Far East Journal of Mathematical Sciences (FJMS)
Volume 74, Number 1, 2013, Pages 119-138
Published Online: March 2013
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

DERIVATIONS OF BOOLEAN ALGEBRAS

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Abstract

In this paper, we introduce a derivation of Boolean algebra, namely Boolean derivation, and we also investigate some related properties. Moreover, we show that the fixed set and the kernel of Boolean algebras are ideals in Boolean algebra. Finally, we prove that the set of Boolean derivations and Boolean algebra are isomorphic.

1. Introduction

Boolean algebras are special lattices which are useful in the study of logic, both digital computer logic and that of human thinking, and of switching circuits. This latter application was initiated by C. E. Shannon, who showed that fundamental properties of electrical circuits of bistable elements can be represented by using Boolean algebras.

The notion of derivation, introduced from the analytic theory, is helpful

© 2013 Pushpa Publishing House
2010 Mathematics Subject Classification: 06E25, 03G05.
Keywords and phrases: derivation, Boolean algebra.
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Submitted by K. K. Azad
Received October 8, 2012

to the research of structure and property in algebraic system. Several authors [2, 3, 7, 11] studied derivations in rings and near rings. In 2004, Jun and Xin [6] applied the notion of derivation in ring and near-ring theory to BCI-algebras. Further, in 2009, Prabprayak and Leerawat [12] also studied the derivation of BCC-algebra. In 2010, Alshehri [1] introduced the notion of derivation for an MV-algebra. In 2008, Xin et al. [13] studied derivation of lattice and obtained some related properties. In 2011, Harmaitree and Leerawat [5] studied *f*-derivation of lattice and investigated some of its properties. In this paper, we introduce a new derivation on Boolean algebra and investigate some related properties.

2. Preliminaries

We first recall some definitions and results which are essential in this paper.

Definition 2.1 [8]**.** A Boolean algebra $(B, \wedge, \vee, ', 0, 1)$ is a nonempty set B with two binary operations $\wedge$ (called "*meet*" or "*and*"), $\vee$ (called "*join*" or "*or*"), a unary operation $'$ (called *complement*), and two elements 0 and 1 such that for all elements x, y, z of B, the following axioms holds:

(i) $x \wedge x = x,\ x \vee x = x$;

(ii) $x \wedge y = y \wedge x,\ x \vee y = y \vee x$;

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z,\ x \vee (y \vee z) = (x \vee y) \vee z$;

(iv) $x = x \wedge (x \vee y),\ x = x \vee (x \wedge y)$;

(v) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

(vi) $x \wedge 1 = x$ and $0 \vee x = x$ for all $x \in B$;

(vii) for all $x \in B$, there is $x' \in B$ such that $x \wedge x' = 0$ and $x \vee x' = 1$.

Example 2.2. Let $B = \{0, a, b, 1\}$ and $\wedge, \vee$ are two binary operations and complement defined as follows:

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

x	x'
0	1
a	b
b	a
1	0

Therefore, $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra.

Example 2.3. Let $B = \{0, a, 1\}$ and $\wedge, \vee$ are two binary operations defined as follows:

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\vee$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

x	x'
0	1
a	a
1	0

Thus $(B, \wedge, \vee, ', 0, 1)$ is not a Boolean algebra.

Remark 1. From now on, B denotes a set with the two binary operations $\wedge$ and $\vee$, with 0 and 1 and the unary operation complement $'$, that is $B = (B, \wedge, \vee, ', 0, 1)$.

Lemma 2.4 [8]**.** *Every Boolean algebras are modular*.

Definition 2.5 [8]**.** Let B be a Boolean algebra. Define a binary relation "$\leq$" on B as follows: $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Remark 2. By Definition 2.1(vi), we have $x \leq 1$ and $0 \leq x$ for all $x \in B$.

Lemma 2.6 [9]**.** *Let B be a Boolean algebra*. *Define the binary relation* "$\leq$" *as Definition* 2.5. *Then* $(B, \leq)$ *is a poset and for any* $x, y \in B$, $x \wedge y$ *is the greatest lower bound* (*g.l.b*) *of* $\{x, y\}$ *and* $x \vee y$ *is the least upper bound* (*l.u.b.*) *of* $\{x, y\}$.

Remark 3. $x < y$ means $x \leq y$ and $x \neq y$.

Lemma 2.7. *Let B be a Boolean algebra and $x, y \in B$. If $x \leq y$ and $y \leq x$, then $x = y$.*

Proof. Let $x, y \in B$ be such that $x \leq y$ and $y \leq x$. Since $x \leq y$ and $y \leq x$, we have $x \wedge y = x$ and $y \wedge x = y$. By Definition 2.1(ii), we get $x = x \wedge y = y \wedge x = y$.

