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ON THE EULER CHARACTERISTIC OF A FINITE T_0-SPACE

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Abstract

Given a finite topological space, there exists an adjacency matrix for the graph associated to the topology, which is called a topogenous matrix of the finite topological space. In this paper, we show that the Euler characteristic of a finite T_0-space is represented by the topogenous matrix.

1. Introduction

A finite set with a topology is called a *finite topological space* or *finite space*. Let X_n denote the finite set consisting of n elements, and $\mathcal{O}_n$ a topology on X_n. We say that a finite topological space $(X_n, \mathcal{O}_n)$ is a finite T_0-space if it satisfies the T_0-separation axiom, that is, for each pair of

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distinct two points, there exists an open set containing one but not the other. We often write X_n for $(X_n, \mathcal{O}_n)$. Of course, a T_1-space is a T_0-space. A topology of a finite space can be represented by a suitable matrix, which is called a *topogenous matrix*.

A finite T_0-space is our target. For this reason, it has the structure of a partially ordered set (a poset, for short). Conversely, one can give any finite poset the structure of a finite T_0-space. In this paper, we focus on a productive relationship between graph theory, matrix algebra, finite T_0-topologies and finite poset theory.

Our main result is the following:

Theorem. *Let X_n be a finite T_0-space consisting of n points, and A be a topogenous matrix of X_n. Then the Euler characteristic of X_n is the sum of entries of the inverse matrix of A.*

The rest of this article is organized as follows: In Section 2, we give a brief introduction to topogenous matrices. In Section 3, we investigate the Euler characteristics of posets and prove the above Theorem. The last section gives some examples.

2. Topogenous Matrices

Let X_n denote a finite topological space consisting of *n* points. Let a set U_j be the minimal open set which contains x_j, that is, U_j is the intersection of all open sets containing x_j. It is easy to see that a set $\{U_j\}_{1 \le j \le n}$ constitutes a basis for the topology of X_n. Then we define a square $n \times n$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in U_i, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is called the *topogenous matrix* of X_n. A topogenous matrix completely determines the topology on a finite topological space. By definition, any diagonal term of a topogenous matrix is one. The following proposition is well-known in finite topology theory, which was discovered by H. Sharp, Jr.

Proposition 2.1. *A matrix* $A = (a_{ij})$ *is a topogenous matrix if and only if A satisfies the following conditions*:

(1) $a_{ij} = 0$ *or* 1.

(2) $a_{ii} = 1$.

(3) $A^2 = A$, *where matrix multiplication involves Boolean arithmetic.*

Now we can define a *preorder* on X_n by

$$x_i \leq x_j \quad \text{if} \quad x_i \in U_j,$$

where a set U_j is the minimal open set which contains x_j, that is, U_j is the intersection of all open sets containing x_j. In other words, every open set containing x_j also contains x_i if and only if $x_i \leq x_j$.

Proposition 2.2. *A* T_0*-space with the above preorder* $\leq$ *is a poset.*

Proof. Let us just verify the antisymmetry. Let $U_i = U_j$ and $x_i \neq x_j$. By T_0-separation axiom, U_i does not contain a point x_j. This is a contradiction. Thus, if $U_i = U_j$, then $x_i = x_j$. Assume that $x_i \leq x_j$ and $x_j \leq x_i$. By definition, it follows that $U_i = U_j$, and so $x_i = x_j$. □

Conversely, we can give any finite poset a topology. Let $(X_n, \leq)$ be a finite poset. We let $U_i = \{y \in X_n \mid y \leq x_i\}_{x_i \in X_n}$. Then we can define a topology on X_n, the open base of which is a set $\{U_i \mid x_i \in X_n\}$. Moreover, each U_i is the minimal open set containing x_i, and so we can deduce that X_n is a finite T_0-space. Consequently, we have:

Proposition 2.3. *A finite T_0-space corresponds to a finite poset.*

Example 2.4. Let $X_3 = \{x_1, x_2, x_3\}$ be a finite space whose topology is $\{\emptyset, \{x_1, x_2, x_3\}, \{x_2, x_3\}, \{x_2\}, \{x_3\}\}$. This space is T_0. Immediately, $U_1 = \{x_1, x_2, x_3\}$, $U_2 = \{x_2\}$ and $U_3 = \{x_3\}$. Therefore $x_2 \leq x_1$ and $x_3 \leq x_1$, but there exists no order relation between x_2 and x_3. Then the topogenous matrix of X_3 is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Example 2.5. Let $X_4 = \{x_1, x_2, x_3, x_4\}$ be a finite space whose topology is $\{\emptyset, \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4\}, \{x_2\}, \{x_2, x_3\}, \{x_2, x_4\}\}$. This space is also T_0. Immediately, $U_1 = \{x_1, x_2, x_3, x_4\}$, $U_2 = \{x_2\}$, $U_3 = \{x_2, x_3\}$ and $U_4 = \{x_2, x_4\}$. The topogenous matrix of X_4 is $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. On the order relation, we see the following Hasse diagram:

