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FUZZY SU-SUBALGEBRAS AND FUZZY SU-IDEALS

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Abstract

In this paper, the notions of fuzzy SU-subalgebra and fuzzy SU-ideal in SU-algebra are introduced and some of their properties are investigated. Moreover, we have discussed the relations between fuzzy SU-subalgebras and fuzzy SU-ideals of SU-algebras.

1. Introduction

The study of BCI/BCK-algebras was initiated by Iseki in 1966 as a generalization of a concept of set-theoretic difference and propositional calculus [1]. In 1983, Hu and Li introduced the notion of a BCH-algebra which is a generalization of BCI/BCK-algebras [2]. Recently, a new algebraic structure was presented as SU-algebra and a concept of ideal in SU-algebra [3]. In 1965, Zadeh defined fuzzy subset of a non-empty set as a collection of objects with grade of membership in continuum, with each object being assigned a value between 0 and 1 by a membership function [4].

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In 1991, Xi applied the concept of fuzzy set in BCK-algebras and defined fuzzy subalgebra on BCK-algebras [5]. In 2011, Mostafa et al. [6] introduced the notion of fuzzy KU-ideals of KU-algebras and they also investigated several basic properties of fuzzy KU-ideals of KU-algebras. The aim of this work is to introduce the concept of fuzzy SU-subalgebras and fuzzy SU-ideals of SU-algebras. Furthermore, we investigate some of their properties.

2. Preliminaries

We give some definitions and results which will be used in other sections.

Definition 2.1 [3]**.** A SU-algebra is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(1) $((x * y) * (x * z)) * (y * z) = 0$,

(2) $x * 0 = x$,

(3) if $x * y = 0$ implies $x = y$

for all $x, y, z \in X$.

From now on, a binary operation "$*$" will be denoted by juxtaposition.

Example 2.2 [3]**.** Let $X = \{0, 1, 2, 3\}$ be a set in which operation $*$ is defined by the following:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then X is a SU-algebra.

Theorem 2.3 [3]**.** *Let X be a SU-algebra. Then the following results hold for all* $x, y, z \in X$:

(1) $xx = 0$,

(2) $xy = yx$,

(3) $0x = x$,

(4) $(xy)z = (xz)y$,

(5) $x(yz) = z(yx)$,

(6) $(xy)z = x(yz)$.

Theorem 2.4 [3]**.** *Let X be a SU-algebra. A nonempty subset I of X is called a SU-subalgebra of X if* $xy \in I$ *for all* $x, y \in I$.

Definition 2.5 [3]**.** Let *X* be a SU-algebra. A nonempty subset *I* of *X* is called an *ideal* of *X* if it satisfies the following properties:

(1) $0 \in I$,

(2) if $(xy)z \in I$ and $y \in I$ imply $xz \in I$

for all $x, y, z \in X$.

Theorem 2.6 [3]**.** *Let X be a SU-algebra. Then X is a BCI-algebra.*

Theorem 2.7 [7]**.** *Let X be a BCI-algebra. A nonempty subset A of X is called an ideal of X if it satisfies the following properties*:

(1) $0 \in A$,

(2) *if* $xy \in A$ *and* $y \in A$ *imply* $x \in A$,

for all $x, y \in X$.

Definition 2.8 [4]**.** Let *X* be a set. A fuzzy set μ in *X* is a function $\mu : X \to [0, 1]$.

Definition 2.9 [7]**.** Let *X* be a BCI-algebra. A fuzzy set μ in *X* is called a *fuzzy BCI-ideal* of *X* if it satisfies the following properties:

(F_1) $\mu(0) \geq \mu(x)$,

(F_2) $\mu(x) \geq \min\{\mu(xy), \mu(y)\}$

for all $x, y \in X$.

3. Fuzzy SU-subalgebras

We first give the definition of fuzzy SU-subalgebra and provide some of its properties.

Definition 3.1. Let X be a SU-algebra. A fuzzy set μ in X is called *fuzzy SU-subalgebra* of X if $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

The set $\text{Im}(\mu) = \{t \in [0, 1] | \mu(x) = t$ for some $x \in X\}$ is called the *image set* of μ.

