JP Journal of Algebra, Number Theory and Applications
Volume 28, Number 2, 2013, Pages 167-179
Published Online: March 2013
Available online at http://pphmj.com/journals/jpanta.htm
Published by Pushpa Publishing House, Allahabad, INDIA

EXPLICIT DETERMINANTS OF THE FIBONACCI RFPLR CIRCULANT AND LUCAS RFPLR CIRCULANT MATRIX

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Abstract

Let A be a row first-plus-last right circulant matrix and C be a row first-plus-last left circulant matrix whose first rows are $(F_1, F_2, \ldots, F_n)$ and $(L_1, L_2, \ldots, L_n)$, respectively, where F_n is the Fibonacci number and L_n is the Lucas number. In this paper, by using the inverse factorization of polynomial of degree n, the explicit

2010 Mathematics Subject Classification: 15A15, 11B39, 11C20.

Keywords and phrases: determinant, Fibonacci, Lucas, RFPLR circulant matrix, FRPLL circulant matrix.

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This project is supported by NSFC (Grant No. 11201212) and the Promotive Research Fund for Excellent Young and Middle-aged Scientists of Shandong Province (Grant No. BS 2012DX004).

Communicated by K. K. Azad

Received September 19, 2012

determinants of matrices A and C are expressed by utilizing only the Fibonacci and Lucas numbers.

1. Introduction

The Fibonacci and Lucas sequences [12] are defined by the following recurrence relations, respectively,

$$F_{n+1} = F_n + F_{n-1}, \text{ where } F_0 = 0,\ F_1 = 1,$$

$$L_{n+1} = L_n + L_{n-1}, \text{ where } L_0 = 2,\ L_1 = 1.$$

The first few values of the sequences are given by the following table $(n \geq 0)$:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
L_n	2	1	3	4	7	11	18	29	47	76

The sequences $\{F_n\}$, $\{L_n\}$ are given by the Binet formulae

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_n = \alpha^n + \beta^n,$$

where α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$.

Definition 1 [4]**.** A row first-plus-last right (RFPLR) circulant matrix with the first row $(a_1, a_2, ..., a_n)$, denoted by $\mathrm{RFPLRcircfr}(a_1, a_2, ..., a_n)$, is meant a square matrix of the form

$$M := \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_n & a_1 + a_n & a_2 & \cdots & a_{n-1} \\ \vdots & a_n + a_{n-1} & a_1 & \ddots & \vdots \\ a_3 & \vdots & \ddots & \ddots & a_2 \\ a_2 & a_3 + a_2 & \cdots & a_n + a_{n-1} & a_1 + a_n \end{pmatrix}_{n \times n}.$$

If the first row of a RFPLR circulant matrix is $(F_1, F_2, ..., F_n)$, then the matrix is called *Fibonacci RFPLR circulant matrix*. Similarly, if the first row of a RFPLR circulant matrix is $(L_1, L_2, ..., L_n)$, then the matrix is called *Lucas RFPLR circulant matrix*.

Note that the RFPLR circulant matrix is an $x^n - x - 1$ circulant matrix [2], and that is neither the extension of circulant matrix [6] nor its special case and they are two different kinds of special matrices. Moreover, it is a FLS r-circulant matrix [4] when $r = 1$, and is a ULS r-circulant matrix [14] when $r = 1$.

We define $\Theta_{(1,1)}$ as the basic RFPLR circulant matrix, that is,

$$\Theta_{(1,1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix}_{n \times n} = \mathrm{RFPLRcircfr}(0, 1, 0, ..., 0).$$

It is easily verified that $g(x) = x^n - x - 1$ has no repeated roots in its splitting field and $g(x) = x^n - x - 1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\Theta_{(1,1)}$. In addition, $\Theta_{(1,1)}$ is nonderogatory and satisfies $\Theta_{(1,1)}^j = \mathrm{RFPLRcircfr}(\underbrace{0, ..., 0}_{j}, 1, \underbrace{0, ..., 0}_{n-j-1})$ and $\Theta_{(1,1)}^n = I_n + \Theta_{(1,1)}$. Then a matrix A can be written in the form

$$A = f(\Theta_{(1,1)}) = \sum_{i=1}^{n} a_i \Theta_{(1,1)}^{i-1} \tag{1}$$

if and only if A is a RFPLR circulant matrix, where the polynomial $f(x) = \sum_{i=1}^{n} a_i x^{i-1}$ is called the *representer* of the RFPLR circulant matrix A.

