Far East Journal of Mathematical Sciences (FJMS)
Volume 73, Number 2, 2013, Pages 289-303
Published Online: March 2013
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

OSCILLATION OF FIRST-ORDER IMPULSIVE ADVANCED DIFFERENTIAL EQUATIONS WITH INTEGRAL JUMP CONDITION

Phollakrit Thiramanus and Jessada Tariboon

Department of Mathematics
Faculty of Applied Science
King Mongkut's University of Technology North Bangkok
Bangkok 10800, Thailand
e-mail: phollakrit@hotmail.com
jessadat@kmutnb.ac.th

Abstract

In this paper, we investigate the oscillation of first-order impulsive advanced differential equations with integral jump condition of the form

$$x'(t) = p(t)x(t+\tau), \quad t \neq t_k,\ t \in [t_0, \infty),$$

$$\Delta x(t_k) = c_k \int_{t_k - \theta_k}^{t_k - \sigma_k} x(s)\,ds, \quad t = t_k,\ k = 1, 2, \ldots,$$

where $0 \leq \sigma_k \leq \theta_k \leq t_k - t_{k-1}$, $c_k \geq 0$, $k = 1, 2, \ldots,$ and $\tau \geq 0$ are given constants.

2010 Mathematics Subject Classification: 34A37, 34C10, 34K11.

Keywords and phrases: oscillation, impulsive differential inequality, impulsive differential equation, advanced argument.

Submitted by K. K. Azad

Received November 23, 2012

1. Introduction

Many evolution processes in real world phenomena and applications are characterized by the fact that at certain moments of time they experience an abrupt change of state. Often these short-term perturbations are treated as having acted instantaneously or in the form of impulses. Impulsive differential equations and impulsive functional differential equations have been developed for mathematical modeling of impulsive problems in physics, medicine, population dynamics, biotechnology, control theory, etc. see [1-5].

The theory of oscillation of impulsive functional differential equations has been studied by a number of authors; see, for example, [6-13] and references therein. However, there are only a few papers that consider the oscillatory behavior of impulsive differential equations with advanced argument, see [14-17].

In this paper, we investigate sufficient conditions for oscillation of all solutions of the first-order impulsive advanced differential equation with integral jump condition of the form

$$\begin{cases} x'(t) = p(t)x(t+\tau), & t \neq t_k,\ t \in [t_0, \infty), \\ \Delta x(t_k) = c_k \int_{t_k-\theta_k}^{t_k-\sigma_k} x(s)ds, & t = t_k,\ k = 1, 2, \ldots. \end{cases} \tag{1.1}$$

Together with equation (1.1), we consider the first-order impulsive advanced differential inequalities with integral jump condition

$$\begin{cases} x'(t) \geq p(t)x(t+\tau), & t \neq t_k,\ t \in [t_0, \infty), \\ \Delta x(t_k) \geq c_k \int_{t_k-\theta_k}^{t_k-\sigma_k} x(s)ds, & t = t_k,\ k = 1, 2, \ldots, \end{cases} \tag{1.2}$$

and

$$\begin{cases} x'(t) \leq p(t)x(t+\tau), & t \neq t_k,\ t \in [t_0, \infty), \\ \Delta x(t_k) \leq c_k \int_{t_k-\theta_k}^{t_k-\sigma_k} x(s)ds, & t = t_k,\ k = 1, 2, \ldots, \end{cases} \tag{1.3}$$

under the following hypotheses:

(H_1) $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ are fixed points with $\lim_{k\to\infty} t_k = \infty$;

(H_2) Function $p \in PC([t_0, \infty), R) = \{p : [t_0, \infty) \to R \mid p(t)$ is continuous for $t \in [t_0, \infty)$, $t \ne t_k$ and $p(t_k^+)$, $p(t_k^-)$ exist with $p(t_k^-) = p(t_k)$, $k = 1, 2, ...\}$;

(H_3) Constants $\tau > 0$ and $c_k \ge 0$, σ_k, θ_k are satisfied $0 \le \sigma_k \le \theta_k \le t_k - t_{k-1}$, for $k = 1, 2, \ldots$.

