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# ON THE HOMOTOPY GROUPS OF AN INVERTIBLE SPECTRUM IN THE $E(2)$-LOCAL CATEGORY AT THE PRIME 3

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## Abstract

Let $L_2$ denote the Bousfield localization functor with respect to the second Johnson-Wilson spectrum $E(2)$. A spectrum $L_2X$ is called invertible if there is a spectrum $Y$ such that $L_2X \wedge Y = L_2S^0$. Then Hovey and Sadofsky showed that every invertible spectrum is a suspension of the sphere spectrum $L_2S^0$ if the prime $p$ is greater than three. At the prime three, Kamiya and the second author constructed an invertible spectrum $P$ other than a suspension of $L_2S^0$, and showed a possibility of existence of another invertible spectrum $Q$ such that every invertible spectrum has a form $\Sigma^k P^{\wedge p} \wedge Q^{\wedge q}$ for integers $k \in \mathbf{Z}$ and $p, q \in \mathbf{Z}/3$, where $X \wedge X \wedge X = L_2S^0$ for $X = P, Q$. In this paper, we consider the homotopy groups of the invertible spectrum $Q$ under the assumption that $Q$ exists, and determine the homotopy

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groups $\pi_*(Q^{\wedge q} \wedge V(1))$ and $\pi_*(P^{\wedge p} \wedge Q^{\wedge q} \wedge V(1))$ for the Smith-Toda spectrum $V(1)$. The results make the authors conjecture that $Q$ does not exist.

## 1. Introduction

Let $\mathcal{S}_p$ for a prime number $p$ and $\mathcal{L}_n$ for an integer $n \geq 0$ denote the category of $p$-local spectra and its full subcategory of $E(n)$-local spectra, respectively, and $L_n : \mathcal{S}_p \to \mathcal{L}_n$ be the Bousfield localization functor with respect to $E(n)$, where $E(n)$ denotes the Johnson-Wilson spectrum with the homotopy groups $\pi_*(E(n)) = E(n)_* = \mathbf{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$. We call a spectrum $X \in \mathcal{L}_n$ *invertible* if there exists a spectrum $Y \in \mathcal{L}_n$ such that $X \wedge Y = L_n S^0$. Hopkins introduced the Picard group $\mathrm{Pic}_{(n)} = \mathrm{Pic}(\mathcal{L}_n)$ consisting of isomorphism classes of invertible spectra in $\mathcal{L}_n$ with multiplication defined by the smash product. Then in [5] (cf. [4]), it is shown by Hovey and Sadofsky that $\mathrm{Pic}_{(n)}$ is a well defined abelian group, that $\mathrm{Pic}_{(n)} \cong \mathbf{Z}$ if $n^2 + n < 2p - 2$ and that $\mathrm{Pic}_{(1)} \cong \mathbf{Z} \oplus \mathbf{Z}/2$ at the prime 2. Note that suspensions $\Sigma^k L_n S^0$ of the sphere spectrum for $k \in \mathbf{Z}$ form a subgroup of $\mathrm{Pic}_{(n)}$ isomorphic to $\mathbf{Z}$.

Let $E_r^{s,t}(X)$ denote the $E_r$-term of the Adams-Novikov spectral sequence converging to $\pi_*(L_n X)$. Then Kamiya and the second author [8] constructed a monomorphism

$$\varphi : \mathrm{Pic}_{(n)}/\mathbf{Z} \subset \bigoplus_{r \geq 2} E_r^{r, r-1}(S^0) = T. \tag{1.1}$$

