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SOME FIXED POINT RESULTS IN LINEAR 2-BANACH SPACES

Y. Dominic and M. Marudai

Department of Mathematics
St. Joseph's College
Tiruchirappalli, Tamil Nadu, India
e-mail: dominic.yesu@gmail.com

Department of Mathematics
Bharathidasan University
Tiruchirappalli, Tamil Nadu, India
e-mail: mmarudai@yahoo.co.in

Abstract

In this paper, we prove the existence and uniqueness of fixed point for generalized (ϕ, ψ) weak contraction in the setting of 2-Banach spaces.

1. Introduction

The concept of linear 2-normed spaces was introduced and investigated by S. Gähler in 1960s. Thereafter, the subject has got great attention of mathematicians around the world. A. White introduced the concept of 2-Banach space. Zofia Lewandowska a Poland mathematician introduced the concept of generalized 2-normed space and proved the Hahn-Banach

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theorem. Like other spaces, the fixed point theory has also been developed in the framework 2-normed and 2-metric spaces. I'seki was the first man who obtained the basic results of fixed points in 2-metric and 2-normed spaces. In this paper, we prove an analogue of the fixed point theorem for generalized (ϕ, ψ) weak contraction in the setting of 2-normed spaces, which generalizes the results of Gangopadhyay et al. [4].

2. Preliminaries

Definition 2.1. Let X be a real linear space of dimension greater than 1. Suppose $\|., .\|$ is a real valued function on $X \times X$ satisfying the following conditions:

(i) $\| x, y \| = 0$ if and only if x and y are linearly dependent vectors.

(ii) $\| x, y \| = \| y, x \|$.

(iii) $\| \lambda x, y \| = |\lambda| \| x, y \|$ for all $\lambda \in \mathbb{R}$ and $x, y \in X$.

(iv) $\| x + y, z \| \leq \| x, z \| + \| y, z \|$ for all x, y and $z \in X$.

Then $\|., .\|$ is called a 2-*norm* on X and $(X, \|., .\|)$ is called *linear* 2-*normed space* over $\mathbb{R}$.

Every 2-normed space X is a locally convex topological vector space. In fact, for a fixed $b \in X$, $p_b = \| x, b \|$; $x \in X$ is a semi-norm on X and the family $P = \{p_b : b \in X\}$ of semi-norms generates a locally convex topology.

Definition 2.2. A sequence $\{x_n\}$ in a 2-normed space X is said to converge to some point x in X if for every $y \in X$,

$$\lim_{n \to \infty} \| x_n - x, y \| = 0.$$

Definition 2.3. Let X be a 2-normed space. A sequence $\{x_n\}$ in X is called *Cauchy sequence* if for any $x \in X$,

$$\lim_{n, m \to \infty} \| x_n - x_m, x \| = 0.$$

Definition 2.4. A 2-normed space X is said to be a 2-*Banach space* if every Cauchy sequence in X converges to some point in X.

Lemma 2.5 [7]. *Let* $\{x_n\}$ *be a convergent sequence in a linear* 2*-normed space X. Then for all* $y \in X$,

$$\lim_{n\to\infty} \| x_n, y \| = \| \lim_{n\to\infty} x_n, y \|.$$

3. Main Results

For all $x, y, u \in X$ and for a self-map T on X, we use the following notation throughout the paper:

$$M(x, y)$$
$$= \max\left\{\| x - y, u \|, \| x - Tx, u \|, \| y - Ty, u \|, \frac{\| x - Ty, u \| + \| y - Tx, u \|}{2}\right\}.$$

Theorem 3.1. *Let* $(X, \|., .\|)$ *be a* 2*-Banach space and* $T : X \to X$ *be a self-map satisfying the condition*

$$\psi(\| Tx - Ty, u \|) \le \psi(M(x, y)) - \phi(M(x, y)), \tag{1}$$

where

(i) $\psi : [0, \infty) \to [0, \infty)$ *is a monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$.

(ii) $\phi : [0, \infty) \to [0, \infty)$ *is a function with* $\phi(t) = 0$ *if and only if* $t = 0$ *and* $\liminf_{n\to\infty} \phi(a_n) > 0$ *if* $\lim_{n\to\infty} a_n > 0$.