Lemma 2.8 [8]**.** *Let B be a Boolean algebra and $x, y, z \in B$. Then the following hold*:

(i) *if $x \leq y$, then $x \wedge z \leq y \wedge z$ and $x \vee z \leq y \vee z$*;

(ii) $(x \wedge y)' = x' \vee y'$ *and* $(x \vee y)' = x' \wedge y'$;

(iii) $x \leq y$ *if and only if* $y' \leq x'$.

Theorem 2.9 [8]**.** *Let B be a Boolean algebra and $x, y \in B$. Then the following conditions are equivalent*:

(i) $x \leq y$;

(ii) $x \wedge y' = 0$;

(iii) $x' \vee y = 1$;

(iv) $x \wedge y = x$;

(v) $x \vee y = y$.

Theorem 2.10 [9]**.** *Let B be a Boolean algebra and $x, y, z \in B$. Then the following hold*:

(i) $x \vee y = 0$ *if and only if* $x = 0$ *and* $y = 0$;

(ii) $x \wedge y = 1$ *if and only if* $x = 1$ *and* $y = 1$;

(iii) $x = 0$ *if and only if* $y = (x \wedge y') \vee (x' \wedge y)$;

(iv) $x = y$ *if and only if* $(x \wedge y') \vee (x' \wedge y) = 0$;

(v) *if $x \vee y = x \vee z$ and $x' \vee y = x' \vee z$, then $y = z$.*

Definition 2.11 [9]**.** If A is any nonempty subset of a Boolean algebra B closed under the operations $\wedge, \vee, '$, then $(A, \wedge_A, \vee_A, '_A, 0, 1)$ is a Boolean algebra, where $\wedge_A, \vee_A, '_A$ are the restrictions of the operations $\wedge, \vee, '$ to the set A. The Boolean algebra A is called a *Boolean subalgebra* of B.

Remark 4. (1) Observe that 0 and 1 must belong to a Boolean subalgebra A of B. For, if $x \in A$, then $x' \in A$. Thus by Definition 2.1(vii), we obtain $0 = x \wedge x' \in A$ and $1 = x \vee x' \in A$.

(2) To show that a subset A of B is closed under the operation $\wedge, \vee, '$, it suffices to show that A is closed either under $\wedge$ and $'$, or under $\vee$ and $'$. For, if A is closed under $\wedge$ and $'$, then by Lemma 2.8(ii) we get for any $x, y \in A$, $x \vee y = (x' \wedge y')' \in A$. Likewise, if A is closed under $\vee$ and $'$, then by Lemma 2.8(ii), we get for any $x, y \in A$, $x \wedge y = (x' \vee y')' \in A$.

Definition 2.12 [8]**.** Let $f : B_1 \to B_2$ be a function from a Boolean algebra B_1 to a Boolean algebra B_2. Then f is called a *Boolean homomorphism* (or *homomorphism*) when:

(i) $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in B_1$;

(ii) $f(x') = (f(x))'$ for all $x \in B_1$.

Definition 2.13. Let $f : B_1 \to B_2$ be a Boolean homomorphism from a Boolean algebra B_1 to a Boolean algebra B_2. Then

(i) f is called an *injective homomorphism* if f is an injective function;

(ii) f is called a *surjective homomorphism* if f is a surjective function;

(iii) f is called an *isomorphism* if f is a bijective function.

Definition 2.14 [8]**.** An ideal is a nonempty subset I of a Boolean algebra B with the properties:

(i) If $x \in I$ and $b \in B$, then $x \wedge b \in I$;

(ii) If $x, y \in I$, then $x \vee y \in I$.

Remark 5. If I_1 and I_2 are ideals of a Boolean algebra B, so is $I_1 \cap I_2$.

3. The Derivations of Boolean Algebras

The following definition introduces the notion of Boolean derivation of Boolean algebras.

Definition 3.1. Let B be a Boolean algebra and $d : B \to B$ be a function. We call d a *Boolean derivation* on B if it satisfies the following conditions: for all $x, y \in B$,

(i) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$;

(ii) $d(x \vee y) = d(x) \vee d(y)$.

Remark 6. We often abbreviate $d(x)$ to dx.

Now we give some examples and some properties for the Boolean derivations of Boolean algebras.

Example 3.2. Let $B = \{0, a, b, 1\}$ and $\wedge, \vee$ are two binary operations defined as Example 2.2. Then B is a Boolean algebra. Define a function d on B by

$$dx = \begin{cases} 0, & \text{if } x = 0, a, \\ b, & \text{if } x = b, 1. \end{cases}$$

Then d is a Boolean derivation of B.