In particular, it is well known that $T = 0$ if $n^2 + n < 2p - 2$ and $T = E_3^{3,2}(S^0) = \mathbf{Z}/2$ if $n = 1$ and $p = 2$, which imply the above results of Hovey and Sadofsky's. At the prime 3, $T = E_5^{5,4}(S^0) = \mathbf{Z}/3 \oplus \mathbf{Z}/3$ by [13] (cf. [14]). In [6], we determined the structure of the homotopy groups $\pi_*(L_1 QM \wedge V(0))$, where $V(0)$ denotes the mod 2 Moore spectrum, and the question mark complex $QM = S^0 \cup_2 e^1 \cup_\eta e^3$ represents the generator of $\mathbf{Z}/2$ in $\mathrm{Pic}_{(1)}$ at the prime two. In this paper, we study homotopy groups of an invertible spectrum in $\mathcal{L}_2$ at the prime three. Let $\alpha$ and $\beta$ denote the generators of $E_5^{5,4}(S^0) = \mathbf{Z}/3 \oplus \mathbf{Z}/3$. In [8], it is shown that $\alpha$ is pulled back to an invertible spectrum $P$ under the monomorphism $\varphi$ of (1.1). We determined the homotopy groups $\pi_*(P^{\wedge p} \wedge V(1))$ for an integer $p \in \mathbf{Z}/3$ in [6]. Here $V(1)$ denotes the first Smith-Toda spectrum, and $X^{\wedge n}$ denotes the $n$-fold smash product of $X$. In [7], we also determined $\pi_*(P^{\wedge p})$ for $p \in \mathbf{Z}/3$. We further, in this paper, assume the existence of an invertible spectrum $Q$ such that $\varphi(Q) = \beta$. Then $\mathrm{Pic}_{(2)} \cong \mathbf{Z} \oplus \mathbf{Z}/3 \oplus \mathbf{Z}/3$. Note that $Q \wedge Q \wedge Q = L_2 S^0$ by the definition corresponding $3\beta = 0$. Since $E_2^{*,*}(P^{\wedge p} \wedge Q^{\wedge q} \wedge V(1)) = E_2^{*,*}(V(1))$ for integers $p, q \in \mathbf{Z}/3$, we can compute the homotopy groups $\pi_*(P^{\wedge p} \wedge Q^{\wedge q} \wedge V(1))$ by studying the differentials $d_5(g_{p,q})$ and $d_9(g_{p,q})$ on the generator $g_{p,q} \in E_2^{0,0}(P^{\wedge p} \wedge Q^{\wedge q} \wedge V(1))$ of the $E_2$-term.

We recall [11] (cf. [3], [10]) that the $E_2$-term $E_2^{*,*}(V(1))$ is isomorphic, as $K(2)_*$-module, to the tensor product of $K(2)_*[b_{10}]$ and the direct sum of

$$\widetilde{F} = K(2)_*\{1, h_{10}, h_{11}, b_{11}\},$$

$$\widetilde{F\zeta_2} = K(2)_*\{\zeta_2, h_{10}\zeta_2, h_{11}\zeta_2, b_{11}\zeta_2\},$$

$$\widetilde{F^*} = K(2)_*\{\xi, \psi_0, \psi_1, b_{11}\xi\}$$

and

$$\widetilde{F^*\zeta_2} = K(2)_*\{\xi\zeta_2, \psi_0\zeta_2, \psi_1\zeta_2, b_{11}\xi\zeta_2\}.$$

Here $K(2)_* = \mathbf{Z}/3[v_2^{\pm 1}]$ and the bidegrees of generators are given as

$$\| v_2 \| = (0, 16), \quad \| h_{10} \| = (1, 4), \quad \| h_{11} \| = (1, 12), \quad \| b_{10} \| = (2, 12),$$
$$\| b_{11} \| = (2, 36), \quad \| \xi \| = (2, 8), \quad \| \psi_0 \| = (3, 16) \quad \text{and} \quad \| \psi_1 \| = (3, 24).$$

In order to describe the homotopy groups of the invertible spectrum $Q^{\wedge q}$ for $q \in \mathbf{Z}/3$, we introduce some modules:

$$H_q = \sum_{j \not\equiv 0,1,5\ (9)} G_2\{v_2^{j-2}h_{11}\} \oplus G_3\{v_2^{j+3}h_{10}\},$$

$$H_q\zeta_2 = \sum_{j \not\equiv 0,1,5\ (9)} G_2\{v_2^{j-2}h_{11}\zeta_2\} \oplus G_3\{v_2^{j+3}h_{10}\zeta_2\},$$

$$H_q^* = \sum_{j \not\equiv 0,1,5\ (9)} G_2\{v_2^{j-1}b_{11}\xi\} \oplus G_3\{v_2^{j+5}\xi\}$$