A function $x \in PC([t_0, \infty), R)$ is said to be a *solution* of equation (1.1) on $[t_0, \infty)$ if the following conditions are satisfied:

(i) $x(t)$ satisfies $x'(t) = p(t)x(t + \tau)$ for $t \in [t_0, \infty)$ and $t \ne t_k$;

(ii) $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = c_k \int_{t_k-\theta_k}^{t_k-\sigma_k} x(s)ds$ for each t_k, and $x(t)$ and $x'(t)$ are left continuous for each t_k, $k = 1, 2, \ldots$.

The solution $x(t)$ of inequality (1.2) is said to be *eventually positive* if there exists $T > t_0$ such that $x(t) > 0$ for $t \ge T$. Analogously, the solution $x(t)$ of inequality (1.3) is said to be *eventually negative* if there exists $T > t_0$ such that $x(t) < 0$ for $t \ge T$.

Definition 1.1. A nontrivial solution of equation (1.1) is said to be *nonoscillatory* if the solution is eventually positive or eventually negative. Otherwise, it is said to be *oscillatory*. Equation (1.1) is said to be *oscillatory* if all solutions are oscillatory.

Denote $t_l = \max\{t_k : t \ge t_k, k = 1, 2, ...\}$. The following lemma will be used in our main results.

Lemma 1.2. *Let* (H_1) *hold. Assume that* $q \in PC([t_0, \infty), R)$ *and* $0 \le \sigma_k \le \theta_k \le t_k - t_{k-1}$, $c_k \ge 0$, $k = 1, 2, ...,$ *are constants. If*

$$\begin{cases} x'(t) \geq q(t), & t \neq t_k,\, t \in [t_0, \infty), \\ \Delta x(t_k) \geq c_k \int_{t_k - \theta_k}^{t_k - \sigma_k} x(s)ds, & t = t_k,\, t_k \in [t_0, \infty), \end{cases} \tag{1.4}$$

then for $t \geq t_0$,

$$x(t) \geq x(t_0)\left(\prod_{t_0 < t_k < t}[1 + c_k(\theta_k - \sigma_k)]\right) + \sum_{t_0 < t_k < t}\left[\prod_{t_k < t_j < t}(1 + c_j(\theta_j - \sigma_j))\right.$$

$$\times\left([1 + c_k(\theta_k - \sigma_k)]\int_{t_{k-1}}^{t_k - \theta_k} q(s)ds\right.$$

$$+ \int_{t_k - \theta_k}^{t_k - \sigma_k}[1 + c_k(t_k - \sigma_k - s)]q(s)ds$$

$$\left.\left. + \int_{t_k - \sigma_k}^{t_k} q(s)ds\right)\right] + \int_{t_l}^{t} q(s)ds. \tag{1.5}$$

Proof. See [18].

2. Main Results

Theorem 2.1. *Let* (H_1)-(H_3) *hold. Assume that there exists a sequence of disjoint intervals* $J_n = (\xi_n, \eta_n)$, $n \in N$ *such that* $\eta_n - \xi_n = 2\tau$ *and* $t_j - \theta_j \in J_n$, *where* $t_j = \min\{t_k : t_k \in J_n\}$. *In addition, suppose that the following conditions hold*:

(M_1) *For each* $n \in N$, $t \in J_n$ *and* $t_k \in J_n$,

$$p(t) \geq 0, \quad c_k \geq 0, \quad k = 1, 2, \ldots. \tag{2.1}$$

(M_2) *There exists* $\nu_1 > 0$ *such that for* $n \geq \nu_1$,

$$\sum_{\xi_n < t_k < \xi_n + \tau}\left(\prod_{t_k < t_j < \xi_n + \tau}(1 + c_j(\theta_j - \sigma_j))\left[(1 + c_k(\theta_k - \sigma_k))\int_{t_{k-1}}^{t_k - \theta_k} p(s)ds\right.\right.$$

$$+ \int_{t_k-\theta_k}^{t_k-\sigma_k} p(s)(1 + c_k(t_k - \sigma_k - s))ds + \int_{t_k-\sigma_k}^{t_k} p(s)ds\Bigg]\Bigg)$$

$$+ \int_{t_l}^{\xi_n+\tau} p(s)ds \geq 1. \tag{2.2}$$

Then:

(D_1) *Inequality* (1.2) *has no eventually positive solutions*.

