_t\}_{t \in \mathbb{R}}$ is continuous if the mapping $f : \mathbb{R} \times S^1 \to S^1$ defined by $(t, z) \mapsto f_t(z)$ is continuous [6, 7].

Let $\{f_t\}_{t \in \mathbb{R}}$, $\{g_t\}_{t \in \mathbb{R}}$, be two flows on the circle S^1. We say that two flows $\{f_t\}_{t\in\mathbb{R}}$ and $\{g_t\}_{t\in\mathbb{R}}$ are topologically conjugate if there exists a homeomorphism $\varphi : S^1 \to S^1$ such that $g_t = \varphi^{-1} \circ f_t \circ \varphi$ for all $t \in \mathbb{R}$.

A continuous flow $\{f_t\}_{t\in\mathbb{R}}$ of homeomorphisms is said to be *positively equicontinuous* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $z, w \in S^1$, $d(z, w) < \delta$ implies $d(f_t(z), f_t(w)) < \varepsilon$ for all $t \geq 0$.

In 1999, Bae et al. [1] studied the continuity of the rotation map which maps from the set of degree one circle maps to the family of subsets of reals, and applied their results to prove that any equicontinuous flow on the circle is topologically conjugate to a rotation flow.

Theorem 3.1 [1]. *Let* $\{f_t\}_{t\in\mathbb{R}}$ *be a positively equicontinuous flow of homeomorphisms on* S^1. *Then either* $\{f_t\}_{t\in\mathbb{R}}$ *is the trivial flow, or* $\{f_t\}_{t\in\mathbb{R}}$ *is topologically conjugate to a continuous flow of rotation maps of* S^1. *Indeed,*

(i) *for all* $t \in \mathbb{R}$, $f_t = id_{S^1}$; *or*

(ii) *there exists a homeomorphism* $\varphi : S^1 \to S^1$ *and a continuous map* $c : \mathbb{R} \to S^1$ *such that*

$$\varphi^{-1} \circ f_t \circ \varphi(z) = c(t)z, \quad \text{for } t \in \mathbb{R} \text{ and } z \in S^1, \tag{$*$}$$

$$c(s + t) = c(s)c(t), \quad \text{for } s, t \in \mathbb{R}. \tag{$**$}$$

According to Theorem 3.1, a positively equicontinuous flow of homeomorphisms on S^1 is topologically conjugate to a continuous flows of rotation maps of S^1. Therefore, by applying Lemma 2.4 and Lemma 2.5, we can show that the flow is continuously reversible.

Theorem 3.2. *A positively equicontinuous flow is continuously reversible. More precisely if* $\{f_t\}_{t\in\mathbb{R}}$ *is a positively equicontinuous flow of homeomorphisms on* S^1, *then there is a continuous involution* $\alpha : S^1 \to S^1$ *such that* $f_{-t} = \alpha \circ f_t \circ \alpha$ *for all* $t \in \mathbb{R}$.

Proof. Let $\{f_t\}_{t\in\mathbb{R}}$ be a positively equicontinuous flow of homeomorphisms on S^1. We may assume that $\{f_t\}_{t\in\mathbb{R}}$ is not a trivial flow. Then by Theorem 3.1, there exists a homeomorphism $\varphi : S^1 \to S^1$ and a continuous map $c : \mathbb{R} \to S^1$ such that (∗) and (∗∗) in Theorem 3.1 hold.

Let γ be the reflexion on S^1 about x-axis, that is, $\gamma(e^{i\theta}) = e^{-i\theta}$. Then by Lemma 2.4 and (∗), we have

$$\gamma\varphi^{-1}f_t\varphi\gamma(z) = c(t)^{-1}z$$

for $t \in \mathbb{R}$ and $z \in S^1$. Also by (∗∗) we have $c(t)^{-1} = c(-t)$, for all $t \in \mathbb{R}$ because $c(0) = 1$. By combining above equalities, we get

$$\gamma\varphi^{-1}f_t\varphi\gamma(z) = c(-t)z = \varphi^{-1}f_{-t}\varphi(z).$$

Therefore as in the proof of Lemma 2.5, by letting $\alpha = \varphi\gamma\varphi^{-1}$, we have $f_{-t} = \alpha f_t\alpha$ and

$$\alpha^2 = \varphi\gamma\varphi^{-1}\varphi\gamma\varphi^{-1} = \varphi\gamma^2\varphi^{-1} = id.$$

Consequently the flow $\{f_t\}_{t\in\mathbb{R}}$ is continuously reversible.

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