(iii) $\phi(a) > \psi(a) - \psi(a-)$ *for any* $a > 0$, *where* $\psi(a-)$ *is the left limit of* ψ *at a. Then T has a unique fixed point.*

Proof. Let x be any element in X and let $x_n = T^n(x)$ for $n = 1, 2, 3 \ldots$.

Suppose $x_n = x_{n+1}$ for some n. Then $x_n = x_{n+1} = T^{n+1}(x) = T(x_n)$. That is, x_n is a fixed point of T. So we assume that $x_n \ne x_{n+1}, \ \forall n \in \mathbb{N}$.

Let $d_n(u) = \| x_n - x_{n+1}, u \| = \| T^n(x) - T^{n+1}(x), u \|$ for each $n \in \mathbb{N}$, $u \in X$.

By (i),

$$\psi(d_n(u)) \leq \psi M(x_{n-1}, x_n) - \phi M(x_{n-1}, x_n),$$

$$\psi(d_n(u)) \leq \psi M(x_{n-1}, x_n) \text{ since } M(x_{n-1}, x_n) \geq 0,$$

that is, $d_n(u) \leq M(x_{n-1}, x_n)$.

Now,

$$\begin{aligned} M(x_{n-1}, x_n) &= M(T^{n-1}(x), T^n(x)) \\ &= \max\Big\{\| T^{n-1}(x) - T^n(x), u \|, \\ &\qquad \| T^{n-1}(x) - T^n(x), u \|, \| T^n(x) - T^{n+1}(x), u \|, \\ &\qquad \frac{\| T^{n-1}(x) - T^{n+1}(x), u \| + \| T^n(x) - T^n(x), u \|}{2}\Big\} \\ &= \max\Big\{\| T^{n-1}(x) - T^n(x), u \|, \| T^n(x) - T^{n+1}(x), u \|, \\ &\qquad \frac{\| T^{n-1}(x) - T^{n+1}(x), u \|}{2}\Big\}. \end{aligned}$$

Note that

$$\| T^{n-1}(x) - T^{n+1}(x), u \| \leq \| T^{n-1}(x) - T^n(x), u \| + \| T^n(x) - T^{n+1}(x), u \|$$

and therefore,

$$\frac{\| T^{n-1}(x) - T^{n+1}(x), u \|}{2}$$

$$\leq \max\{\| T^{n-1}(x) - T^n(x), u \|, \| T^n(x) - T^{n+1}(x), u \|\}.$$

Therefore,

$$M(x_{n-1}, x_n) \leq \max\{\| T^{n-1}(x) - T^n(x), u \|, \| T^n(x) - T^{n+1}(x), u \|\}.$$

Suppose

$$\max\{\| T^{n-1}(x) - T^n(x), u \|, \| T^n(x) - T^{n+1}(x), u \|\}$$

$$= \| T^n(x) - T^{n+1}(x), u \| = d_n(u).$$

Then, from (i), $\psi(d_n(u)) \leq \psi(d_n(u)) - \phi(d_n(u))$.

It is a contradiction to (iii) since $\phi(d_n(u)) > 0$.

That is, $M(x_{n-1}, x_n) = d_{n-1}(u)$.

Therefore, $d_n(u) \leq M(x_{n-1}, x_n) = d_{n-1}(u)$ for all $n \in \mathbb{N}$.

Hence, $\{d_n(u)\}$ is a non-increasing sequence and each $d_n(u) \geq 0$.

Therefore,

$$\lim_{n\to\infty} d_n(u) = l \geq 0.$$

Next, we show that $l = 0$ suppose, $l > 0$.

If there exists an n such that $d_{n-1}(u) = l$, then $l = d_n(u) \leq M(d_{n-1}(u))$ $= l$,

$$\psi(d_n(u)) \leq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)),$$

$$\psi(l) \leq \psi(l) - \phi(l).$$

It is a contradiction. Suppose $d_n(u) > l$ for all $n \geq 1$. Then

$$\psi(d_n(u)) \leq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)),$$

$$\psi(l+) \leq \psi(l+) - \liminf_{n\to\infty} \phi(M(x_{n-1}, x_n)).$$

Again, it is a contradiction to (iii).

Therefore,

$$\lim_{n\to\infty} d_n(u) = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Suppose not, then there exists an $\varepsilon > 0$, for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with n_k is the smallest index such that $n_k > m_k > k > 1$, $\| x_{m_k} - x_{n_k}, u \| \geq \varepsilon$ and $\| x_{m_k} - x_{n_k - 1} \| < \varepsilon$ for all $u \in X$.