$$H_q^*\zeta_2 = \begin{cases} \sum_{j \not\equiv 0,1,5\ (9)} G_2\{v_2^{j-1}b_{11}\xi\zeta_2\} \oplus G_3\{v_2^{j+5}\xi\zeta_2\} & \text{for } q = 0 \\ \sum_{j \equiv 3,4,8\ (9)} \mathbf{Z}/3\{v_2^{j-1}b_{11}\xi\zeta_2\} \oplus \mathbf{Z}/3\{v_2^{j+5}\xi\zeta_2\} & \\ \oplus \sum_{j \equiv 2,6,7\ (9)} G_2\{v_2^{j-1}b_{11}\xi\zeta_2\} \oplus G_3\{v_2^{j+5}\xi\zeta_2\} & \text{for } q = 1,\ 2 \end{cases} \tag{1.2}$$

and

$$\overline{I_q} = \begin{cases} \sum_{j \equiv 0,1,5\ (9)} G_5\{v_2^{j}\} \oplus G_4\{v_2^{j+3}b_{11}\} & \text{for } q = 0, \\ \sum_{j \equiv 0,1,5\ (9)} G_3\{(v_2^{j}b_{10}^2)^{\sim},\ (v_2^{j+3}b_{11}b_{10})^{\sim}\} & \text{for } q = 1,\ 2, \end{cases}$$

$$\overline{I_q\zeta_2} = \sum_{j \equiv 0,1,5\ (9)} G_5\{v_2^{j}\zeta_2\} \oplus G_4\{v_2^{j+3}b_{11}\zeta_2\},$$

$$\overline{I_q^*} = \sum_{j \equiv 0,1,5\ (9)} G_4\{v_2^{j+2}\psi_0\} \oplus G_5\{v_2^{j+6}\psi_1\}$$

$$\overline{I_q^*\zeta_2} = \sum_{j \equiv 0,1,5\ (9)} G_4\{v_2^{j+2}\psi_0\zeta_2\} \oplus G_5\{v_2^{j+6}\psi_1\zeta_2\}, \tag{1.3}$$

in which $G_k = \mathbf{Z}/3[b_{10}]/(b_{10}^k)$. Note that an element of the form $(xb_{10})^{\sim}$ is not divisible by $b_{10}$. Put

$$\overline{F_q} = H_q \oplus \overline{I_q}, \qquad \overline{F_q\zeta_2} = H_q\zeta_2 \oplus \overline{I_q\zeta_2},$$

$$\overline{F_q^*} = H_q^* \oplus \overline{I_q^*} \quad \text{and} \quad \overline{F_q^*\zeta_2} = H_q^*\zeta_2 \oplus \overline{I_q^*\zeta_2}.$$

Then $\pi_*(Q^{\wedge q} \wedge V(1))$ for $q = 0$, which is $\pi_*(L_2V(1))$, is shown in [11] (cf. [3], [1]) to be the direct sum of the four subgroups $\overline{F_0}$, $\overline{F_0\zeta_2}$, $\overline{F_0^*}$, and $\overline{F_0^*\zeta_2}$.

**Theorem I.** *At the prime three, the homotopy groups* $\pi_*(Q^{\wedge q} \wedge V(1))$ *for* $q \in \mathbf{Z}/3$ *are isomorphic to the direct sum of the subgroups* $\overline{F_q}$, $\overline{F_q \zeta_2}$, $\overline{F_q^*}$ *and* $\overline{F_q^* \zeta_2}$.

Note that this theorem shows an isomorphism $\pi_*(Q^{\wedge 1} \wedge V(1)) \cong \pi_*(Q^{\wedge 2} \wedge V(1))$ while $\pi_*(Q^{\wedge 0} \wedge V(1)) \ncong \pi_*(Q^{\wedge 1} \wedge V(1))$. This is the reason why the authors are skeptical about the existence of $Q$, though the authors of [2] seem to believe the existence.

Recall [6] the equivalence $v_2^3$ shown by determining the homotopy groups of $P^{\wedge p} \wedge V(1)$.

**Theorem II** [6]**.** *There exists a homotopy equivalence* $v_2^3 : \Sigma^{48} L_2 V(1) \simeq P^{\wedge 1} \wedge V(1)$.

Theorems I and II give rise to the homotopy groups of an invertible spectrum in the $E(2)$-local category $\mathcal{L}_2$ smashing with the Smith-Toda spectrum $V(1)$ at the prime 3. Indeed, each invertible spectrum has the form $\Sigma^k P^{\wedge p} \wedge Q^{\wedge q}$ for integers $k \in \mathbf{Z}$ and $p, q \in \mathbf{Z}/3$.