Example 3.3. Let $B = \{0, a, b, 1\}$ and $\wedge, \vee$ are two binary operations defined as Example 2.2. Then B is a Boolean algebra. Define a function d on B by

$$dx = \begin{cases} 0, & \text{if } x = 0, 1, b, \\ a, & \text{if } x = a. \end{cases}$$

Then d is not a Boolean derivation of B.

Example 3.4. Let B be a Boolean algebra.

(i) If d is a function defined by $dx = 0$ for all $x \in B$, then d is a Boolean derivation on B, which is called the *zero derivation*.

(ii) If d is an identity function on a Boolean algebra B, then d is a Boolean derivation on B, which is called the *identity derivation*.

Example 3.5. Let B be a Boolean algebra and $a \in B$. Define a function $d_a : B \to B$ by $d_a(x) = x \wedge a$ for all $x \in B$. Then d_a is a Boolean derivation on B.

Theorem 3.6. *Let B be a Boolean algebra and d be Boolean derivation on B. Then the following hold: for any element $x, y \in B$,*

(1) $dx \leq x$;

(2) $d0 = 0$ *and* $d1 \leq 1$;

(3) $dx \wedge x' = x \wedge dx' = 0$;

(4) $dx \wedge dy \leq d(x \wedge y) \leq d(x \vee y)$;

(5) *if I is an ideal of B, then* $dI \subseteq I$.

Proof. Let $x, y \in B$.

(1) By Definition 2.1(i), we have $dx = d(x \wedge x) = (dx \wedge x) \vee (x \wedge dx)$ $= dx \wedge x$ and so $dx \leq x$.

(2) From (1), we get $d1 \leq 1$ and $d0 \leq 0$. Since Remark 2, we have $0 \leq d0$. Thus, by Lemma 2.7, we get $d0 = 0$.

(3) From (2) and Definition 2.1(vii), we have $0 = d0 = d(x \wedge x') =$ $(dx \wedge x') \vee (x \wedge dx')$. Thus by Theorem 2.10(i), we get $dx \wedge x' = x \wedge dx' = 0$.

(4) From (1) and Lemma 2.8(i), we have

$$dx \wedge dy \leq x \wedge dy \leq (dx \wedge y) \vee (x \wedge dy) = d(x \wedge y).$$

We know that $dx \wedge y \leq dx$ and $x \wedge dy \leq dy$. Thus

$$d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy) \leq dx \vee dy = d(x \vee y).$$

(5) Let $a \in dI$. Then $a = db$ for some $b \in I$. From (1), we have $a = db \leq b$. Since I is an ideal of B, we have $a = a \wedge b \in I$. Thus $a \in I$ and so $dI \subseteq I$.

By Theorem 3.6(2), we can get the following corollary.

Corollary 3.7. *Let B be a Boolean algebra and* $d : B \to B$ *be function. If* $d0 \neq 0$, *then d is not a Boolean derivation on B.*

Example 3.8. Let B be a Boolean algebra and $b \in B$ be such that $b \neq 0$. Define a function $d_b : B \to B$ by $d_b(x) = x \vee b$ for all $x \in B$. Note that $d_b(0) = 0 \vee b = b \neq 0$. Therefore, d_b is not a Boolean derivation on B by Corollary 3.7.

Definition 3.9. Let B be a Boolean algebra. A function $d : B \to B$ is said to be *regular* if $d0 = 0$.

Corollary 3.10. *Every Boolean derivations are regular.*

Theorem 3.11. *Let d be Boolean derivation on a Boolean algebra B. If* $x, y \in B$ *are such that* $x \leq y$, *then the following hold*:

(1) $d(x \wedge y') = 0$;

(2) $dy' \leq x'$;

(3) $dx \wedge dy' = 0$.

Proof. Let $x, y \in B$ be such that $x \leq y$.

(1) By Theorem 2.9 implies that $x \wedge y' = 0$. Thus by Theorem 3.6(2), we have $d(x \wedge y') = d0 = 0$.

(2) By Lemma 2.8(iii) implies that $y' \leq x'$. Thus by Theorem 3.6(1), we have $dy' \leq y' \leq x'$.

(3) By Theorem 3.6(1), we have $dx \leq x \leq y$. Thus by Lemma 2.8(i), we have $dx \wedge dy' \leq y \wedge dy'$.