(D_2) *Inequality* (1.3) *has no eventually negative solutions*.

(D_3) *Every solution of equation* (1.1) *is oscillatory*.

Proof. Firstly, we prove that inequality (1.2) has no eventually positive solution. Suppose, to the contrary, that there exists a solution $x(t)$ of (1.2) such that $x(t) > 0$ for $t \geq T$. Since $\xi_n \to \infty$ as $n \to \infty$, there exists $\nu_0 > 0$ such that for $n \geq \nu_0$ we have $\xi_n > T$. From (1.2) and (2.1), it follows that $x'(t) \geq 0$ and $\Delta x(t_k) = c_k \int_{t_k-\theta_k}^{t_k-\sigma_k} x(s)ds \geq 0$ for $t, t_k \in J_n$, i.e., $x(t)$ is a nondecreasing in $t \in J_n$, $n \geq \nu_0$.

Let $\nu = \max\{\nu_0, \nu_1\}$ and $n \geq \nu$. By using Lemma 1.2, we get

$$x(\xi_n + \tau) \geq x(\xi_n) \prod_{\xi_n < t_k < \xi_n+\tau} (1 + c_k(\theta_k - \sigma_k))$$

$$+ \sum_{\xi_n < t_k < \xi_n+\tau} \Bigg[\prod_{t_k < t_j < \xi_n+\tau} (1 + c_j(\theta_j - \sigma_j))$$

$$\times \Bigg((1 + c_k(\theta_k - \sigma_k)) \int_{t_{k-1}}^{t_k-\theta_k} p(s)x(s+\tau)ds$$

$$+ \int_{t_k-\theta_k}^{t_k-\sigma_k} (1 + c_k(t_k - \sigma_k - s))\, p(s)x(s+\tau)ds$$

$$+\int_{t_k-\sigma_k}^{t_k} p(s)x(s+\tau)ds\Bigg)\Bigg] + \int_{t_l}^{\xi_n+\tau} p(s)x(s+\tau)ds. \tag{2.3}$$

If $t \in (\xi_n, \xi_n + \tau)$, then $(t+\tau) \in (\eta_n - \tau, \eta_n)$. From $x(t)$ is a nondecreasing function in $(\eta_n - \tau, \eta_n)$, it follows that

$$\begin{aligned} x(\xi_n + \tau) \geq x(\xi_n) & \prod_{\xi_n < t_k < \xi_n + \tau} (1 + c_k(\theta_k - \sigma_k)) \\ & + x(\xi_n + \tau)\Bigg\{\sum_{\xi_n < t_k < \xi_n + \tau}\Bigg[\prod_{t_k < t_j < \xi_n + \tau} (1 + c_j(\theta_j - \sigma_j)) \\ & \times\Bigg((1 + c_k(\theta_k - \sigma_k))\int_{t_{k-1}}^{t_k - \theta_k} p(s)ds \\ & + \int_{t_k - \theta_k}^{t_k - \sigma_k} (1 + c_k(t_k - \sigma_k - s))\, p(s)ds \\ & + \int_{t_k - \sigma_k}^{t_k} p(s)ds\Bigg)\Bigg] + \int_{t_l}^{\xi_n + \tau} p(s)ds\Bigg\}. \end{aligned} \tag{2.4}$$

Therefore,

$$\begin{aligned} 0 \geq x(\xi_n) & \prod_{\xi_n < t_k < \xi_n + \tau} (1 + c_k(\theta_k - \sigma_k)) \\ & + x(\xi_n + \tau)\Bigg\{\sum_{\xi_n < t_k < \xi_n + \tau}\Bigg[\prod_{t_k < t_j < \xi_n + \tau} (1 + c_j(\theta_j - \sigma_j)) \\ & \times\Bigg((1 + c_k(\theta_k - \sigma_k))\int_{t_{k-1}}^{t_k - \theta_k} p(s)ds \\ & + \int_{t_k - \theta_k}^{t_k - \sigma_k} (1 + c_k(t_k - \sigma_k - s))\, p(s)ds \end{aligned}$$

$$+\int_{t_k-\sigma_k}^{t_k} p(s)ds\Big)\Big] + \int_{t_l}^{\xi_n+\tau} p(s)ds - 1\Big\}. \tag{2.5}$$