**Corollary III.** *The homotopy groups* $\pi_*(P^{\wedge p} \wedge Q^{\wedge q} \wedge V(1))$ *for* $p, q \in \mathbf{Z}/3$ *are isomorphic to* $v_2^{3p}\pi_*(Q^{\wedge q} \wedge V(1))$.

In Sections 2 and 3, we compute the differentials $d_5$ and $d_9$, which give us the $E_9$- and $E_{13}$-terms of the Adams-Novikov spectral sequence converging to the homotopy groups of $Q^{\wedge q} \wedge V(1)$, respectively, and show the Theorem I.

## 2. The Adams-Novikov $E_9$-term for the Invertible Spectrum $Q^{\wedge q}$

Let $E_r^{*,*}(X)$ denote the $E_r$-term of the Adams-Novikov spectral sequence for $\pi_*(L_2 X)$. Then $E_2^{*,*}(X)$ is an $E_2^{*,*}(S^0)$-module with the action induced from the pairing $X \wedge S^0 \xrightarrow{\simeq} X$. Let $Q$ be an invertible spectrum such that $\varphi(Q) = \beta = v_2^{-1}\xi b_{10}\zeta_2 \in E_5^{5,4}(S^0)$ for $\varphi$ in (1.1). Then

$E_2^{*,*}(Q^{\wedge q})$ is isomorphic to $E_2^{*,*}(S^0)$ as an $E_2^{*,*}(S^0)$-module on the generator $g_q = g_{0,q} \in E_2^{0,0}(Q^{\wedge q})$ and $d_5(g_q) = q\beta g_q \in E_5^{5,4}(Q^{\wedge q})$ by the definition of $\varphi$ [8]. We determine the $E_9$-term converging to the homotopy groups $\pi_*(Q^{\wedge q} \wedge V(1))$ for the Smith-Toda spectrum $V(1)$ and an integer $q = 1, 2$ by computing $d_5(g_q)$ with help of the relations of $E_2^{*,*}(Q^{\wedge q} \wedge V(1)) = E_2^{*,*}(V(1))$ given in [11, Proposition 5.9]:

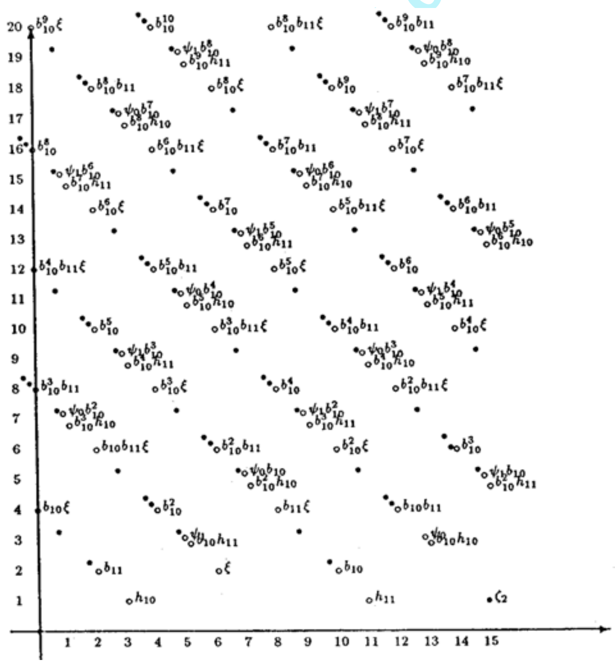
$$h_{10}h_{11} = 0, \ h_{10}\xi = 0, \ h_{11}\xi = 0,$$

$$v_2^2 h_{10}b_{10} = h_{11}b_{11}, \ v_2 h_{11}b_{10} = -h_{10}b_{11},$$

$$b_{11}\xi = v_2 h_{10}\psi_1 = v_2 h_{11}\psi_0, \ b_{10}\xi = -h_{10}\psi_0 = v_2^{-1}h_{11}\psi_1,$$

$$v_2^3 b_{10}^2 = -b_{11}^2, \ b_{10}\psi_1 = -v_2^{-1}b_{11}\psi_0 \ \text{ and } \ b_{10}\psi_0 = v_2^{-2}b_{11}\psi_1. \tag{2.1}$$

For conveniences, we write down the chart of the $E_2$-term $E_2^{*,*}(Q^{\wedge q} \wedge V(1)) = E_2^{*,*}(V(1))$, which is drawn up to multiple of $v_2$. In other words, $E_2^{*,*}(V(1)) \cong K(2)_* \otimes$ (the chart) for $K(2)_* = \mathbf{Z}/3[v_2, v_2^{-1}]$.