Definition 2 [4]**.** A row first-plus-last left (RFPLL) circulant matrix with the first row $(a_1, a_2, ..., a_n)$, denoted by $\text{RFPLLcircfr}(a_1, a_2, ..., a_n)$, is meant a square matrix of the form

$$N := \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_1 + a_n & a_1 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ a_{n-1} & a_1 + a_n & \cdots & a_{n-2} + a_{n-3} & a_{n-2} \\ a_1 + a_n & a_2 + a_1 & \cdots & a_{n-1} + a_{n-2} & a_{n-1} \end{pmatrix}_{n \times n}.$$

If the first row of a RFPLL circulant matrix is $(F_1, F_2, ..., F_n)$, then the matrix is called *Fibonacci RFPLL circulant matrix*. Similarly, if the first row of a RFPLL circulant matrix is $(L_1, L_2, ..., L_n)$, then the matrix is called *Lucas RFPLL circulant matrix*.

Lemma 1 [14]**.** *Let* $A = RFPLRcircfr(a_1, a_2, ..., a_n)$. *Then we have*

$$\lambda_i = f(\varepsilon_i) = \sum_{j=1}^{n} a_j \varepsilon_i^{j-1}$$

and

$$\det A = \prod_{i=1}^{n} \sum_{j=1}^{n} a_j \varepsilon_i^{j-1},$$

where ε_i $(i = 1, 2, ..., n)$ *are the roots of the equation*

$$x^n - x - 1 = 0. \tag{2}$$

Recently, there are many interests in properties and generalization of some special matrices with famous numbers. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [3]. Lin gave the determinant of the Fibonacci-Lucas quasi-cyclic matrices [8]. Lind presented the determinants of circulant and skew-circulant involving Fibonacci numbers in [9]. Shen et al. [12] discussed the determinant of circulant matrix involving Fibonacci and Lucas numbers. Akbulak and Bozkurt [1] gave the norms of Toeplitz involving Fibonacci and

Lucas numbers. The authors [7, 11] discussed some properties of Fibonacci and Lucas matrices. Melham [10] gave some formulae involving Fibonacci and Pell numbers. Stanimirović et al. gave generalized Fibonacci and Lucas matrix in [13]. Zhang and Zhang [15] investigated the Lucas matrix and some combinatorial identities. In this paper, we define matrices of forms: $A = \text{RFPLRcircfr}(a_1, a_2, ..., a_n)$ and $C = \text{RFPLLcircfr}\ (a_1, a_2, ..., a_n)$ are $n \times n$ matrices whose first rows are $(F_1, F_2, ..., F_n)$ and $(L_1, L_2, ..., L_n)$, respectively, by using the inverse factorization of polynomial of degree *n*, the explicit determinants of these matrices are given only by the Fibonacci and Lucas numbers.

2. Determinant of the Fibonacci RFPLR Circulant and Fibonacci RFPLL Circulant Matrix

Lemma 2.

$$\prod_{i=1}^{n}(y - \varepsilon_i z + \varepsilon_i^2 x) = y^n + x^{n-1}(y + z) - y[(ax)^{n-1} + (bx)^{n-1}]$$

$$- [(ax)^n + (bx)^n] + x^n,$$

where

$$a = \frac{z + \sqrt{z^2 - 4xy}}{2x},$$

$$b = \frac{z - \sqrt{z^2 - 4xy}}{2x},$$

and ε_i $(i = 1, 2, ..., n)$ *satisfy equation* (2), $x, y, z \in \mathbf{R}$, $x \neq 0$.