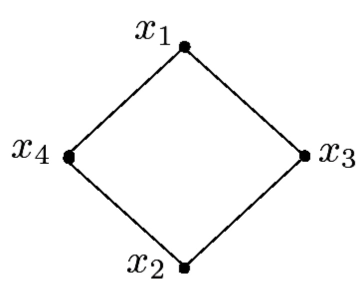

Figure 1

The Hasse diagram of a poset $\mathcal{P}$ is a digraph, whose vertices are the points of $\mathcal{P}$ and whose edges are the ordered pairs (x, y) such that $x < y$ and there exists no $z \in \mathcal{P}$ such that $x < z < y$. In the graphical representation of a Hasse diagram, we will abbreviate an arrow from x to y, but write a segment with y over x. Moreover, we also abbreviate a loop at each point of $\mathcal{P}$.

As shown above, graphs for T_0-topologies can be streamlined to Hasse diagrams. To convert back from a Hasse diagram to a directed graph, we insert an arrow on every line segment that points upward, and we invoke transitivity as needed to add extra edges. Any finite directed graph with n nodes, given in some fixed order, is equivalent to an $n \times n$ *adjacency matrix* $M = (m_{ij})$ consisting of zeroes and ones, where $m_{ij} = 1$ if and only if there is an edge from node x_i to node x_j. The following proposition can be understood easily from the viewpoint of Hasse diagrams.

Proposition 2.6. *Let X_n be a finite T_0-space. Then the topogenous matrix of X_n is an adjacency matrix for the graph associated to the topology.*

A *permutation matrix* is a square matrix such that it contains only zeros and ones, with a unique one in every row and column. The identity matrix is a special case of a permutation matrix.

Proposition 2.7. *A topogenous matrix A of a finite T_0-space is equivalent to a triangular matrix, that is, there exists a permutation matrix P such that ${}^tPAP$ is a triangular matrix.*

Proof. Let $(X_n, \mathcal{O}_n)$ be a finite T_0-space with a topogenous matrix A. Let $X_n = \{x_1, x_2, ..., x_n\}$, and let U_i be the minimal open set which contains x_i. Put n_i as the number of the element of U_i. We rearrange X_n as $X_n = \{x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}\}$ such that if $i \leq j$, then $n_{\sigma(i)} \leq n_{\sigma(j)}$, where σ is some permutation on $\{1, 2, ..., n\}$. We define a matrix $B = (b_{ij})$ by $b_{ij} = a_{\sigma(i)\sigma(j)}$. If $i < j$, then we have $x_{\sigma(j)} \notin U_{\sigma(i)}$, and $b_{ij} = 0$. Thus, B is a triangular matrix. Moreover, we define $P = (p_{ij})$ by $p_{ij} = \begin{cases} 1 & \sigma(j) = i, \\ 0 & \text{otherwise.} \end{cases}$ Then P is a permutation matrix and we obtain that $B = {}^tPAP$. □

Each diagonal term of ${}^tPAP$ equals one, and thus

Corollary 2.8. *A topogenous matrix of a finite* T_0*-space is nonsingular*.

In Example 2.4, we rearrange X_3 as $X_3 = \{x_2, x_3, x_1\}$. Next we define a permutation matrix $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ by the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. For the topogenous matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have a triangular matrix $B = {}^tPAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

The following proposition is a remarkable result.

Proposition 2.9. *Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *be two* T_0*-topologies in a finite set* X_n, *and let* A_1 *and* A_2 *be the topogenous matrices of* $(X_n, \mathcal{O}_1)$ *and* $(X_n, \mathcal{O}_2)$, *respectively*. *Moreover, let* G_1 *and* G_2 *be the directed graphs associated to the topologies of* $\mathcal{O}_1$ *and* $\mathcal{O}_2$, *respectively*. *Then the following conditions are equivalent*:

(1) $(X_n, \mathcal{O}_1)$ *and* $(X_n, \mathcal{O}_2)$ *are homeomorphic*.