Definition 3.2. Let X be a SU-algebra and μ be a fuzzy SU-subalgebra of X. The set $\mu_t = \{x \in X | \mu(x) \geq t\}$, where $t \in [0, 1]$ is fixed, is called a *level SU-subalgebra* of μ. Clearly, $\mu_t \subseteq \mu_s$ whenever $s, t \in [0, 1]$ with $t > s$.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a set in which operation $*$ is defined as Example 2.2. Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(0) = 1$, $\mu(1) = 0.5$ and $\mu(2) = \mu(3) = 0$. Then μ is a fuzzy SU-subalgebra of X.

Lemma 3.4. *Let X be a SU-algebra. If μ is a fuzzy SU-subalgebra of X, then $\mu(0) \geq \mu(x)$ for all $x \in X$.*

Proof. Let $x \in X$. Since $xx = 0$, $\mu(0) = \mu(xx) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$. Thus, $\mu(0) \geq \mu(x)$.

Theorem 3.5. *Let X be a SU-algebra and μ be a fuzzy set of X. Then μ_t is a SU-subalgebra of X for any $t \in [0, 1]$ and $\mu_t \neq \phi$ if and only if μ is a fuzzy SU-subalgebra of X.*

Proof. Let $t \in [0, 1]$ and $\mu_t \neq \phi$. Let μ_t be a SU-subalgebra of X. Assuming μ is not a fuzzy SU-subalgebra of X, there exist $x, y \in X$ such that $\mu(xy) < \min\{\mu(x), \mu(y)\}$. Letting $\alpha = \frac{1}{2}(\mu(xy) + \min\{\mu(x), \mu(y)\})$, we have $\mu(xy) < \alpha < \min\{\mu(x), \mu(y)\}$ which implies $\alpha \in [0, 1]$, $x \in \mu_\alpha$, $y \in \mu_\alpha$ and $xy \notin \mu_\alpha$, then $\mu_\alpha \neq \phi$. Hence μ_α is SU-subalgebra of X, we have $xy \in \mu_\alpha$, which is a contradiction. Thus, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Therefore, μ is a fuzzy SU-subalgebra of X.

Conversely, let μ be a fuzzy SU-subalgebra of X. For any $x, y \in \mu_t$, then $\mu(xy) \geq \min\{\mu(x), \mu(y)\} \geq t$, we have $xy \in \mu_t$. Thus, μ_t is a SU-subalgebra of X.

Theorem 3.6. *Let X be a SU-algebra and A be a SU-subalgebra of X. Then for any $t \in (0, 1]$, there exists a fuzzy SU-subalgebra μ of X such that $\mu_t = A$.*

Proof. Let A be a SU-subalgebra of X and μ be a fuzzy set of X defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in A; \\ 0, & \text{if } x \notin A; \end{cases}$$

where $t \in (0, 1]$ is fixed.

We will show that μ is fuzzy SU-subalgebra of X. Let $x, y \in X$. If $x, y \in A$, then we have $xy \in A$ and $\mu(x) = \mu(y) = \mu(xy) = t$. Hence $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$. Assume that either x or y is not in A. We have $\min\{\mu(x), \mu(y)\} = 0$. Hence $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$. Then μ is fuzzy SU-subalgebra of X. It is clear that $\mu_t = A$ which completes the proof.

Theorem 3.7. *Let X be a SU-algebra and μ be a fuzzy SU-subalgebra of X. If μ_s, μ_t for some $0 \leq s < t \leq 1$ are level SU-subalgebras of μ, then $\mu_s = \mu_t$ if and only if $\{x \in X \mid s \leq \mu(x) < t\} = \phi$.*

Proof. Let μ_s, μ_t be level SU-subalgebras of μ for some $0 \leq s < t \leq 1$. Let $\mu_s = \mu_t$. Suppose $\{x \in X \mid s \leq \mu(x) < t\} \neq \phi$. There exists $y \in X$ such that $s \leq \mu(y) < t$, then $y \in \mu_s$ but $y \notin \mu_t$. Hence $\mu_s \neq \mu_t$, which is a contradiction.