Since $dy' \leq y'$ by Lemma 2.8(i), we have $y \wedge dy' \leq y \wedge y' = 0$. Hence $dx \wedge dy' \leq 0$. From Remark 2, we have $0 \leq dx \wedge dy'$. By Lemma 2.7, we get $dx \wedge dy' = 0$.

Theorem 3.12. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then, the following hold*:

(1) $dx \wedge dx' = 0$;

(2) $dx' = (dx)'$ *if and only if d is the identity derivation on B.*

Proof. (1) It follows directly from Theorem 3.11(3).

(2) Assume that $dx' = (dx)'$. From Theorem 3.6(3), we have $x \wedge (dx)' = x \wedge dx' = 0$. Since Theorem 2.9, we have $x \leq dx$. By Theorem 3.6(1), we get $dx \leq x$. Therefore, by Lemma 2.7, we have $dx = x$. Conversely, if d is the identity derivation on B, then $dx' = x' = (dx)'$.

Definition 3.13. Let B be a Boolean algebra and d be a Boolean derivation on B. Then

(i) d is called an *order derivation* if $x \leq y$ implies $dx \leq dy$;

(ii) d is called an *injective derivation* if d is an injective function;

(iii) d is called a *surjective derivation* if d is a surjective function.

Theorem 3.14. *A Boolean derivation d on a Boolean algebra B is an order derivation*.

Proof. Let $x, y \in B$ and $x \leq y$. Since d is a Boolean derivation, we have $dy = d(x \vee y) = dx \vee dy$. That is $dx \leq dy$. Thus, d is an order derivation.

Theorem 3.15. *Let B be a Boolean algebra and d be a Boolean derivation on B. If* $dx = dx'$ *for all* $x \in B$, *then d is zero derivation*.

Proof. Let $x \in B$, from Remark 2, we have $x \leq 1$. By Theorem 3.14, we get d is an order derivation. Hence $dx \leq d1$. Since $dx = dx'$, we have $d1 = d0 = 0$. Thus $dx \leq d1 = 0$. From Remark 2, we have $0 \leq dx$, by Lemma 2.7, we get $dx = 0$ and so d is zero derivation.

Theorem 3.16. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then $dx = x \wedge d1$ for all $x \in B$.*

Proof. Let $x \in B$. Note that $x \leq 1$ and

$$dx = d(x \wedge 1) = (dx \wedge 1) \vee (x \wedge d1) = dx \vee (x \wedge d1),$$

that is $x \wedge d1 \leq dx$. By Theorem 3.14, we have d is an order derivation. Hence $dx \leq d1$. From Theorem 3.6(1), we get $dx \leq x$. Thus $dx \leq x \wedge d1$. By Lemma 2.7, we have $dx = x \wedge d1$.

The following corollary is an immediate consequence of Theorem 3.16.

Corollary 3.17. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then the following hold*:

(1) *if $d1 \leq x$, then $dx = d1$*;

(2) *if $x \leq d1$, then $dx = x$.*

Theorem 3.18. *Let B be a Boolean algebra and d be a Boolean derivation on B. Let $x, y \in B$ be such that $y \leq x$. If $dx = x$, then $dy = y$.*

Proof. Let $x, y \in B$ be such that $y \leq x$ and $dx = x$. By Theorem 3.6(1), we have $dy \leq y \leq x$. Then

$$dy = d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy) = (x \wedge y) \vee dy = y \vee dy = y.$$

Theorem 3.19. *Let B be a Boolean algebra and d be a Boolean derivation on B. Let $x_1, x_2, \ldots, x_n \in B$ be such that $x_1 \leq x_2 \leq \cdots \leq x_n$. If $dx_n = x_n$, then $dx_i = x_i$ for all positive integer $i \leq n$.*

Proof. For $n = 2$. If $x_1 \leq x_2$ and $dx_2 = x_2$, then by Theorem 3.18, we have $dx_1 = x_1$. Let $n \geq 3$ be positive integer and assume that if $x_1 \leq x_2 \leq$

$\cdots \leq x_n$ and $dx_n = x_n$, then $dx_i = x_i$ for all $i < n$. Suppose that $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1}$ and $dx_{n+1} = x_{n+1}$. Since $x_n \leq x_{n+1}$ and $dx_{n+1} = x_{n+1}$ by Theorem 3.18, we get $dx_n = x_n$. Now we have if $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1}$ and $dx_{n+1} = x_{n+1}$, then $dx_i = x_i$ for all positive integer $i \leq n+1$.

By Theorem 3.18, we can get the following corollary.