Proof.

$$\prod_{i=1}^{n}(y - \varepsilon_i z + \varepsilon_i^2 x) = x^n \prod_{i=1}^{n}\left(\varepsilon_i^2 - \frac{z}{x}\varepsilon_i + \frac{y}{x}\right) = x^n \prod_{i=1}^{n}(\varepsilon_i - a)(\varepsilon_i - b)$$

$$= x^n \prod_{i=1}^{n}(a - \varepsilon_i)(b - \varepsilon_i),$$

where

$$a + b = \frac{z}{x}, \quad ab = \frac{y}{x},$$

$$a = \frac{z + \sqrt{z^2 - 4xy}}{2x}, \quad b = \frac{z - \sqrt{z^2 - 4xy}}{2x}.$$

By the ε_i $(i = 1, 2, ..., n)$ satisfy equation (2), we have

$$x^n - x - 1 = \prod_{i=1}^{n} (x - \varepsilon_i).$$

So

$$\begin{aligned}
&\prod_{i=1}^{n} (y - \varepsilon_i z + \varepsilon_i^2 x) \\
&= x^n (a^n - a - 1)(b^n - b - 1) \\
&= x^n [(ab)^n + ab - ab(a^{n-1} + b^{n-1}) + (a + b) - (a^n + b^n) + 1] \\
&= x^n \left[\left(\frac{y}{x}\right)^n + \frac{y}{x} + \frac{z}{x} - \frac{y}{x}(a^{n-1} + b^{n-1}) - (a^n + b^n) + 1 \right] \\
&= y^n + x^{n-1}(y + z) - y[(ax)^{n-1} + (bx)^{n-1}] - [(ax)^n + (bx)^n] + x^n.
\end{aligned}$$

Theorem 1. *Let* $A = RFPLRcircfr(F_1, F_2, ..., F_n)$. *Then*

$$\det A = \frac{(1 - F_{n+1})^n + (F_{n+1} - 1)(g_1^{n-1} + h_1^{n-1}) - (g_1^n + h_1^n) + (-F_n)^{n-1}}{1 - L_{n+1} + (-1)^{n-1}},$$

where

$$g_1 = \frac{F_{n+2} + \sqrt{F_{n-1}^2 + 4F_n}}{2},$$

$$h_1 = \frac{F_{n+2} - \sqrt{F_{n-1}^2 + 4F_n}}{2}.$$

Proof. From Lemma 1, the determinant of A is

$$\det A = \prod_{i=1}^{n} (F_1 + F_2\varepsilon_i + \cdots + F_n\varepsilon_i^{n-1})$$

$$= \prod_{i=1}^{n} \left(\frac{\alpha - \beta}{\alpha - \beta} + \frac{\alpha^2 - \beta^2}{\alpha - \beta}\varepsilon_i + \cdots + \frac{\alpha^n - \beta^n}{\alpha - \beta}\varepsilon_i^{n-1} \right)$$

$$= \prod_{i=1}^{n} \frac{(F_1 - F_{n+1}) - \varepsilon_i(F_{n+1} + F_n) + \varepsilon_i^2(-F_n)}{1 - \varepsilon_i - \varepsilon_i^2}.$$

According to Lemma 2, we can get

$$\det A = \frac{\begin{array}{c}(1 - F_{n+1})^n + (-F_n)^{n-1}(1 + F_n) - (1 - F_{n+1})(g_1^{n-1} + h_1^{n-1}) \\ - (g_1^n + h_1^n) + (-F_n)^n\end{array}}{1 - L_{n+1} + (-1)^{n-1}}$$

$$= \frac{(1 - F_{n+1})^n + (F_{n+1} - 1)(g_1^{n-1} + h_1^{n-1}) - (g_1^n + h_1^n) + (-F_n)^{n-1}}{1 - L_{n+1} + (-1)^{n-1}}.$$

Using the method in Theorem 1 similarly, we also have

Theorem 2. *Let* $B = RFPLRcircfr(F_n, F_{n-1}, \ldots, F_1)$. *Then*

$$\det B = \frac{(-F_n)^n + F_n(g_2^{n-1} + h_2^{n-1}) - (g_2^n + h_2^n) + F_{n-1}}{(-1)^n - L_{n-2} + 1},$$

where

$$g_2 = \frac{F_{n+1} - 1 + \sqrt{(F_{n+1} - 1)^2 + 4F_n}}{2},$$

$$h_2 = \frac{F_{n+1} - 1 - \sqrt{(F_{n+1} - 1)^2 + 4F_n}}{2}.$$

Theorem 3. *Let* $C = RFPLLcircfr(F_1, F_2, ..., F_n)$. *Then*

$$\det C = \frac{(-F_n)^n + F_n(g_2^{n-1} + h_2^{n-1}) - (g_2^n + h_2^n) + F_{n-1}}{(-1)^n - L_{n-2} + 1}(-1)^{\frac{n(n-1)}{2}},$$