(2) A_1 *and* A_2 *are equivalent*, *that is*, *there exists a permutation matrix* Q *such that* $A_2 = {}^tQA_1Q$.

(3) G_1 *and* G_2 *are isomorphic*.

Proof. (1) $\Rightarrow$ (2) Let U_i be the minimal open set which contains x_i on the topology $\mathcal{O}_1$, and V_i be the minimal open set which contains x_i on the topology $\mathcal{O}_2$ for each element x_i of X_n $(i = 1, 2, ..., n)$. Let a map f from $(X_n, \mathcal{O}_1)$ to $(X_n, \mathcal{O}_2)$ be a homeomorphism by

$$f(x_i) = x_{\sigma(i)} \quad (i = 1, 2, ..., n),$$

where σ is some permutation on $\{1, 2, ..., n\}$. It follows that f induces a mapping $f(U_i) = V_{\sigma(i)}$ $(i = 1, 2, ..., n)$, which preserves the inclusion relation. If A_1 is noted as (a_{ij}), we have $A_2 = (a_{\sigma(i)\sigma(j)})$. Moreover, we define $Q = (\delta_{i\sigma(j)})$, where $\delta_{i\sigma(j)}$ is the Kronecker's delta. Then we obtain that $A_2 = {}^tQA_1Q$.

(2) $\Rightarrow$ (3) Assume that there exists a permutation matrix $Q = (\delta_{i\sigma(j)})$ such that $A_2 = {}^tQA_1Q$. Define $f : G_1 \to G_2$ by

$$f(x_i) = x_{\sigma(i)},$$

then f is an isomorphism from G_1 to G_2.

(3) $\Rightarrow$ (1) Let a map g from G_1 to G_2 be an isomorphism. Since $g(x_i)$ is a unique element of X_n, we can put $g(x_i) = x_{\tau(i)}$, where τ is a permutation on $\{1, 2, ..., n\}$. Then g induces a homeomorphism from $(X_n, \mathcal{O}_1)$ to $(X_n, \mathcal{O}_2)$. □

3. Euler Characteristics of Posets

Next we shall investigate the Euler characteristic of a poset. See our general reference Aigner [1] for details. Let $\mathcal{P}$ be a finite poset and $\mathcal{P} \times \mathcal{P}$ be the set of all pairs (x, y) with $x, y \in \mathcal{P}$. Let $\mathbb{Q}$ be the field of rational numbers. We let

$$A_{\mathbb{Q}}(\mathcal{P}) := \{f : \mathcal{P} \times \mathcal{P} \to \mathbb{Q} \mid x \not\leq y \Rightarrow f(x, y) = 0\},$$

Then $A_{\mathbb{Q}}(\mathcal{P})$ is a vector space over $\mathbb{Q}$ in the usual way, where $f, g \in A_{\mathbb{Q}}(\mathcal{P})$, $r \in \mathbb{Q}$,

$$(f + g)(x, y) := f(x, y) + g(x, y),$$

$$(rf)(x, y) := rf(x, y).$$

We define the multiplication $f * g$ of $f, g \in A_{\mathbb{Q}}(\mathcal{P})$ by

$$(f * g)(x, y) := \sum_{x \leq z \leq y} f(x, z) g(z, y).$$

Notice that the right-hand side is well-defined by the finiteness of $\mathcal{P}$. Obviously, $(f * g) \in A_{\mathbb{Q}}(\mathcal{P})$ again. Also, we have an element δ of $A_{\mathbb{Q}}(\mathcal{P})$ such that $\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$ and call δ the *Kronecker function* of $\mathcal{P}$.

Proposition 3.1. *$A_{\mathbb{Q}}(\mathcal{P})$ is an associative $\mathbb{Q}$-algebra with Kronecker function δ as the unit element.*

Proof. Let us first verify the associative law. For $f, g, h \in A_{\mathbb{Q}}(\mathcal{P})$, we have

$$\begin{aligned}
(f * (g * h))(x, y) &= \sum_{x \leq z \leq y} f(x, z)(g * h)(z, y) \\
&= \sum_{x \leq z \leq y} f(x, z) \left(\sum_{z \leq w \leq y} g(z, w) h(w, y) \right) \\
&= \sum_{x \leq w \leq y} \left(\sum_{x \leq z \leq w} f(x, z) g(z, w) \right) h(w, y) \\
&= \sum_{x \leq w \leq y} (f * g)(x, w) h(w, y) \\
&= ((f * g) * h)(x, y).
\end{aligned}$$

By the definition of Kronecker function, it follows that $f * \delta = \delta * f = f$.