Here each little circle denotes $\mathbf{Z}/3$ generated by the indicated element, and the dot on the upper left of a circle is $\mathbf{Z}/3$ whose generator is the multiplication of the element and $\zeta_2$.

**Lemma 2.2.** *The differential* $d_5$ *of the Adams-Novikov spectral sequence converging to* $\pi_*(Q^{\wedge q} \wedge V(1))$ *acts trivially except for*

$$d_5(v_2^j X g_q) = v_2^{j-1} Y_k g_q \quad \textit{if} \quad j \equiv 3k,\ 3k+1,\ 3k+5 \quad (9),$$

*where the elements* $X$ *and* $Y_k$ *for* $k \in \mathbf{Z}/3$ *are given in the following table*:

| $X$ | $Y_0$ | $Y_1$ | $Y_2$ |
|---|---|---|---|
| $1$ | $q\xi b_{10}\zeta_2$ | $-v_2^{-1}h_{11}b_{10}^2 + q\xi b_{10}\zeta_2$ | $v_2^{-1}h_{11}b_{10}^2 + q\xi b_{10}\zeta_2$ |
| $\zeta_2$ | $0$ | $-v_2^{-1}h_{11}b_{10}^2\zeta_2$ | $v_2^{-1}h_{11}b_{10}^2\zeta_2$ |
| $b_{11}$ | $v_2h_{10}b_{10}^3 + qb_{11}\xi b_{10}\zeta_2$ | $qb_{11}\xi b_{10}\zeta_2$ | $-v_2h_{10}b_{10}^3 + qb_{11}\xi b_{10}\zeta_2$ |
| $b_{11}\zeta_2$ | $v_2h_{10}b_{10}^3\zeta_2$ | $0$ | $-v_2h_{10}b_{10}^3\zeta_2$ |
| $v_2^2\psi_0$ | $0$ | $-b_{11}\xi b_{10}^2$ | $b_{11}\xi b_{10}^2$ |
| $v_2^2\psi_0\zeta_2$ | $0$ | $-b_{11}\xi b_{10}^2\zeta_2$ | $b_{11}\xi b_{10}^2\zeta_2$ |
| $\psi_1$ | $-\xi b_{10}^3$ | $\xi b_{10}^3$ | $0$ |
| $\psi_1\zeta_2$ | $-\xi b_{10}^3\zeta_2$ | $\xi b_{10}^3\zeta_2$ | $0$ |



**Proof.** Since the identity map $Q^{\wedge q} \wedge V(1) \xrightarrow{=} (Q^{\wedge q} \wedge V(1))$ gives a natural pairing, we have a derivation formula

$$d_5(xy) = d_5(x)y + (-1)^{t-s} x d_5(y) \tag{2.3}$$

for $x \in E_2^{s,t}(Q^{\wedge q})$ and $y \in E_2^{s',t'}(V(1))$ [10, Theorem 2.3.3]. Then the formula (2.3) and the equations on $d_5$ given in [11, Propositions 8.4, 9.9, 9.10] show us the desired differentials. Note that we set $\lambda$ in [11] to be 1 in this paper.

By Lemma 2.2, we have

**Lemma 2.4.** *In the* $E_2$*-term of the Adams-Novikov spectral sequence converging to* $\pi_*(Q^{\wedge q} \wedge V(1))$, *the elements* $(v_2^j b_{10}^2)^{\sim} g_q = (v_2^j b_{10}^2 + q v_2^j \psi_1 \zeta_2) g_q$ *for* $j \equiv 0, 1, 5$ *and* $(v_2^j b_{11} b_{10})^{\sim} g_q = (v_2^j b_{11} b_{10} + q v_2^{j+2} \psi_0 \zeta_2) g_q$ *for* $j \equiv 3, 4, 8$ *are* $d_5$*-cycles.*