Corollary 3.20. *Let B be a Boolean algebra and $d : B \to B$ be a function. If there exists a pair $x, y \in B$ such that $y \leq x$, $dx = x$ and $dy \neq y$, then d is not a Boolean derivation on B.*

Proof. Assume that there exists a pair $x, y \in B$ such that $y \leq x$, $dx = x$ and $dy \neq y$. Suppose that d is a Boolean derivation on B. By Theorem 3.6(1), we have $dy \leq y \leq x$. Then

$$dy = d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy) = (x \wedge y) \vee dy = y \vee dy,$$

it follows that $y \leq dy$. By Lemma 2.7, we get $dy = y$, which is a contradiction. Thus d is not a Boolean derivation on B.

Example 3.21. Let $B = \{0, a, b, 1\}$ and $\wedge, \vee$ are two binary operations defined as Example 2.2. Then B is a Boolean algebra. Define a function d on B by

$$dx = \begin{cases} a, & \text{if } x = 0, 1, a, \\ b, & \text{if } x = b. \end{cases}$$

Note that $db = b$ and $0 \leq b$, but $d0 = a \neq 0$. Therefore, d is not a Boolean derivation of B by Corollary 3.20.

Example 3.22. Let $B = \{0, a, b, 1\}$ and $\wedge, \vee$ are two binary operations defined as Example 2.2. Then B is a Boolean algebra. Define a function d on B by

$$dx = \begin{cases} 1, & \text{if } x = 1, a, \\ a, & \text{if } x = b, \\ 0, & \text{if } x = 0. \end{cases}$$

Note that $d1 = 1$ and $b \leq 1$, but $db = a \neq b$. Therefore, d is not a Boolean derivation of B by Corollary 3.20.

Theorem 3.23. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then* $dx = x \wedge d(x \vee y)$ *for all* $x, y \in B$.

Proof. Let $x, y \in B$. By Theorem 3.6(1), we have $dx \leq x \leq x \vee y$. By Definition 2.1(iv), we have $x = (x \vee y) \wedge x$. So

$$dx = d((x \vee y) \wedge x) = (d(x \vee y) \wedge x) \vee ((x \vee y) \wedge dx)$$

$$= (d(x \vee y) \wedge x) \vee dx,$$

that is, $d(x \vee y) \wedge x \leq dx$. By Theorem 3.14, we have d is an order derivation. Hence $dx \leq d(x \vee y)$. From Theorem 3.6(1), we get $dx \leq x$, so $dx \leq d(x \vee y) \wedge x$. Thus, by Lemma 2.7, we have $dx = x \wedge d(x \vee y)$.

Theorem 3.24. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then*

(1) $d1 = 1$ *if and only if d is the identity derivation*;

(2) $d1 = 0$ *if and only if d is the zero derivation*.

Proof. Let $x \in B$.

(1) Assume that $d1 = 1$. By Remark 2, we have $x \leq 1$. By Theorem 3.18, we get $dx = x$. Thus d is the identity derivation. Conversely, if d is the identity derivation, then we have $d1 = 1$.

(2) Suppose that $d1 = 0$. By Theorem 3.16, we get $dx = x \wedge d1 = x \wedge 0 = 0$. Thus d is the zero derivation. Conversely, it is trivial.

Theorem 3.25. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then* $d(x \wedge y) = dx \wedge dy$ *for all* $x, y \in B$.

Proof. Suppose that $x, y \in B$. By Theorem 3.16, we get

$$dx \wedge dy = (x \wedge d1) \wedge (y \wedge d1) = (x \wedge y) \wedge d1 = d(x \wedge y).$$

Converse of Theorem 3.25 need not be true as the following example.

Example 3.26. Let B be a Boolean algebra and $b \in B$ be such that $b \neq 0$. Define a function $d_b : B \to B$ by $d_b(x) = x \vee b$ for all $x \in B$. Note that for all $x, y \in B$, we have

$$d_b(x \wedge y) = (x \wedge y) \vee b = (x \vee b) \wedge (y \vee b) = d_b(x) \wedge d_b(y).$$

But d_b is not a Boolean derivation on B by Example 3.8.

Theorem 3.27. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then $d(x_1 \wedge x_2 \wedge \cdots \wedge x_n) = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ for all $x_1, x_2, ..., x_n \in B$ and for integer $n \geq 2$.*

Proof. For $n = 2$. By Theorem 3.25, we get $d(x_1 \wedge x_2) = dx_1 \wedge dx_2$. Let $n \geq 3$ be positive integer and assume that $d(x_1 \wedge x_2 \wedge \cdots \wedge x_n) = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. Suppose that $a = x_1 \wedge x_2 \wedge \cdots \wedge x_n$. Then

$$\begin{aligned} d(x_1 \wedge x_2 \wedge \cdots \wedge x_n \wedge x_{n+1}) &= d(a \wedge x_{n+1}) \\ &= da \wedge dx_{n+1} = d(x_1 \wedge x_2 \wedge \cdots \wedge x_n) \wedge x_{n+1} \\ &= x_1 \wedge dx_2 \wedge \cdots \wedge dx_n \wedge dx_{n+1}. \end{aligned}$$

Theorem 3.28. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then the following are equivalent: for any $x, y \in B$,*

(1) $d(x \wedge y) = dx \wedge y$;

(2) $d(x \wedge y) = dx \wedge dy$.