$\square$

Proposition 3.2. *An element $f \in A_{\mathbb{Q}}(\mathcal{P})$ is a unit if and only if $f(x, x) \neq 0$ for all $x \in \mathcal{P}$.*

Proof. If f is a unit, there exists an element $g \in A_{\mathbb{Q}}(\mathcal{P})$ such that $f * g = \delta$ (or $g * f = \delta$). Then for all $x \in \mathcal{P}$,

$$1 = (f * g)(x, x) = (g * f)(x, x) = f(x, x)g(x, x).$$

Thus $f(x, x) \neq 0$. Conversely, let $f(x, x) \neq 0$ for all $x \in \mathcal{P}$. We define the left inverse inductively by

$$f^{-1}(x, x) = \frac{1}{f(x, x)},$$

$$f^{-1}(x, y) = \frac{1}{f(y, y)}\left(-\sum_{x \leq z < y} f^{-1}(x, z) f(z, y)\right).$$

In the same way, the right inverse is defined as follows:

$$f^{-1}(x, x) = \frac{1}{f(x, x)},$$

$$f^{-1}(x, y) = \frac{1}{f(x, x)}\left(-\sum_{x < x \leq y} f(x, z) f^{-1}(z, y)\right).$$

Two inverses are the same by associativity. □

We denote the set of all pairs (x, y) such that $x \leq y$ for $x, y \in \mathcal{P}$ by $I(\mathcal{P})$. Then we have an element ζ of $A_{\mathbb{Q}}(\mathcal{P})$ such that $\zeta(x, y) = 1$ for all $(x, y) \in I(\mathcal{P})$, which is called the *zeta function* of $\mathcal{P}$. The element ζ has a multiplicative inverse in $A_{\mathbb{Q}}(\mathcal{P})$, denoted by μ.

Proposition 3.3. *For* $(x, y) \in I(\mathcal{P})$, *one has*

$$\mu(x, x) = 1,$$

$$\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z) = -\sum_{x < z \leq y} \mu(z, y) \text{ if } x < y.$$

Proof. By replacing f with ζ in the proof of the previous proposition, we get the assertion. □

The following corollary is an immediate consequence of the above proposition.

Corollary 3.4. $\sum_{x \leq z \leq y} \mu(x, z) = \sum_{x \leq z \leq y} \mu(z, y) = \delta(x, y).$

This function μ from $\mathcal{P} \times \mathcal{P}$ to $\mathbb{Q}$ is called the *Möbius function* of $\mathcal{P}$. Moreover, $\zeta - \delta$ is also an element of $A_{\mathbb{Q}}(\mathcal{P})$ satisfying

$$(\zeta - \delta)(x, y) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{if } x = y. \end{cases}$$

We put $\eta = \zeta - \delta$. Notice that the value of μ is some integer and η is so again.

Proposition 3.5. *For* $x, y \in \mathcal{P}$, *we have* $\mu(x, y) = \sum_{k \geq 0} (-1)^k \eta^k (x, y)$, *where* $\eta^0 = \delta$.

Proof. Recall that μ is the inverse of ζ. Hence

$$\begin{aligned}
\mu(x, y) &= \zeta^{-1}(x, y) \\
&= (\delta + (\zeta - \delta))^{-1}(x, y) \\
&= (\delta - (\zeta - \delta) + (\zeta - \delta)^2 - \cdots)(x, y) \\
&= \delta(x, y) - (\zeta - \delta)(x, y) + (\zeta - \delta)^2(x, y) - \cdots \\
&= \eta^0(x, y) - \eta^1(x, y) + \eta^2(x, y) - \cdots \\
&= \sum_{k \geq 0} (-1)^k \eta^k (x, y),
\end{aligned}$$

which yields the result. □

Recall that for a poset $\mathcal{P}$, the set $S \subset \mathcal{P}$ is called a *chain* if S is totally ordered with respect to the partial ordering on $\mathcal{P}$. Define the *length* $\ell(\mathcal{P})$ to be the length of a longest chain of $\mathcal{P}$, where the length of a chain is one less than its number of elements. In particular, the length of the empty poset is –1. The *order complex* $\Delta(\mathcal{P})$ of a poset $\mathcal{P}$ is defined to be the abstract simplicial complex whose vertices are all elements of $\mathcal{P}$ and whose simplices are all finite chains of $\mathcal{P}$, including the empty chain. Clearly, $\dim \Delta(\mathcal{P}) = \ell(\mathcal{P})$. We denote the geometric realization of an abstract simplicial complex Δ by $\| \Delta \|$. The Euler characteristic $\chi(\mathcal{P})$ of $\mathcal{P}$ is defined to be the Euler characteristic $\chi(\Delta(\mathcal{P}))$.