Conversely, let $\{x \in X \mid s \leq \mu(x) < t\} = \phi$. It is obvious that $\mu_t \subseteq \mu_s$. If $x \in \mu_s$, then we have $\mu(x) \geq s$. Since $\{x \in X \mid s \leq \mu(x) < t\} = \phi$, we have $\mu(x) \geq t$, $x \in \mu_t$, thus, $\mu_s \subseteq \mu_t$. Therefore, $\mu_s = \mu_t$.

Remark. If $t_1 = \mu(0)$, then μ_{t_1} is the smallest level SU-subalgebra. Hence we have $\mu_{t_1} \subset \mu_{t_2} \subset \mu_{t_3} \subset \cdots \subset \mu_{t_n} = X$, where $\mathrm{Im}(\mu) = \{t_1, t_2, t_3, ..., t_n\}$ with $t_1 > t_2 > t_3 > \cdots > t_n$, where n is positive integer.

Note that in Example 3.3, if $t_1 = 1$, then we have $\mu_{t_1} = \{0\}$. If $t_2 = 0.5$, then we have $\mu_{t_2} = \{0, 1\}$. If $t_3 = 0$, then we have $\mu_{t_3} = \{0, 1, 2, 3\} = X$. Therefore, $\mu_{t_1} \subset \mu_{t_2} \subset \mu_{t_3} = X$.

Corollary 3.8. *Let X be a SU-algebra and* μ *be a fuzzy SU-subalgebra of X. If* $\mathrm{Im}(\mu) = \{t_1, t_2, t_3, ..., t_n\}$ *with* $t_1 > t_2 > t_3 > \cdots > t_n$, *then the set* $\{\mu_{t_i} \mid 1 \leq i \leq n\}$ *is the set of all level SU-subalgebras of* μ.

Proof. Let $\beta \in [0, 1]$ and $\beta \notin \mathrm{Im}(\mu)$. We will show that μ_β belongs to the set $\{\mu_{t_i} \mid 1 \leq i \leq n\}$. If $t_1 < \beta$, then $\mu_\beta \subseteq \mu_{t_1}$. Since μ_{t_1} is smallest level SU-subalgebra, we have $\mu_\beta = \phi$. If $t_i < \beta < t_{i+1}$, then $\{x \in X \mid \beta \leq \mu(x) < t_{i+1}\} = \phi$. From Theorem 3.7, $\mu_{t_{i+1}} = \mu_\beta$. If $\beta < t_n$, then $\mu_{t_n} \subseteq \mu_\beta$. Since $\mu_{t_n} = X$, we have $\mu_\beta = X$. Hence $\mu_{t_n} = \mu_\beta$. Therefore, for any $\beta \in [0, 1]$, the level SU-subalgebra is one of $\{\mu_{t_i} \mid 1 \leq i \leq n\}$.

Theorem 3.9. *Let X be a SU-algebra and* μ *be a fuzzy SU-subalgebra of X with finite image. If* $\mu_s = \mu_t$ *for some* $s, t \in \mathrm{Im}(\mu)$, *then* $s = t$.

Proof. Let $x \in X$ and $\mu_s = \mu_t$ for some $s, t \in \mathrm{Im}(\mu)$. We will show that $s = t$. Assume $s < t$. Since $s \in \mathrm{Im}(\mu)$, there exists $x \in X$ such that

$\mu(x) = s < t$. We have $x \in \mu_s$ and $x \notin \mu_t$. Hence $\mu_s \neq \mu_t$, which is a contradiction. Assume $t < s$. Since $t \in \mathrm{Im}(\mu)$, there exists $x \in X$ such that $\mu(x) = t < s$. We have $x \in \mu_t$ and $x \notin \mu_s$. Hence $\mu_t \neq \mu_s$, which is a contradiction. Therefore, $s = t$.

4. Fuzzy SU-ideals of SU-algebras

In this section, we introduce the notions of fuzzy SU-ideal and discuss the related properties.