Now,

$$\begin{aligned} \varepsilon &\leq \| x_{m_k} - x_{n_k}, u \| \\ &\leq \| x_{m_k} - x_{n_k - 1}, u \| + \| x_{n_k - 1} - x_{n_k}, u \| \\ &< \| x_{m_k} - x_{n_k - 1}, u \| + d_{n_k - 1}(u). \end{aligned}$$

Letting *k* tends to infinity, we get

$$\varepsilon \leq \lim_{k\to\infty} \| x_{m_k} - x_{n_k}, u \| < \varepsilon + 0.$$

That is,

$$\lim_{k\to\infty} \| x_{m_k} - x_{n_k}, u \| = \varepsilon.$$

Again,

$$\| x_{m_k} - x_{n_k - 1}, u \| \leq \| x_{m_k} - x_{n_k}, u \| + \| x_{n_k} - x_{n_k - 1}, u \|, \tag{2}$$

$$\| x_{m_k} - x_{n_k}, u \| \leq \| x_{m_k} - x_{n_k - 1}, u \| + \| x_{n_k - 1} - x_{n_k}, u \|. \tag{3}$$

That is,

$$| \| x_{m_k} - x_{n_k}, u \| - \| x_{m_k} - x_{n_k - 1}, u \| | \leq \| x_{n_k - 1} - x_{n_k}, u \|. \tag{4}$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} \| x_{m_k} - x_{n_k - 1}, u \| = \varepsilon.$$

Similarly, we can prove that

$$\lim_{k\to\infty} \| x_{m_k-1} - x_{n_k}, u \| = \lim_{k\to\infty} \| x_{m_k-1} - x_{n_k-1}, u \| = \varepsilon,$$

$$\lim_{k\to\infty} \| x_{m_k} - x_{n_k+1}, u \| = \varepsilon.$$

Now,

$$M(x_{m_k-1}, x_{n_k-1}) = \max\Big\{\| x_{m_k-1} - x_{n_k-1}, u \|,$$

$$\| x_{m_k-1} - x_{m_k}, u \|, \| x_{n_k-1} - x_{n_k}, u \|,$$

$$\frac{\| x_{m_k-1} - x_{n_k}, u \| + \| x_{m_k} - x_{n_k-1}, u \|}{2}\Big\}.$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}) = \varepsilon.$$

If there is a subsequence $\{k(p)\}$ of $\{k\}$ such that

$$\varepsilon < \| x_{m(k(p))} - x_{n(k(p))}, u \|,$$

then from (1), we get

$$\psi(\| x_{m(k(p))} - x_{n(k(p))}, u \|)$$

$$\leq \psi(M(x_{m(k(p))-1}, x_{n(k(p))-1})) - \phi(M(x_{m(k(p))-1}, x_{n(k(p))-1})).$$

Letting $p \to \infty$, we get

$$\psi(\varepsilon+) \leq \psi(\varepsilon+) - \liminf_{p\to\varepsilon} \phi(\varepsilon)$$

which is a contradiction to (iii). Repeat the discussion for if there is a subsequence $\{k(p)\}$ of $\{k\}$ such that

$$\varepsilon < \| x_{m(k(p))+1} - x_{n(k(p))}, u \|$$

and if there is a subsequence $\{k(p)\}$ of $\{k\}$ such that

$$\varepsilon < \| x_{m(k(p))} - x_{n(k(p))+1}, u \|,$$

then we arrive at a contradiction to (iii). Therefore, we can suppose that there exists a k_1 such that

$$\| x_{m(k(p))} - x_{n(k(p))}, u \| = \varepsilon;$$

$$\| x_{m(k(p))} - x_{n(k(p))+1}, u \| = \varepsilon; \quad \| x_{m(k(p))+1} - x_{n(k(p))}, u \| \leq \varepsilon$$

for all $k \geq k_1$. Then $M(x_{m(k)}, x_{n(k)}) = \varepsilon$ for $k \geq k_3$, where $k_3 = \max\{k_1, k_2\}$ is such that $\| x_n - x_{n+1}, u \| < \varepsilon$ for all $k \geq k_2$. Substituting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (i), we have

$$\psi(\| x_{m+1} - x_{n+1}, u \|) \leq \psi(\varepsilon) - \phi(\varepsilon)$$

for any $k \geq k_3$. Letting $k \to \infty$, we obtain $\psi(\varepsilon-) \leq \psi(\varepsilon) - \phi(\varepsilon)$. Again, it is a contradiction to (iii). Hence $\{x_n\}$ is a Cauchy sequence. Since X is a Banach space, there exists a $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Next, we will show that z is the fixed point of T.