Proposition 3.6. $\chi(\mathcal{P}) = \sum_{x, y \in \mathcal{P}} \mu(x, y)$.

Proof. We also set $\eta = \zeta - \delta$. Remark that $\eta^k(x, y)$ equals the number of the chain joining *x* and *y* whose length is *k*, where $k \geq 1$. In the case of $k = 0$, we define $\eta^0(x, y) = \delta(x, y)$. Then the number of the vertices of $\mathcal{P}$ is $\sum_{x, y \in \mathcal{P}} \delta(x, y)$. Let α_k be the number of *k*-simplices. By Proposition 3.5, we compute as follows:

$$\begin{aligned}
\sum_{x, y \in \mathcal{P}} \mu(x, y) &= \sum_{x, y \in \mathcal{P}} \left(\sum_{k \geq 0} (-1)^k \eta^k(x, y) \right) \\
&= \sum_{x, y \in \mathcal{P}} \left(\eta^0(x, y) + \sum_{k > 0} (-1)^k \eta^k(x, y) \right) \\
&= \sum_{x, y \in \mathcal{P}} \left(\delta(x, y) + \sum_{k > 0} (-1)^k \eta^k(x, y) \right) \\
&= \sum_{x, y \in \mathcal{P}} \delta(x, y) + \sum_{x, y \in \mathcal{P}} \left(\sum_{k > 0} (-1)^k \eta^k(x, y) \right) \\
&= \sum_{x, y \in \mathcal{P}} \delta(x, y) + \sum_{k > 0} \left(\sum_{x, y \in \mathcal{P}} (-1)^k \eta^k(x, y) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_0 + \sum_{k>0}\left(\sum_{x,\, y\in\mathcal{P}} (-1)^k \eta^k(x,\ y)\right) \\
&= \alpha_0 + \sum_{k>0}(-1)^k\left(\sum_{x,\, y\in\mathcal{P}} \eta^k(x,\ y)\right) \\
&= \sum_{k\geq 0}(-1)^k \alpha_k \\
&= \chi(\mathcal{P}),
\end{aligned}
$$

which completes the proof. □

Moreover, we consider two square matrices $M = (\mu(x,\ y))_{(x,\, y)\in\mathcal{P}\times\mathcal{P}}$ and $Z = (\zeta(x,\ y))_{(x,\, y)\in\mathcal{P}\times\mathcal{P}}$. The former is called the *Möbius matrix* of $\mathcal{P}$, and the latter is called the *zeta matrix* of $\mathcal{P}$.

Proposition 3.7. *$MZ = ZM = I$, where I is the usual identity matrix.*

Proof. By definition, $MZ = \left(\sum_{z\in\mathcal{P}} \mu(x,\ z)\zeta(z,\ y)\right)_{(x,\, y)\in\mathcal{P}\times\mathcal{P}}$. We now compute:

$$
\begin{aligned}
\sum_{z\in\mathcal{P}} \mu(x,\ z)\zeta(z,\ y) &= \sum_{x\leq z\leq y} \mu(x,\ z)\zeta(z,\ y) \\
&= \sum_{x\leq z\leq y} \mu(x,\ z)\cdot 1 \\
&= \sum_{x\leq z\leq y} \mu(x,\ z) \\
&= \delta(x,\ y).
\end{aligned}
$$

Hence $MZ = I$. Similarly $ZM = I$. □

Therefore, both M and Z are nonsingular. By definition, diagonal entries of both matrices are all ones.

Definition 3.8. The Euler characteristic of a finite T_0-space X_n, which is as usual denoted by $\chi(X_n)$, is the Euler characteristic of a poset $(X_n, \leq)$, where a partial ordering $\leq$ has been produced in Section 2.

We can now prove the result mentioned in the introduction.