Proof. Let $x, y \in B$.

(1) $\Rightarrow$ (2) Assume that (1) holds. By Theorem 3.6(4), we have $dx \wedge dy \leq d(x \wedge y)$. From (1), we have $dx \wedge y = d(x \wedge y) = d(y \wedge x) = dy \wedge x$. Since $dx \wedge y \leq dx$ and $dy \wedge x \leq dy$, we can get $d(x \wedge y) = dx \wedge y = dy \wedge x \leq dx \wedge dy$, that is, $d(x \wedge y) \leq dx \wedge dy$. Thus, by Lemma 2.7, we have $d(x \wedge y) = dx \wedge dy$.

(2) $\Rightarrow$ (1) Suppose that (2) holds. We know that $dx \wedge dy \leq dx$ and $dx \wedge dy \leq dy \leq y$, thus $dx \wedge dy \leq dx \wedge y$. From (2), we can get $d(x \wedge y) \leq dx \wedge y$. Since $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$, that is, $dx \wedge y \leq d(x \wedge y)$. Hence, by Lemma 2.7, we have $d(x \wedge y) = dx \wedge y$.

From Theorem 3.25 and Theorem 3.28, we have the following theorem.

Theorem 3.29. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then $d(x \wedge y) = dx \wedge y$ for all $x, y \in B$.*

Theorem 3.30. *Let B be a Boolean algebra and d_1, d_2 be two Boolean derivations on B. Define $d_1 \circ d_2(x) = d_1(d_2 x)$ for all $x \in B$. Then $d_1 \circ d_2$ is a Boolean derivation on B.*

Proof. Let $x, y \in B$. Since d_1, d_2 are two Boolean derivations on B and by Theorem 3.29, we have

$$\begin{aligned} d_1 \circ d_2(x \wedge y) = d_1(d_2(x \wedge y)) &= d_1((d_2 x \wedge y) \vee (x \wedge d_2 y)) \\ &= d_1(d_2 x \wedge y) \vee d_1(x \wedge d_2 y) \\ &= (d_1(d_2 x) \wedge y) \vee (x \wedge d_1(d_2 y)) \\ &= (d_1 \circ d_2(x) \wedge y) \vee (x \wedge d_1 \circ d_2(y)). \end{aligned}$$

Moreover, we get

$$\begin{aligned} d_1 \circ d_2(x \vee y) = d_1(d_2(x \vee y)) &= d_1(d_2 x) \vee d_1(d_2 y) \\ &= d_1 \circ d_2(x) \vee d_1 \circ d_2(y). \end{aligned}$$

Hence $d_1 \circ d_2$ is a Boolean derivation on B.

Theorem 3.31. *Let B be a Boolean algebra and $d_1, d_2, \ldots, d_n$ be Boolean derivations on B. Define $d_1 \circ d_2 \circ \cdots \circ d_n(x) = d_1(d_2 \cdots (d_n x))$ for all $x \in B$. Then $d_1 \circ d_2 \circ \cdots \circ d_n$ is a Boolean derivation on B for positive integer $n \geq 2$.*

Proof. For $n = 2$. By Theorem 3.30, we get $d_1 \circ d_2$ is a Boolean derivation on B. Let $n \geq 3$ be positive integer and assume that $d_1 \circ d_2 \circ \cdots \circ d_n$ is a Boolean derivation on B. Let $D_n = d_1 \circ d_2 \circ \cdots \circ d_n$. Thus D_n is a Boolean derivation on B. Since d_{n+1} is a Boolean derivation on B and by Theorem 3.30, we have $d_1 \circ d_2 \circ \cdots \circ d_n \circ d_{n+1} = D_n \circ d_{n+1}$ is a Boolean derivation on B.