By (1),

$$\psi(\| x_{n+1}(x) - Tz, u \|) \leq \psi(M(x_n, z)) - \phi(M(x_n, z)),$$

where

$$M(x_n, z) = \max\left\{ \| x_n - z, u \|, \| x_n - x_{n+1}, u \|, \| z - Tz, u \|, \right.$$

$$\left. \frac{\| x - Tz, u \| + \| z - x_{n+1}, u \|}{2} \right\}.$$

Suppose $z \neq Tz$. Then $\| z - Tz, u \| > 0$ for some $u \in X$,

$$\lim_{n\to\infty} M(x_n, z) = \lim_{n\to\infty} \max\left\{ \| x_n - z, u \|, \frac{\| x_n - x_{n+1}, u \| + \| z - Tz, u \|}{2}, \right.$$

$$\left.\frac{\| x_n - Tz, u \| + \| z - x_{n+1}, u \|}{2}\right\}$$

$$= \frac{\| z - Tz, u \|}{2} > 0.$$

That is, we have

$$\lim_{n \to \infty} M(x_n, z) = \frac{\| z - Tz, u \|}{2} > 0$$

using this and letting $n \to \infty$ in [3],

$$\lim_{n \to \infty} \psi(\| x_{n+1} - Tz, u \|) \leq \lim_{n \to \infty} \psi(M(x_n, z)) - \liminf_{n \to \infty} \phi(M(x_n, z)).$$

That is,

$$\psi(\| z - Tz, u \|) \leq \psi\left(\frac{\| z - Tz, u \|}{2}\right) - \liminf_{n \to \infty} \phi\left(\frac{\| z - Tz, u \|}{2}\right),$$

that is,

$$\psi(\| z - Tz, u \|) \leq \psi\left(\frac{\| z - Tz, u \|}{2}\right) \text{ since } \liminf_{n \to \infty} \phi\left(\frac{\| z - Tz, u \|}{2}\right) > 0.$$

It is a contradiction to ψ is nondecreasing. Hence $z = Tz$. Next, we show that z is unique. Suppose there is a $y \in X$ such that $y = Ty$. Then we have

$$M(y, z) = \max\left\{\| y - z, u \|, \| y - Ty, u \|, \| z - Tz, u \|, \right.$$

$$\left.\frac{\| y - Tz, u \| + \| z - Ty, u \|}{2}\right\}$$

$$= \| y - z, u \|.$$

Then from (1),

$$\psi(\| y - z, u \|) \leq \psi(\| y - z, u \|) - \phi(\| y - z, u \|).$$

It is a contradiction. Hence z is a unique fixed point for T. □

Corollary 3.1. *Let X be a 2-Banach space and let $T : X \to X$ be a self-mapping satisfying the condition $\| Tx - Ty, u \| \leq M(x, y) - \phi(M(x, y))$ for all $x, y, u \in X$ and where ϕ is non-decreasing function from $\mathbb{R}^+$ into itself such that $\phi(t) < t$ for each $t > 0$. Then T has a unique fixed point in X.*

Proof. In the previous theorem, taking $\psi(t) = t$ and $\phi(t) = \frac{t}{2}$, we get the result. □

Corollary 3.2. *Let X be a 2-Banach space and let $T : X \to X$ be a self-mapping satisfying the condition $\| Tx - Ty, u \| \leq \phi(M(x, y))$ for all $x, y, u \in X$ and where ϕ is non-decreasing function from $\mathbb{R}^+$ into itself such that $\phi(t) < t$ for each $t > 0$. Then T has a unique fixed point* in *X.*

Proof. In the previous corollary, taking $\phi(t) = \frac{t}{2}$, we get the result. □

As a particular case of this corollary, we get the main results of Gangopadhyay et al. appeared in [4].