**Proof.** The lemma follows from the equations

$$d_5(v_2^j b_{10}^2 g_q) = q v_2^{j-1} \xi b_{10}^3 \zeta_2 g_q,$$

$$d_5(q v_2^j \psi_1 \zeta_2 g_q) = -q v_2^{j-1} \xi b_{10}^3 \zeta_2 g_q,$$

$$d_5(v_2^j b_{11} b_{10} g_q) = q v_2^{j-1} b_{11} \xi b_{10}^2 \zeta_2 g_q$$

and

$$d_5(q v_2^{j+2} \psi_0 \zeta_2 g_q) = -q v_2^{j-1} b_{11} \xi b_{10}^2 \zeta_2 g_q$$

shown in Lemma 2.2.

Lemmas 2.2 and 2.4 show the structure of the $E_q$-term of the Adams-Novikov spectral sequence.

**Proposition 2.5.** *For* $q = 1, 2,$ *the subgroup* $F_q \oplus F_q \zeta_2$ *(resp.* $F_q^* \oplus F_q^* \zeta_2$*) of the* $E_9$*-term* $E_9^{*,*}(Q^{\wedge q} \wedge V(1))$ *originating from* $\widetilde{F} \oplus \widetilde{F\zeta_2}$ *(resp.* $\widetilde{F^*} \oplus \widetilde{F^* \zeta_2}$*) is* $H_q \oplus I_q \oplus H_q \zeta_2 \oplus I_q \zeta_2$ *(resp.* $H_q^* \oplus I_q^* \oplus H_q^* \zeta_2 \oplus I_q^* \zeta_2$*). Here* $H_q$, $H_q \zeta_2$, $H_q^*$ *and* $H_q^* \zeta_2$ *are the modules of* (1.2), *and* $I_q$, $I_q \zeta_2$, $I_q^*$ *and* $I_q^* \zeta_2$ *are modules defined by*

$$I_q = \sum_{j \equiv 0, 1, 5 \ (9)} G\{(v_2^j b_{10}^2)^{\sim}, v_2^{j-2} h_{11}, v_2^{j+3} h_{10}, (v_2^{j+3} b_{11} b_{10})^{\sim}\},$$

$$I_q \zeta_2 = \sum_{j \equiv 0, 1, 5 \ (9)} G\{v_2^j \zeta_2, v_2^{j-2} h_{11} \zeta_2, v_2^{j+3} h_{10} \zeta_2, v_2^{j+3} b_{11} \zeta_2\},$$

$$I_q^* = \sum_{j \equiv 0, 1, 5 \ (9)} G\{v_2^{j+2} \psi_0, v_2^{j-1} b_{11} \xi, v_2^{j+5} \xi, v_2^{j+6} \psi_1\}$$

*and*

$$I_q^* \zeta_2 = \sum_{j \equiv 0, 1, 5 \ (9)} G\{v_2^{j+2} \psi_0 \zeta_2, v_2^{j-1} b_{11} \xi \zeta_2, v_2^{j+5} \xi \zeta_2, v_2^{j+6} \psi_1 \zeta_2\},$$

*in which G denotes* $\mathbf{Z}/3[b_{10}]$.

**3. The Adams-Novikov $E_{13}$-term for the Invertible Spectrum $Q^{\wedge q}$**

In this section, we compute the Adams-Novikov differential $d_9$ and determine the $E_{13}$-term for the homotopy groups $\pi_*(Q^{\wedge q} \wedge V(1))$, and prove Theorem I by showing that the $E_\infty$-term is isomorphic to the $E_{13}$-term.

Consider the spectra $N^1$ and $M^2$ defined by the cofiber sequences

$$L_2S^0 \to L_0S^0 \to N^1 \xrightarrow{j_0} \Sigma L_2S^0 \quad \text{and} \quad N^1 \to L_1N^1 \to M^2 \xrightarrow{j_1} \Sigma N^1.$$

In [13, Lemma 6.2], it is shown that

$$d_9(v_2^4h_{10}/3v_1) = v_2b_{10}^5/3v_1 \in E_2^*(M^2).$$

Sending this to $E_2^*(S^0)$ under the map $(j_0j_1)_*$, we see that

$$d_9(\alpha_1\beta_4) = \beta_1^6 \in E_2^*(S^0), \tag{3.1}$$

since $\alpha_1 = h_{10}$, $\beta_1 = (j_0j_1)_*(v_2/3v_1) = b_{10}$ and $\beta_4 = (j_0j_1)_*(v_2^4/3v_1)$ by definition (cf. [9]).