From (2.5) for $n \geq \nu,$ we have that

$$\sum_{\xi_n<t_k<\xi_n+\tau}\Bigg(\prod_{t_k<t_j<\xi_n+\tau}(1+c_j(\theta_j-\sigma_j))\Big[(1+c_k(\theta_k-\sigma_k))\int_{t_{k-1}}^{t_k-\theta_k} p(s)ds$$

$$+\int_{t_k-\theta_k}^{t_k-\sigma_k} p(s)(1+c_k(t_k-\sigma_k-s))ds + \int_{t_k-\sigma_k}^{t_k} p(s)ds\Big]\Bigg)$$

$$+\int_{t_l}^{\xi_n+\tau} p(s)ds < 1, \tag{2.6}$$

which contradicts (2.2). In order to prove that (1.3) has no eventually negative solution it suffices to note that if $x(t)$ is a solution of (1.3), then $-x(t)$ is a solution of (1.2). From (M_1) and (M_2), it follows that equation (1.1) has neither an eventually positive nor an eventually negative solution. Therefore, from Definition 1.1 the solution of (1.1) is oscillatory. The proof is complete. □

Theorem 2.2. *Let* (H_1)-(H_3) *hold. Assume that there exists a sequence of disjoint intervals* $J_n = (\xi_n, \eta_n),$ $n \in N$ *such that* $\eta_n - \xi_n \geq 2\tau$ *and* $t_j - \theta_j \in J_n,$ *where* $t_j = \min\{t_k : t_k \in J_n\}.$ *In addition, suppose that the following conditions hold*:

(M_3) *For any* $n \in N,$ $t \in J_n$ *and* $t_k \in J_n,$

$$p(t) \geq 0, \quad c_k \geq 0, \quad k = 1, 2, \ldots. \tag{2.7}$$

(M_4) *There exists a constant K and integer* $\nu_1 > 0$ *such that for any* $n \geq \nu_1$ *and* $t \in (\xi_n, \eta_n - \tau),$

$$\sum_{t \le t_k \le t+\tau} \left[\prod_{t_k \le t_j \le t+\tau} (1 + c_j(\theta_j - \sigma_j)) \left((1 + c_k(\theta_k - \sigma_k)) \int_{t_{k-1}}^{t_k - \theta_k} p(s) ds \right. \right.$$

$$\left. \left. + \int_{t_k - \theta_k}^{t_k - \sigma_k} p(s)(1 + c_k(t_k - \sigma_k - s)) ds + \int_{t_k - \sigma_k}^{t_k} p(s) ds \right) \right]$$

$$+ \int_{t_l}^{t+\tau} p(s) ds \ge K > e^{-1}. \tag{2.8}$$

(M_5) *There exist a constant* $\phi > 0$ *and an integer* $\nu_2 > 0$ *such that for any* $n \ge \nu_2$ *there exists* $t_n^* \in [\xi_n, \xi_n + \tau]$ *such that*

$$A_n(t_n^*) B_n(t_n^*) \ge \phi, \tag{2.9}$$

where

$$A_n(t) = \sum_{\xi_n \le t_k \le t} \left[\prod_{t_k \le t_j \le t} (1 + c_j(\theta_j - \sigma_j)) \right.$$

$$\times \left((1 + c_k(\theta_k - \sigma_k)) \int_{t_{k-1}}^{t_k - \theta_k} p(s) ds \right.$$

$$+ \int_{t_k - \theta_k}^{t_k - \sigma_k} p(s)(1 + c_k(t_k - \sigma_k - s)) ds$$

$$\left. \left. + \int_{t_k - \sigma_k}^{t_k} p(s) ds \right) \right] + \int_{t_l}^{t} p(s) ds,$$

and

$$B_n(t) = \sum_{t \le t_k \le \xi_n + \tau} \left[\prod_{t_k \le t_j \le \xi_n + \tau} (1 + c_j(\theta_j - \sigma_j)) \right.$$

$$\times \left((1 + c_k(\theta_k - \sigma_k)) \int_{t_{k-1}}^{t_k - \theta_k} p(s) ds \right.$$

$$+\int_{t_k-\theta_k}^{t_k-\sigma_k} p(s)(1+c_k(t_k-\sigma_k-s))ds$$

$$\left.\left. + \int_{t_k-\sigma_k}^{t_k} p(s)ds\right)\right] + \int_{t_l}^{\xi_n+\tau} p(s)ds.$$