Corollary 3.3. *Let X be a 2-Banach space and let $T : X \to X$ be a self-mapping satisfying the condition $\| Tx - Ty, u \| \leq \phi(N(x, y))$ for all $x, y, u \in X$ and where ϕ is upper semicontinuous from $\mathbb{R}^+$ into itself such that $\phi(t) < t$ for each $t > 0$. Then T has a unique fixed point in X, where*

$$N(x, y) = \max\left\{\| x - y, u \|, \frac{\| x - Tx, u \| + \| y - Ty, u \|}{2}, \right.$$

$$\left. \frac{\| x - Ty, u \| + \| y - Tx, u \|}{2}\right\}.$$

Proof. Note the fact for all $x, y, u \in X$,

$$N(x, y) \leq M(x, y),$$

$$\phi(N(x, y)) \leq \phi(M(x, y)),$$

$$\| Tx - Ty, u \| \leq \phi(N(x, y)) \leq \phi(M(x, y)),$$

then by Corollary 3.2, T has a unique fixed point. □

Remark 3.2. Note that if ψ and ϕ are continuous montone nondecreasing functions satisfying the condition in (1), then they satisfy the condition in (iii) of the main theorem.

Theorem 3.3. *Let X be a linear 2-Banach space. Let $\{T_n\}$ be a sequence of self-mapping defined on X satisfying*

$$\psi(\| T_n x - T_n y, u \|) \leq \psi(M_n(x, y)) - \phi(M_n(x, y)),$$

where

(i) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$.

(ii) $\phi : [0, \infty) \to [0, \infty)$ *is a continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$ *and*

$$M_n(x, y) = \max\left\{\| x - y, u \|, \| x - T_n x, u \|, \| y - T_n y, u \|, \frac{\| x - T_n y, u \| + \| y - T_n x, u \|}{2}\right\}.$$

And if $T = \lim_{n\to\infty} T_n$, *then* T *has a unique fixed point* z *such that* $\lim_{n\to\infty} z_n = z$.

Proof. Since ψ and 2-norm are continuous and T_n is convergent to T, we have for all $x, y, u \in X$,

$$\lim_{n\to\infty} \| T_n x - T_n y, u \| = \| \lim_{n\to\infty}(T_n x - T_n y), u \| = \| Tx - Ty, u \|$$

and

$$\lim_{n\to\infty} M_n(x, y) = \lim_{n\to\infty} \max\left\{\| x - y, u \|, \| x - T_n x, u \|, \| y - T_n y, u \|,\right.$$

$$\left.\frac{\| x - T_n y, u \| + \| y - T_n x, u \|}{2}\right\}$$

$$= \max\left\{\| x - y, u \|, \| x - Tx, u \|, \| y - Ty, u \|,\right.$$

$$\left.\frac{\| x - Ty, u \| + \| y - Tx, u \|}{2}\right\}$$

$$= M(x, y).$$

Therefore,

$$\begin{aligned}\psi(\| Tx - Ty, u \|) &= \lim_{n\to\infty} \psi(\| T_n x - T_n y, u \|)\\ &\leq \lim_{n\to\infty} \psi(M_n(x, y)) - \phi(M_n(x, y))\\ &= \psi(M(x, y)) - \phi(M(x, y)).\end{aligned}$$

That is, T satisfies the required conditions of Theorem 3.1.

Therefore, T has a unique fixed point $z \in X$.

We show that $z_n \to z$.

Suppose $\| z - z_n, u \| > 0$ for all $n \in \mathbb{N}$.

Consider

$$\| z - z_n, u \| = \| Tz - T_n z_n, u \| \leq \| Tz - T_n z, u \| + \| T_n z - T_n z_n, u \|,$$

that is,

$$|\| z - z_n, u \| - \| Tz - T_n z, u \|| \leq \| T_n z - T_n z_n, u \|$$

and ψ is a nondecreasing function, we have

$$\begin{aligned}\psi(|\| z - z_n, u \| - \| Tz - T_n z, u \||) &\leq \psi(\| T_n z - T_n z_n, u \|)\\ &\leq \psi(M_n(z, z_n)) - \phi(M_n(z, z_n))\end{aligned}$$

taking limit as $n \to \infty$, we get

$$\psi(\| z - z_n, u \|) \leq \psi(M_n(z, z_n)) - \phi(M_n(z, z_n))$$
$$= \psi(\| z - z_n, u \|) - \phi(\| z - z_n, u \|),$$

it is a contradiction, therefore we conclude that $z_n \to z$. □

Theorem 3.4. *Let X be a linear* 2*-Banach space. Let* $\{T_n\}$ *be a sequence of self-mappings defined on X satisfying*

$$\psi(\| T_i x - T_j y, u \|) \leq \psi(M(x, y)) - \phi(M(x, y))$$

for all i, j *and* $i \neq j$,

$$M(x, y)$$
$$= \max\left\{\| x - y, u \|, \| x - T_i x, u \|, \| y - T_j y, u \|, \frac{\| x - T_j y, u \| + \| y - T_i x, u \|}{2}\right\},$$

where

(i) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$.