where

$$g_2 = \frac{F_{n+1} - 1 + \sqrt{(F_{n+1} - 1)^2 + 4F_n}}{2},$$

$$h_2 = \frac{F_{n+1} - 1 - \sqrt{(F_{n+1} - 1)^2 + 4F_n}}{2}.$$

Proof. The matrix C can be written as

$$C = \begin{pmatrix} F_1 & F_2 & \cdots & F_{n-1} & F_n \\ F_2 & F_3 & \cdots & F_1 + F_n & F_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-1} & F_1 + F_n & \cdots & F_{n-2} + F_{n-3} & F_{n-2} \\ F_1 + F_n & F_2 + F_1 & \cdots & F_{n-1} + F_{n-2} & F_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} F_n & F_{n-1} & \cdots & F_2 & F_1 \\ F_1 & F_n + F_1 & \cdots & F_3 & F_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-2} & F_{n-3} + F_{n-2} & \cdots & F_n + F_1 & F_{n-1} \\ F_{n-1} & F_{n-2} + F_{n-1} & \cdots & F_1 + F_2 & F_n + F_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= B\Gamma.$$

Hence, we have

$$\det C = \det B \det \Gamma,$$

where $B = \text{RFPLRcircfr}(F_n, F_{n-1}, ..., F_1)$, and its determinant is obtained from Theorem 2,

$$\det B = \frac{(-F_n)^n + F_n(g_2^{n-1} + h_2^{n-1}) - (g_2^n + h_2^n) + F_{n-1}}{(-1)^n - L_{n-2} + 1},$$

where

$$g_2 = \frac{F_{n+1} - 1 + \sqrt{(F_{n+1} - 1)^2 + 4F_n}}{2},\ h_2 = \frac{F_{n+1} - 1 - \sqrt{(F_{n+1} - 1)^2 + 4F_n}}{2}.$$

In addition,

$$\det \Gamma = (-1)^{\frac{n(n-1)}{2}}.$$

So

$$\det C = \frac{(-F_n)^n + F_n(g_2^{n-1} + h_2^{n-1}) - (g_2^n + h_2^n) + F_{n-1}}{(-1)^n - L_{n-2} + 1}(-1)^{\frac{n(n-1)}{2}}.$$

3. Determinant of the Lucas RFPLR Circulant and Lucas RFPLL Circulant Matrix

Theorem 4. *Let* $A = RFPLRcircfr(L_1, L_2, \ldots, L_n)$. *Then*

$$\det A = \frac{\begin{array}{c}(1 - L_{n+1})^n + (L_{n+1} - 1)(g_3^{n-1} + h_3^{n-1}) - (g_3^n + h_3^n) \\ + 3(-L_n)^{n-1} + 2(-L_n)^n\end{array}}{1 - L_{n+1} + (-1)^{n-1}},$$

where

$$g_3 = \frac{L_{n+2} + 2 + \sqrt{(L_{n+2} + 2)^2 - 4L_n(L_{n+1} - 1)}}{-2L_n},$$

$$h_3 = \frac{L_{n+2} + 2 - \sqrt{(L_{n+2} + 2)^2 - 4L_n(L_{n+1} - 1)}}{-2L_n}.$$

Proof. From Lemma 1, we have

$$\begin{aligned}\det A &= \prod_{i=1}^{n}(L_1 + L_2\varepsilon_i + \cdots + L_n\varepsilon_i^{n-1}) \\ &= \prod_{i=1}^{n}[(\alpha + \beta) + (\alpha^2 + \beta^2)\varepsilon_i + \cdots + (\alpha^n + \beta^n)\varepsilon_i^{n-1}] \\ &= \prod_{i=1}^{n}\frac{(1 - L_{n+1}) - (L_{n+2} + 2)\varepsilon_i - L_n\varepsilon_i^2}{1 - \varepsilon_i - \varepsilon_i^2}.\end{aligned}$$