**Lemma 3.2.** *The action of the differential $d_9$ of the Adams-Novikov spectral sequence converging to $\pi_*(Q^{\wedge q} \wedge V(1))$ for $q = 1, 2$ is trivial except for*

$$d_9(X_jg_q) = Y_jg_q \quad \textit{for} \quad j \equiv 0, 1, 5 \ \ (9),$$

*where the elements $X_j$ and $Y_j$ are those in the table*:

| $X_j$ | $Y_j$ | $X_j$ | $Y_j$ |
|---|---|---|---|
| $v_2^{j+3}h_{10}$ | $(v_2^jb_{10}^2)^\sim b_{10}^3$ | $v_2^{j+3}h_{10}\zeta_2$ | $v_2^jb_{10}^5\zeta_2$ |
| $v_2^{j+7}h_{11}$ | $(v_2^{j+3}b_{11}b_{10})^\sim b_{10}^3$ | $v_2^{j+7}h_{11}\zeta_2$ | $v_2^{j+3}b_{11}b_{10}^4\zeta_2$ |
| $v_2^{j+5}\xi$ | $v_2^{j+2}\psi_0b_{10}^4$ | $v_2^{j+5}\xi\zeta_2$ | $v_2^{j+2}\psi_0b_{10}^4\zeta_2$ |
| $v_2^{j+8}b_{11}\xi$ | $v_2^{j+6}\psi_1b_{10}^5$ | $v_2^{j+8}b_{11}\xi\zeta_2$ | $v_2^{j+6}\psi_1b_{10}^5\zeta_2$ |

*Here $(v_2^jb_{10}^2)^\sim$ and $(v_2^{j+3}b_{11}b_{10})^\sim$ are the elements of Lemma* 2.4.

**Proof.** Let $\kappa$ be one of the generators of $I_q$, $I_q\zeta_2$, $I_q^*$ and $I_q^*\zeta_2$ such that $h_{10}\kappa \neq 0$. Then by (2.1), we see that $\kappa$ is one of the elements $(v_2^j b_{10}^2)^\sim$, $(v_2^{j+3}b_{11}b_{10})^\sim$, $v_2^j\zeta_2$, $v_2^{j+3}b_{11}\zeta_2$, $v_2^{j+2}\psi_0$, $v_2^{j+6}\psi_1$, $v_2^{j+2}\psi_0\zeta_2$ and $v_2^{j+6}\psi_1\zeta_2$ for $j \equiv 0, 1, 5\ (9)$.

Since $\beta_4 \equiv v_2^3 b_{10} \bmod (3, v_1)$, $\beta_1 = b_{10}$ and $\alpha_1 = h_{10}$, the derivation formula shows $d_9(v_2^3 h_{10} b_{10}\kappa) = b_{10}^6\kappa + v_2^3 h_{10} b_{10} d_9(\kappa)$ by (3.1). Since $\kappa$ and $v_2^3 h_{10}\kappa$ are $G = \mathbf{Z}/3[b_{10}]$-free generators of the $E_9$-term, we have

$$d_9(v_2^3 h_{10}\kappa) = b_{10}^5\kappa + v_2^3 h_{10} d_9(\kappa). \tag{3.3}$$

From the chart of the $E_2(= E_5)$-term in the previous section, we obtain the following table:

| $\kappa$ | $C$ | $\kappa$ | $C$ |
|---|---|---|---|
| $(v_2^j b_{10}^2)^\sim$ | $b_{10}^7,\ \psi_1 b_{10}^5\zeta_2,\ h_{11}b_{10}^6\zeta_2$ | $(v_2^{j+3}b_{11}b_{10})^\sim$ | $b_{11}b_{10}^6,\ \psi_0 b_{10}^5\zeta_2,\ h_{10}b_{10}^6\zeta_2$ |
| $v_2^{j+2}\psi_0$ | $\psi_0 b_{10}^5,\ h_{10}b_{10}^6,\ b_{11}\xi b_{10}^4\zeta_2$ | $v_2^{j+6}\psi_1$ | $\psi_1 b_{10}^5,\ h_{11}b_{10}^6,\ \xi b_{10}^5\zeta_2$ |
| $v_2^{j+2}\psi_0\zeta_2$ | $b_{11}b_{10}^6,\ \psi_0 b_{10}^5\zeta_2,\ h_{10}b_{10}^6\zeta_2$ | $v_2^{j+6}\psi_1\zeta_2$ | $b_{10}^7,\ \psi_1 b_{10}^5\zeta_2,\ h_{11}b_{10}^6\zeta_2$ |
| $v_2^j\zeta_2$ | $b_{10}^5\zeta_2$ | $v_2^{j+3}b_{11}\zeta_2$ | $b_{10}^5 b_{11}\zeta_2$ |