Theorem 3.9. *Let X_n be a finite T_0-space consisting n points, and A be a topogenous matrix of X_n. Then the Euler characteristic of X_n is the sum of entries of the inverse matrix of A.*

Proof. The zeta matrix of a poset $(X_n, \leq)$ equals the adjacency matrix for the graph associated to the topology. By Proposition 2.6, the topogenous matrix of a finite T_0-space X_n is the zeta matrix of a poset $(X_n, \leq)$.

By Proposition 3.6, the Euler characteristic of $\mathcal{P}$ is the sum of entries of the Möbius matrix of $\mathcal{P}$. Since the Euler characteristic of a finite T_0-space X_n is defined to be the Euler characteristic $\chi((X_n, \leq))$, after all, it is the sum of entries of the Möbius matrix of $(X_n, \leq)$. Remark that the Möbius matrix is the inverse matrix of the zeta matrix, and the proof is complete. □

For any $n \times n$ matrix *A*, we denote the sum of all entries of *A* by $s(A)$. Another representation on the Euler characteristic of X_n is the following. This was first proved in [8].

Proposition 3.10 [8, Theorem]**.** *Let X_n be a finite T_0-space consisting n points, and A be a topogenous matrix of X_n. Let $\alpha_q = s((A - I)^q)$ $(0 \leq q \leq n-1)$. Then we have*

$$\chi(X_n) = \sum_{q=0}^{n-1} (-1)^q \alpha_q,$$

where $(A - I)^0 = I$.

Proof. The assertion follows from Proposition 3.5 and Proposition 3.6. □

4. Examples

In this section, we present some examples of application of Theorem 3.9. By Proposition 2.7 and Proposition 2.9, it is no loss to assume that any topogenous matrix is a triangular matrix.

Example 4.1. We let $X_2 = \{x_1, x_2\}$, and $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let X_2 be a finite T_0-space determined by A. Since the inverse matrix of A is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $\chi(X_2) = 2$. Indeed, $\Delta((X_2, \leq))$ is the disjoint union of two elements.

Example 4.2. We let $X_2 = \{x_1, x_2\}$, and $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Let X_2 be a finite T_0-space determined by A. Since the inverse matrix of A is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, we have $\chi(X_2) = 1$. Indeed, $\| \Delta((X_2, \leq)) \|$ is homeomorphic to a segment.

Example 4.3. We let $X_3 = \{x_1, x_2, x_3\}$, and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, which is equivalent to the topogenous matrix in Example 2.4. Let X_3 be a finite T_0-space determined by A. Since the inverse matrix of A is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$, we have $\chi(X_3) = 1$. The following Hasse diagram indicates that $\| \Delta((X_3, \leq)) \|$ is homeomorphic to a segment.

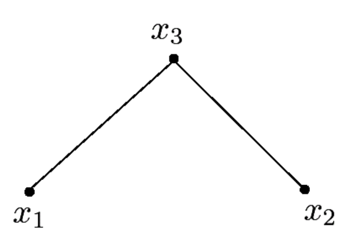

Figure 2

Example 4.4. We let $X_3 = \{x_1, x_2, x_3\}$, and $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Let X_3 be a finite T_0-space determined by A. The inverse matrix of A is $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, and so $\chi(X_3) = 1$. Since $\ell((X_3, \leq)) = 2$, it follows that $\dim \Delta((X_3 \leq)) = 2$. Then $\| \Delta((X_3, \leq)) \|$ is homeomorphic to a 2-disc D^2.

Example 4.5. We let $X_4 = \{x_1, x_2, x_3, x_4\}$, and $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, which is equivalent to the topogenous matrix in Example 2.5. Let X_4 be a finite T_0-space determined by A. Since the inverse matrix of A is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, we have $\chi(X_4) = 1$. Indeed, $\| \Delta((X_4, \leq)) \|$ is homeomorphic to the one-point union of two 2-disks, that is, the wedge of two 2-disks.

Example 4.6. We let $X_4 = \{x_1, x_2, x_3, x_4\}$, and $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. Let X_4 be a finite T_0-space determined by A. Since the inverse matrix of A is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$, we have $\chi(X_4) = 1$. In this case, $\| \Delta((X_4, \leq)) \|$ is homeomorphic to a 3-disk D^3.

Example 4.7. We let $X_n = \{x_1, x_2, ..., x_n\}$, and $A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ $(n \times n$ matrix$)$. Let X_n be a finite T_0-space determined by A. Since the inverse matrix of A is $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$, we have $\chi(X_n) = 1$. Then $\| \Delta((X_n, \leq)) \|$ is homeomorphic to a $(n-1)$-disk D^{n-1}.

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