According Lemma 2, we get

$$\det A = \frac{\begin{array}{c}(1-L_{n+1})^n + (L_{n+1}-1)(g_3^{n-1}+h_3^{n-1}) - (g_3^n + h_3^n) \\ +(3-L_n)(-L_n)^{n-1} + (-L_n)^n\end{array}}{1-L_{n+1}+(-1)^{n-1}}$$

$$= \frac{\begin{array}{c}(1-L_{n+1})^n + (L_{n+1}-1)(g_3^{n-1}+h_3^{n-1}) - (g_3^n + h_3^n) \\ +3(-L_n)^{n-1} + 2(-L_n)^n\end{array}}{1-L_{n+1}+(-1)^{n-1}},$$

where

$$g_3 = \frac{L_{n+2}+2+\sqrt{(L_{n+2}+2)^2 - 4L_n(L_{n+1}-1)}}{-2L_n},$$

$$h_3 = \frac{L_{n+2}+2-\sqrt{(L_{n+2}+2)^2 - 4L_n(L_{n+1}-1)}}{-2L_n}.$$

Using the method in Theorem 4 similarly, we also have

Theorem 5. *Let* $B = RFPLRcircfr(L_n, L_{n-1}, \ldots, L_1)$. *Then*

$$\det B = \frac{(2-L_n)^n + (L_n-2)(g_4^{n-1}+h_4^{n-1}) - (g_4^n+h_4^n) + L_{n-1}}{(-1)^n - L_{n-2}+1},$$

where

$$g_4 = \frac{L_{n+1}-3+\sqrt{(L_{n+1}-3)^2+4(L_n-2)}}{2},$$

$$h_4 = \frac{L_{n+1}-3-\sqrt{(L_{n+1}-3)^2+4(L_n-2)}}{2}.$$

Theorem 6. *Let* $C = RFPLLcircfr(L_1, L_2, \ldots, L_n)$. *Then*

$$\det C = \frac{(2-L_n)^n + (L_n-2)(g_4^{n-1}+h_4^{n-1}) - (g_4^n+h_4^n) + L_{n-1}}{(-1)^n - L_{n-2}+1}(-1)^{\frac{n(n-1)}{2}},$$

where

$$g_4 = \frac{L_{n+1} - 3 + \sqrt{(L_{n+1} - 3)^2 + 4(L_n - 2)}}{2},$$

$$h_4 = \frac{L_{n+1} - 3 - \sqrt{(L_{n+1} - 3)^2 + 4(L_n - 2)}}{2}.$$

Proof. The matrix C can be written as

$$C = \begin{pmatrix} L_1 & L_2 & \cdots & L_{n-1} & L_n \\ L_2 & L_3 & \cdots & L_1 + L_n & L_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{n-1} & L_1 + L_n & \cdots & L_{n-2} + L_{n-3} & L_{n-2} \\ L_1 + L_n & L_2 + L_1 & \cdots & L_{n-1} + L_{n-2} & L_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} L_n & L_{n-1} & \cdots & L_2 & L_1 \\ L_1 & L_n + L_1 & \cdots & L_3 & L_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{n-2} & L_{n-3} + L_{n-2} & \cdots & L_n + L_1 & L_{n-1} \\ L_{n-1} & L_{n-2} + L_{n-1} & \cdots & L_1 + L_2 & L_n + L_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & ⋰ & ⋰ & ⋰ & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= B\Gamma.$$

Thus, we have

$$\det C = \det B \det \Gamma,$$

which matrix $B = \mathrm{RFPLRcircfr}(L_n, L_{n-1}, \ldots, L_1)$ and its determinant can be obtained from Theorem 5,

$$\det B = \frac{(2 - L_n)^n + (L_n - 2)(g_4^{n-1} + h_4^{n-1}) - (g_4^n + h_4^n) + L_{n-1}}{(-1)^n - L_{n-2} + 1},$$

where

$$g_4 = \frac{L_{n+1} - 3 + \sqrt{(L_{n+1} - 3)^2 + 4(L_n - 2)}}{2},$$

$$h_4 = \frac{L_{n+1} - 3 - \sqrt{(L_{n+1} - 3)^2 + 4(L_n - 2)}}{2}.$$

In addition,

$$\det \Gamma = (-1)^{\frac{n(n-1)}{2}}.$$

So the determinant of matrix C is

$$\det C = \det B \det \Gamma$$

$$= \frac{(2 - L_n)^n + (L_n - 2)(g_4^{n-1} + h_4^{n-1}) - (g_4^n + h_4^n) + L_{n-1}}{(-1)^n - L_{n-2} + 1} (-1)^{\frac{n(n-1)}{2}}.$$

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