Theorem 3.32. *Let B be a Boolean algebra and d be a Boolean derivation on B. Define $d^2x = d(dx)$ for all $x \in B$. Then we have $d^2 = d$.*

Proof. Let $x \in B$. By Theorem 3.6(1), we get $d^2x = d(dx) \leq dx \leq x$. Then

$$d^2x = d(dx) = d(x \wedge dx) = (dx \wedge dx) \vee (x \wedge d^2x) = dx \vee d^2x = dx.$$

Theorem 3.33. *Let B be a Boolean algebra and d be a Boolean derivation on B. Define $d^nx = \underbrace{d(d \cdots d(dx))}_{n}$ for all $x \in B$. Then we have $d^n = d$ for integer $n \geq 2$.*

Proof. Assume that $x \in B$. For $n = 2$. By Theorem 3.32, we get $d^2x = dx$. Let $n \geq 3$ be positive integer and assume that $d^nx = \underbrace{d(d \cdots d(dx))}_{n} = dx$. Then

$$d^{n+1}x = \underbrace{d(d \cdots d(dx))}_{n+1} = d(d^nx) = d(dx) = d^2x = dx.$$

Remark 7. If d is a Boolean derivation on B, then by Theorem 3.33, we get d^n is a Boolean derivation on B for positive integer $n \geq 2$.

Theorem 3.34. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then the following conditions are equivalent*:

(1) *d is the identity derivation*;

(2) *d is an injective derivation*;

(3) *d is a surjective derivation*.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let d be an injective derivation. If there is $a \in B$ such that $da \neq a$, then $da < a$. Denote $a_1 = da$, by Theorem 3.6(1), we have $da_1 \leq a_1 < a$. Thus

$$da_1 = d(a_1 \wedge a) = (da_1 \wedge a) \vee (a_1 \wedge da) = da_1 \vee a_1 = a_1 = da.$$

Since $a_1 \neq a$, this contradicts that d is an injective derivation.

(1) $\Rightarrow$ (3) is straightforward.

(3) $\Rightarrow$ (1) Assume that d is a surjective derivation, that is, $dB = B$. Thus for any $x \in B$, there is $y \in B$ such that $x = dy$. By Theorem 3.32, we have $dx = d(dy) = d^2y = dy = x$. This shows that d is the identity derivation.

Let B be Boolean algebra and d be a Boolean derivation on B. Denote $Fix(B, d) = \{x \in B \mid dx = x\}$.

Remark 8. (1) By Theorem 3.6(2), we have $0 \in Fix(B, d)$.

(2) By Theorem 3.18, we see that if $x \in Fix(B, d)$ and $y \leq x$, then $y \in Fix(B, d)$.

(3) By Theorem 3.32, we have $d(dx) = d^2x = dx$. Thus $dx \in Fix(B, d)$.

Theorem 3.35. *Let B be a Boolean algebra and $d_1, d_2, ..., d_n$ be Boolean derivations on B. Then $d_i = d_j$ if and only if $Fix(B, d_i) = Fix(B, d_j)$ for positive integers $i, j \leq n$.*

Proof. It is clear that if $d_i = d_j$ implies $Fix(B, d_i) = Fix(B, d_j)$. Conversely, let $Fix(B, d_i) = Fix(B, d_j)$ for some $i, j \leq n$ and $x \in B$. By Remark 8(3), we have $d_ix \in Fix(B, d_i) = Fix(B, d_j)$ and so $d_jd_ix = d_ix$.

Similarly, we can get $d_i d_j x = d_j x$. From Theorem 3.6(1), we know that $d_i x \leq x$ and by Theorem 3.14, we get d_i and d_j are order derivations on B. Thus we have $d_j d_i x \leq d_j x = d_i d_j x$ and so $d_j d_i x \leq d_i d_j x$. Symmetrically we can also get $d_i d_j x \leq d_j d_i x$. By Lemma 2.7, we get $d_i d_j x = d_j d_i x$. It follows that $d_i x = d_j d_i x = d_i d_j x = d_j x$, that is $d_i = d_j$.

Theorem 3.36. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then* $Fix(B, d)$ *is an ideal of B.*

Proof. Let $x, y \in Fix(B, d)$ and $b \in B$. Since $x, y \in Fix(B, d)$, we have $dx = x$ and $dy = y$. Thus $d(x \vee y) = dx \vee dy = x \vee y$, this shows that $x \vee y \in Fix(B, d)$. Next, we will show that $x \wedge b \in Fix(B, d)$. By Theorem 3.6(1) and Lemma 2.8(i), we can get $x \wedge db \leq x \wedge b$. Then

$$d(x \wedge b) = (dx \wedge b) \vee (x \wedge db) = (x \wedge b) \vee (x \wedge db) = x \wedge b,$$

this shows that $x \wedge b \in Fix(B, d)$. Hence $Fix(B, d)$ is an ideal of B.