(M_6) *There exists* $\nu_3 > 0$ *such that for any* $n \geq \nu_3$ *the inequality*

$$\eta_n - \xi_n > (m_0 + 1)\tau, \tag{2.10}$$

is valid, where

$$m_0 = \min\{m \in N : \phi(eK)^m > 1\}. \tag{2.11}$$

Then:

(D_4) *Inequality* (1.2) *has no eventually positive solutions.*

(D_5) *Inequality* (1.3) *has no eventually negative solutions*.

(D_6) *Every solution of equation* (1.1) *is oscillatory*.

Proof. Suppose, to the contrary, that the inequality (1.2) has a solution $x(t)$ such that for T large enough we have $x(t) > 0,\ t \geq T$. Since $\xi_n \to \infty$ as $n \to \infty$, there exists $\nu_0 > 0$ such that $\xi_n > T,\ n \geq \nu_0,\ k = 1, 2, \ldots$ and then from (1.2) and (2.7) it follows that $x(t)$ is a nondecreasing function in $J_n,\ n \geq \nu_0$.

Now, we set $\nu = \max(\nu_0, \nu_1, \nu_2, \nu_3)$. Then for any $n \geq \nu$ the solution $x(t)$ is a nondecreasing function in J_n. From (1.2) and Lemma 1.2, we have that

$$x(t_n^{*+}) \geq x(\xi_n) \prod_{\xi_n < t_k < t_n^*} (1 + b_k(\theta_k - \sigma_k))$$

$$+ \sum_{\xi_n < t_k < t_n^*} \left[\prod_{t_k < t_j < t_n^*} (1 + b_j(\theta_j - \sigma_j)) \right.$$

$$\times\Bigg((1+b_k(\theta_k-\sigma_k))\int_{t_{k-1}}^{t_k-\theta_k} p(s)x(s+\tau)ds$$

$$+\int_{t_k-\theta_k}^{t_k-\sigma_k}(1+b_k(t_k-\sigma_k-s))\,p(s)x(s+\tau)ds$$

$$+\int_{t_k-\sigma_k}^{t_k} p(s)x(s+\tau)ds\Bigg)\Bigg]+\int_{t_l}^{t_n^*} p(s)x(s+\tau)ds.$$

Since $x(t+\tau)$ is nondecreasing in (ξ_n, t_n^*) and $x(\xi_n) > 0$,

$$x(t_n^{*+}) \geq x(\xi_n+\tau^+)\Bigg\{\sum_{\xi_n<t_k<t_n^*}\Bigg[\prod_{t_k<t_j<t_n^*}(1+c_j(\theta_j-\sigma_j))$$

$$\times\Bigg((1+c_k(\theta_k-\sigma_k))\int_{t_{k-1}}^{t_k-\theta_k} p(s)ds$$

$$+\int_{t_k-\theta_k}^{t_k-\sigma_k}(1+b_k(t_k-\sigma_k-s))\,p(s)ds$$

$$+\int_{t_k-\sigma_k}^{t_k} p(s)ds\Bigg)\Bigg]+\int_{t_l}^{t_n^*} p(s)ds\Bigg\}.$$

That is

$$x(t_n^{*+}) \geq x(\xi_n+\tau^+)A_n(t_n^*). \tag{2.12}$$

Similarly, we have

$$x(\xi_n+\tau^+) \geq x(t_n^{*+})\prod_{t_n^*<t_k<\xi_n+\tau}(1+c_k(\theta_k-\sigma_k))$$

$$+\sum_{t_n^*<t_k<\xi_n+\tau}\Bigg[\prod_{t_k<t_j<\xi_n+\tau}(1+b_j(\theta_j-\sigma_j))$$

$$\times\Bigg((1+c_k(\theta_k-\sigma_k))\int_{t_{k-1}}^{t_k-\theta_k} p(s)x(s+\tau)ds$$

$$+\int_{t_k-\theta_k}^{t_k-\sigma_k}(1+c_k(t_k-\sigma_k-s))\,p(s)x(s+\tau)ds$$

$$+\int_{t_k-\sigma_k}^{t_k} p(s)x(s+\tau)ds\Bigg)\Bigg]+\int_{t_l}^{\xi_n+\tau} p(s)x(s+\tau)ds.$$