(ii) $\phi : [0, \infty) \to [0, \infty)$ *is a continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$.

Then T_n *has a unique common fixed point.*

Proof. Let x be any element in X and let $x_n = T_n(x_{n-1})$. Substituting $i = n$ and $j = n + 1$, we get

$$\psi(\| x_n - x_{n+1}, u \|)$$
$$= \psi(\| T_n(x_{n-1}) - T_{n+1}(x_n), u \|)$$
$$\leq \psi\Bigg(\max\Bigg\{\| x_{n-1} - x_n, u \|, \| x_{n-1} - T_n(x_{n-1}), u \|, \| x_n - T_{n+1}(x_n), u \|,$$
$$\frac{\| x_{n-1} - T_{n+1} x_n, u \| + \| x_n - T_n(x_{n-1}), u \|}{2}\Bigg\}\Bigg)$$

$$- \phi\left(\max\left\{ \| x_{n-1} - x_n, u \|, \| x_{n-1} - T_n(x_{n-1}), u \|, \| x_n - T_{n+1}(x_n), u \|, \right.\right.$$

$$\left.\left. \frac{\| x_{n-1} - T_{n+1}x_n, u \| + \| x_n - T_n(x_{n-1}), u \|}{2} \right\}\right)$$

$$\leq \psi\left(\max\left\{ \| x_{n-1} - x_n, u \|, \| x_n - x_{n+1}, u \|, \frac{\| x_{n-1} - x_{n+1}, u \|}{2} \right\}\right)$$

$$\leq \psi(\max\{\| x_{n-1} - x_n, u \|, \| x_n - x_{n+1}, u \|\}).$$

Suppose

$$\max\{\| x_{n-1} - x_n, u \|, \| x_n - x_{n+1}, u \|\} = \| x_n - x_{n+1}, u \|.$$

Then $\psi(\| x_n - x_{n+1}, u \|) \leq \psi(\| x_n - x_{n+1}, u \|) - \phi(\| x_n - x_{n+1}, u \|)$. It is a contradiction since ϕ is nondecreasing and $\| x_n - x_{n+1}, u \| > 0$.

Therefore, $\max\{\| x_{n-1} - x_n, u \|, \| x_n - x_{n+1}, u \|\} = \| x_{n-1} - x_n, u \|$ and $\{\| x_{n-1} - x_n, u \|\}$ is a decreasing and bounded below sequence in *X*.

Therefore, it converges to some $l \geq 0$ but we can prove that $l = 0$. As in the theorem, we can prove that $\{x_n\}$ is a Cauchy sequence in *X*. Since *X* is a 2-Banach space, $\{x_n\}$ converges to a point $z \in X$. We will show that *z* is the common fixed point of T_m for each *m*. Suppose *z* is not the common fixed point. Then $\| T_m(z) - z, u \| > 0$. Consider

$$\psi(\| T_m z - x_{n+1}, u \|) = \psi(\| T_m z - T_{n+1}(x_n), u \|)$$

$$\leq \psi(M(z, x_n)) - \phi(M(z, x_n)),$$

where

$$M(z, x_n) = \max\left\{ \| x_n - z, u \|, \| x_n - T_{n+1}(x_n), u \|, \| z - T_m(z), u \| \right.$$

$$\left. \frac{\| x_n - T_m z, u \| + \| z - T_{n+1}(x_n), u \|}{2} \right\}$$

$$= \max\left\{\| x_n - z, u \|, \| x_n - x_{n+1}, u \|, \| z - T_m(z), u \|, \frac{\| x_n - T_m z, u \| + \| z - x_{n+1}, u \|}{2}\right\}.$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(z, x_n) = \| z - T_m(z), u \|,$$

$$\psi(\| z - T_m z, u \|) \leq \psi(\| z - T_m z, u \|) - \phi(\| z - T_m z, u \|), \tag{5}$$

it is a contradiction, therefore $T_m z = z$ for all m. □

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