From Theorem 3.31, Theorem 3.33 and Theorem 3.36, we have the following corollary.

Corollary 3.37. *Let B be a Boolean algebra and* $n \geq 2$ *be positive integer. Then*

(1) *if d is a Boolean derivation on B, then* $Fix(B, d^n)$ *is an ideal of B*;

(2) *if* $d_1, d_2, \ldots, d_n$ *are Boolean derivations on B, then* $Fix(B, d_1 \circ d_2 \circ \cdots \circ d_n)$ *is an ideal of B.*

Let B be a Boolean algebra and d be a Boolean derivation on B. Then, by the kernel of d we mean the set of all elements x of B such that $dx = 0$. The kernel of d will be denoted by $Ker(B, d)$.

Remark 9. By Theorem 3.6(2), we have $d0 = 0$. That is $0 \in Ker(B, d)$.

Theorem 3.38. *Let B be a Boolean algebra and d be a Boolean derivation on B. Then* $Ker(B, d)$ *is an ideal of B.*

Proof. Let $x, y \in Ker(B, d)$ and $b \in B$. Since $x, y \in Ker(B, d)$, we have $dx = 0$ and $dy = 0$. Thus $d(x \vee y) = dx \vee dy = 0$, that is, $x \vee y \in Ker(B, d)$. Next, we want to show that $x \wedge b \in Ker(B, d)$. By Theorem 3.25, we have $d(x \wedge b) = dx \wedge db = 0 \wedge db = 0$, this shows that $x \wedge b \in Ker(B, d)$. Hence $Ker(B, d)$ is an ideal of B.

From Theorems 3.31, 3.33 and 3.38, we have the following corollary.

Corollary 3.39. *Let B be a Boolean algebra and $n \geq 2$ be positive integer. Then*

(1) *if d is a Boolean derivation on B, then $Ker(B, d^n)$ is an ideal of B*;

(2) *if $d_1, d_2, ..., d_n$ are Boolean derivations on B, then $Ker(B, d_1 \circ d_2 \circ \cdots \circ d_n)$ is an ideal of B.*

Let B be a Boolean algebra and $a \in B$. Let d_a be Boolean derivation defined by $d_a x = x \wedge a$ for all $x \in B$. Define $D(B) = \{d_a \mid a \in B\}$. In the following we investigate the relation between B and $D(B)$.

For any $d_a, d_b \in D(B)$, define two binary orperations "." and "+" by $(d_a \cdot d_b)x = (d_a x) \wedge (d_b x)$ and $(d_a + d_b)x = (d_a x) \vee (d_b x)$ for all $x \in B$. Then, we have

$$(d_a \cdot d_b)x = (d_a x) \wedge (d_b x) = (x \wedge a) \wedge (x \wedge b) = x \wedge (a \wedge b) = d_{a \wedge b}(x)$$

and

$$(d_a + d_b)x = (d_a x) \vee (d_b x) = (x \wedge a) \vee (x \wedge b) = x \wedge (a \vee b) = d_{a \vee b}(x),$$

that is, $d_a \cdot d_b = d_{a \wedge b}$ and $d_a + d_b = d_{a \vee b}$. Hence the definitions of "." and "+" are well-defined. It is not difficult to prove that $(D(B), \cdot, +, ', d_0, d_1)$ form a Boolean algebra, so we omit the proof of this.

Theorem 3.40. *Let B be a Boolean algebra. Then B is isomorphic to $D(B)$.*

Proof. Define a function $\varphi : B \to D(B)$ by $\varphi(a) = d_a$ for all $a \in B$. Then we can easily see that φ is a bijective function. Next we need to show that φ is Boolean homomorphism. Assume that $a, b \in B$. Then, we have $\varphi(a \wedge b) = d_{a \wedge b} = d_a \cdot d_b = \varphi(a) \cdot \varphi(b)$ and $\varphi(a \vee b) = d_{a \vee b} = d_a + d_b$ $= \varphi(a) + \varphi(b)$. Thus $\varphi(a \wedge b) = \varphi(a) \cdot \varphi(b)$ and $\varphi(a \vee b) = \varphi(a) + \varphi(b)$. Now $\varphi(a) \cdot \varphi(a') = d_a \cdot d_{a'} = d_0$ and $\varphi(a) + \varphi(a') = d_a + d_{a'} = d_1$, that is, $\varphi(a)' = \varphi(a')$. By Definition 2.12, we get φ is Boolean homomorphism. Thus B is isomorphic to $D(B)$.

Acknowledgements

We wish to thank the anonymous referees for their valuable comments and suggestions. Moreover, this work was supported by a grant from Kasetsart University.

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