Definition 4.1. Let X be a SU-algebra. A fuzzy set μ in X is called *fuzzy SU-ideal* of X if it satisfies the following conditions:

(SF_1) $\mu(0) \geq \mu(x)$,

(SF_2) $\mu(xz) \geq \min\{\mu((xy)z), \mu(y)\}$

for all $x, y, z \in X$.

Example 4.2. Let $X = \{0, 1, 2, 3\}$ be a set in which operation $*$ is defined as Example 2.2. A fuzzy set is defined as Example 3.3, then μ is a fuzzy SU-ideal of X.

Definition 4.3. *Let X be a SU-algebra and* μ *be a fuzzy SU-ideal of X. The set* $\mu_t = \{x \in X \mid \mu(x) \geq t\}$, *where* $t \in [0, 1]$ *is fixed, is called a level SU-ideal of* μ. *Clearly,* $\mu_t \subseteq \mu_s$ *whenever* $s, t \in [0, 1]$ *with* $t > s$.

Theorem 4.4. *Let X be a SU-algebra. If* μ_1, μ_2 *are fuzzy SU-ideals of X, then* $\overline{\mu_2}$ *is a fuzzy SU-ideal of X, where* $\overline{\mu_2}(x) = \min\{\mu_1(x), \mu_2(x)\}$ *for all* $x \in X$.

Proof. Let μ_1, μ_2 be fuzzy SU-ideals of X. Obviously, $\overline{\mu_2}$ is a fuzzy set of X. So $\overline{\mu_2}(0) = \min\{\mu_1(0), \mu_2(0)\} \geq \min\{\mu_1(x), \mu_2(x)\} = \overline{\mu_2}(x)$. Thus, $\overline{\mu_2}(0) \geq \overline{\mu_2}(x)$ for all $x \in X$. Now we will show that $\overline{\mu_2}(xz) \geq \min\{\overline{\mu_2}((xy)z), \overline{\mu_2}(y)\}$ for all $x, y, z \in X$. We have

$$\overline{\mu_2}(xz) = \min\{\mu_1(xz), \mu_2(xz)\}$$
$$\geq \min\{\min\{\mu_1((xy)z), \mu_1(y)\}, \min\{\mu_2((xy)z), \mu_2(y)\}\}$$
$$= \min\{\min\{\mu_1((xy)z), \mu_2((xy)z)\}, \min\{\mu_1(y), \mu_2(y)\}\}$$
$$= \min\{\overline{\mu_2}((xy)z), \overline{\mu_2}(y)\}.$$

Thus, $\overline{\mu_2}(xz) \geq \min\{\overline{\mu_2}((xy)z), \overline{\mu_2}(y)\}$ for all $x, y, z \in X$. Therefore, $\overline{\mu_2}$ is a fuzzy SU-ideal of *X*.

In general, we get the following result.

Corollary 4.5. *Let X be a SU-algebra. If* $\mu_1, \mu_2, ..., \mu_n$ *are fuzzy SU-ideals of X, then* $\overline{\mu_n}$ *is a fuzzy SU-ideal of X, where* $\overline{\mu_n}(x) = \min\{\mu_1(x), \mu_2(x), ..., \mu_n(x)\}$ *for all* $x \in X$ *and n is positive integer.*

Theorem 4.6. *Let X be a SU-algebra and* μ *be a fuzzy set of X. Then* μ_t *is a SU-ideal of X for any* $t \in [0, 1]$ *and* $\mu_t \neq \phi$ *if and only if* μ *is a fuzzy SU-ideal of X.*