Then, we obtain

$$x(\xi_n+\tau^+)\geq x(t_n^*+\tau^+)B_n(t_n^*). \tag{2.13}$$

Hence, from (2.12) and (2.13), it follows that

$$\frac{x(t_n^*+\tau^+)}{x(t_n^{*+})}\leq\frac{1}{A_n(t_n^*)B_n(t_n^*)}$$

$$\leq\frac{1}{\phi},\quad n\geq\nu. \tag{2.14}$$

From (1.2) and Lemma 1.2, we have that

$$x(t+\tau)\geq x(t)\prod_{t<t_k<t+\tau}(1+c_k(\theta_k-\sigma_k))$$

$$+\sum_{t<t_k<t+\tau}\Bigg[\prod_{t_k<t_j<t+\tau}(1+c_j(\theta_j-\sigma_j))$$

$$\times\Bigg((1+c_k(\theta_k-\sigma_k))\int_{t_{k-1}}^{t_k-\theta_k} p(s)x(s+\tau)ds$$

$$+\int_{t_k-\theta_k}^{t_k-\sigma_k}(1+c_k(t_k-\sigma_k-s))\,p(s)x(s+\tau)ds$$

$$+\int_{t_k-\sigma_k}^{t_k} p(s)x(s+\tau)ds\Bigg)\Bigg]+\int_{t_l}^{t+\tau} p(s)x(s+\tau)ds,$$

for $n\geq\nu$ and $t\in(\xi_n,\,\eta_n-\tau)$. Since, $1+c_k(\theta_k-\sigma_k)\geq 1$ and $x(s+\tau)\geq$

$x(t+\tau^{+})$ for $s \in (t, t+\tau)$, we have

$$x(t+\tau) \geq x(t) + x(t+\tau)\left\{ \sum_{t<t_k<t+\tau} \left[\prod_{t_k<t_j<t+\tau} (1+c_j(\theta_j - \sigma_j)) \right.\right.$$

$$\times \left((1+c_k(\theta_k - \sigma_k)) \int_{t_{k-1}}^{t_k-\theta_k} p(s)ds \right.$$

$$+ \int_{t_k-\theta_k}^{t_k-\sigma_k} (1+c_k(t_k - \sigma_k - s))\, p(s)ds$$

$$\left.\left.\left. + \int_{t_k-\sigma_k}^{t_k} p(s)ds \right) \right] + \int_{t_l}^{t+\tau} p(s)ds \right\}.$$

Thus,

$$x(t) \leq x(t+\tau)\left\{ 1 - \left[\sum_{t<t_k<t+\tau} \left[\prod_{t_k<t_j<t+\tau} (1+c_j(\theta_j - \sigma_j)) \right.\right.\right.$$

$$\times \left((1+c_k(\theta_k - \sigma_k)) \int_{t_{k-1}}^{t_k-\theta_k} p(s)ds \right.$$

$$+ \int_{t_k-\theta_k}^{t_k-\sigma_k} (1+c_k(t_k - \sigma_k - s))\, p(s)ds$$

$$\left.\left.\left.\left. + \int_{t_k-\sigma_k}^{t_k} p(s)ds \right) \right] + \int_{t_l}^{t+\tau} p(s)ds \right] \right\}.$$

Using the fact that $1 - x \leq e^{-x}$, $x \in R$, we obtain

$$x(t) \leq x(t+\tau)\exp\left\{ - \left[\sum_{t<t_k<t+\tau} \left[\prod_{t_k<t_j<t+\tau} (1+b_j(\theta_j - \sigma_j)) \right.\right.\right.$$

$$\times \left((1+b_k(\theta_k - \sigma_k)) \int_{t_{k-1}}^{t_k-\theta_k} p(s)ds \right.$$

$$+ \int_{t_k-\theta_k}^{t_k-\sigma_k} (1 + b_k(t_k - \sigma_k - s))\, p(s)\,ds$$

$$\left.\left.+ \int_{t_k-\sigma_k}^{t_k} p(s)\,ds\right)\right] + \int_{t_l}^{t+\tau} p(s)\,ds\Bigg]\Bigg\}.$$

Consequently, for each $n \geq \nu$ and $t \in (\xi_n, \eta_n - \tau)$,

$$\frac{x(t+\tau)}{x(t)} \geq e^K \geq eK.$$

Repeating the above argument, we get to

$$\frac{x(t+\tau)}{x(t)} \geq (eK)^{m_0}, \tag{2.15}$$

for any $n \geq \nu$ and $t \in (\xi_n, \eta_n - m_0\tau)$.