Proof. Let $t \in [0, 1]$ be such that $\mu_t \neq \phi$. Let μ_t be a SU-ideal of *X* and let $x, y, z \in X$. Assuming $\mu(0) \geq \mu(x)$ is not true, there exists $y \in X$ such that $\mu(0) < \mu(y)$ and letting $\beta = \frac{1}{2}(\mu(0) + \mu(y))$, we have $\mu(0) < \beta < \mu(y)$ which implies $\beta \in [0, 1]$, $y \in \mu_\beta$ and $0 \notin \mu_\beta$, then $\mu_\beta \neq \phi$. Hence μ_β is SU-ideal of *X*, we have $0 \in \mu_\beta$, which is a contradiction. Thus, $\mu(0) \geq \mu(x)$ for all $x \in X$. Assuming $\mu(xz) \geq \min\{\mu((xy)z), \mu(y)\}$ is not true, there exist $a, b, c \in X$ such that $\mu(ac) < \min\{\mu((ab)c), \mu(b)\}$ and letting $\alpha = \frac{1}{2}(\mu(ac) + \min\{\mu((ab)c), \mu(b)\})$, we have $\mu(ac) < \alpha < \min\{\mu((ab)c), \mu(b)\}$ which implies $\alpha \in [0, 1]$, $(ab)c \in \mu_\alpha$, $b \in \mu_\alpha$ and $ac \notin \mu_\alpha$, then $\mu_\alpha \neq \phi$. Hence μ_α is SU-ideal of *X*, we have $ac \in \mu_\alpha$, which is a contradiction. Thus, $\mu(xz) \geq \min\{\mu((xy)z), \mu(y)\}$ for all $x, y, z \in X$. Therefore, μ is a fuzzy SU-ideal of *X*.

Conversely, let μ be a fuzzy SU-ideal of X and let $x, y, z \in X$. Since $\mu_t \neq \phi$, there exists $y \in X$ such that $y \in \mu_t$. Since μ is a fuzzy SU-ideal of X, $\mu(0) \geq \mu(y) \geq t$. Hence $0 \in \mu_t$. Letting $(xy)z \in \mu_t$ and $y \in \mu_t$, we have $\mu(xz) \geq \min\{\mu((xy)z), \mu(y)\} \geq t$, hence $xz \in \mu_t$. Therefore, μ_t is a SU-ideal of X.

Similarly as Theorem 3.7, we prove

Theorem 4.7. *Let X be a SU-algebra and* μ *be a fuzzy SU-ideal of X. If* μ_s, μ_t *for some* $0 \leq s < t \leq 1$ *are level SU-ideals of* μ, *then* $\mu_s = \mu_t$ *if and only if* $\{x \in X \mid s \leq \mu(x) < t\} = \phi$.

Theorem 4.8. *Let X be a SU-algebra and* μ *be a fuzzy set of X. Then* μ *is a fuzzy SU-subalgebra of X if and only if* μ *is a fuzzy SU-ideal of X.*

Proof. Let μ be a fuzzy SU-subalgebra of X. Let $x, y, z \in X$. By Lemma 3.4, we have $\mu(0) \geq \mu(x)$. Since μ is a fuzzy SU-subalgebra of X, $\mu(xz) \geq \min\{\mu(xy)z, \mu(y)\}$. Thus, μ is a fuzzy SU-ideal of X. Conversely, assume μ is a fuzzy SU-ideal of X. Let $x, y, z \in X$. By Theorem 2.3, we have $(xy)y = x(yy) = x$. We put z with y in (SF_2), we have $\mu(xy) \geq \min\{\mu((xy)y), \mu(y)\} = \min\{\mu(x), \mu(y)\}$. Thus, μ is a fuzzy SU-subalgebra of X.

Theorem 4.9. *Let X be a SU-algebra and* μ *be a fuzzy set of X. If* μ *is a fuzzy SU-ideal of X, then* μ *is a fuzzy BCI-ideal of X.*

Proof. Let $x, y, z \in X$. Assuming μ is a fuzzy SU-ideal of X, we have $\mu(0) \geq \mu(x)$. We put $z = 0$ in (SF_2), we have $\mu(x) \geq \min\{\mu(xy), \mu(y)\}$. Thus, μ is a fuzzy BCI-ideal of X.

Corollary 4.10. *Let X be a SU-algebra and* μ *be a fuzzy set of X. If* μ *is a fuzzy SU-subalgebra of X, then* μ *is a fuzzy BCI-ideal of X.*

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