Since $\xi_n + \tau < \eta_n - m_0\tau$, (2.15) is valid for any $n \geq \nu$ and $t = t_n^* \in [\xi_n, \xi_n + \tau]$, i.e.,

$$\frac{x(t_n^* + \tau^+)}{x(t_n^{*+})} \geq (eK)^{m_0}. \tag{2.16}$$

Then from (2.14) and (2.16), we get

$$\frac{1}{\phi} \geq (eK)^{m_0}$$

which contradicts (2.11).

The proof of assertions (D_5) and (D_6) is carried out as in Theorem 2.1.

□

Acknowledgements

This research is financially supported by the Office of the Higher Education Commission of Thailand, and King Mongkut's University of Technology North Bangkok, Thailand.

References

[1] D. D. Bainov and P. S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Longman Scientific & Technical, Harlow, 1993.

[2] D. D. Bainov and P. S. Simeonov, Impulsive Differential Equations: Asymptotic Properties of the Solutions, World Scientific, Singapore, 1995.

[3] M. Benchohra, J. Henderson and S. Ntouyas, Impulsive Differential Equations and Inclusions, Hindawi Publishing Corporation, New York, 2006.

[4] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[5] A. M. Samoilenko and N. A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[6] R. P. Agarwal and F. Karakoç, A survey on oscillation of impulsive delay differential equations, Comput. Math. Appl. 60 (2010), 1648-1685.

[7] L. P. Gimenes and M. Federson, Oscillation by impulses for a second-order delay differential equation, Comput. Math. Appl. 52 (2006), 819-828.

[8] J. R. Graef, J. H. Shen and I. P. Stavroulakis, Oscillation of impulsive neutral delay differential equations, J. Math. Anal. Appl. 268 (2002), 310-333.

[9] K. Guan and J. Shen, Oscillation criteria for a first-order impulsive neutral differential equation of Euler form, Comput. Math. Appl. 58 (2009), 670-677.

[10] M. Huang and W. Feng, Force oscillations for second order delay differential equations with impulses, Comput. Math. Appl. 59 (2010), 18-30.

[11] J. Shen and Z. Zou, Oscillation criteria for first-order impulsive differential equations with positive and negative coefficients, J. Comput. Appl. Math. 217 (2008), 28-37.

[12] J. Yan, Oscillation properties of a second-order impulsive delay differential equation, Comput. Math. Appl. 47 (2004), 253-258.

[13] J. Yan, A. Zhao and Q. Zhang, Oscillation properties of nonlinear impulsive delay differential equations and applications to population models, J. Math. Anal. Appl. 322 (2006), 359-370.

[14] D. D. Bainov, M. B. Dimitrova and P. S. Simeonov, Sufficient conditions for oscillation of the solutions of a class of impulsive differential equations with advanced argument, Note Mat. 14(1) (1994), 139-145.

[15] D. D. Bainov, M. B. Dimitrova and A. B. Dishliev, Oscillation of the solutions of nonlinear impulsive differential equations of the first order with advanced argument, J. Appl. Anal. 5(2) (1999), 261-275.

[16] M. K. Grammatikopoulos, M. B. Dimitrova and V. I. Donev, Oscillations of first order impulsive differential equations with variable coefficients and advanced argument, AIP Conference Proceedings 946 (2007), 206-214.

[17] J. Yan, Oscillation of first-order impulsive differential equations with advanced argument, Comput. Math. Appl. 42 (2001), 1353-1363.

[18] P. Thiramanus and J. Tariboon, Impulsive differential and impulsive integral inequalities with integral jump conditions, J. Inequal. Appl. (2012), 2012:25.