Here the column name $C$ stands for candidates of $v_2^3 h_{10} d_9(\kappa)$ up to a multiple of $v_2$. Using the relations in (2.1), the possible non-trivial differentials are as follows:

$$d_9((v_2^j b_{10}^2)^\sim) = k_1 v_2^{j-4} b_{11} b_{10}^5 \zeta_2,$$

$$d_9((v_2^{j+3} b_{11} b_{10})^\sim) = k_2 v_2^{j+2} b_{10}^6 \zeta_2,$$

$$d_9(v_2^{j+2}\psi_0) = k_3 v_2^{j-1} b_{10}^6 + k_4 v_2^{j-1}\psi_1 b_{10}^4 \zeta_2,$$

$$d_9(v_2^{j+6}\psi_1) = k_5 v_2^{j+2} b_{11} b_{10}^5 + k_6 v_2^{j+4}\psi_0 b_{10}^4 \zeta_2,$$

$$d_9(v_2^{j+2}\psi_0\zeta_2) = k_7 v_2^{j-1} b_{10}^6 \zeta_2$$

and

$$d_9(v_2^{j+6}\psi_1\zeta_2) = k_8 v_2^{j+2} b_{11} b_{10}^5 \zeta_2$$

for scalars $k_i \in \mathbf{Z}/3$. All generators of the $E_5$-term on the right hand sides of the equations die in $E_9$-term by Lemma 2.2 except for

$$d_9(v_2^3\psi_0\zeta_2) = k_7 b_{10}^6 \zeta_2 \quad \text{and} \quad d_9(v_2^7\psi_1\zeta_2) = k_8 v_2^3 b_{11} b_{10}^5 \zeta_2.$$

The relation (3.3) for $\kappa = \zeta_2$ (resp. $\kappa = v_2^3 b_{11}\zeta_2$) is $d_9(v_2^3 h_{10}\zeta_2) = b_{10}^5\zeta_2$ (resp. $d_9(v_2^6 h_{10} b_{11}\zeta_2) = v_2^3 b_{11} b_{10}^5 \zeta_2$), and so we replace $v_2^3\psi_0\zeta_2$ (resp. $v_2^7\psi_1\zeta_2$) with $v_2^3\psi_0\zeta_2 - k_7 v_2^3 h_{10} b_{10}\zeta_2$ (resp. $v_2^7\psi_1\zeta_2 - k_8 v_2^6 h_{10} b_{11}\zeta_2$). After the replacement, we see that $d_9(\kappa) = 0$, and so $d_9(v_2^3 h_{10}\kappa) = b_{10}^5\kappa$. It gives rise to the table of the theorem by the relations (2.1).

Therefore, we obtain the $E_{13}$-term of the Adams-Novikov spectral sequence.

**Proposition 3.4.** *In the homotopy group* $\pi_*(Q^{\wedge q} \wedge V(1))$ *for each* $q = 1, 2,$ *the subgroup* $\overline{F_q} \oplus \overline{F_q\zeta_2}$ *(resp.* $\overline{F_q^*} \oplus \overline{F_q^*\zeta_2}$*) of the* $E_{13}$*-term originating from* $F_q \oplus F_q\zeta_2$ *(resp.* $F_q^* \oplus F_q^*\zeta_2$*) is* $H_q \oplus \overline{I_q} \oplus H_q\zeta_2 \oplus \overline{I_q\zeta_2}$ *(resp.* $H_q^* \oplus \overline{I_q^*} \oplus H_q^*\zeta_2 \oplus \overline{I_q^*\zeta_2}$*). Here the modules* $H_q$, $H_q\zeta_2$, $H_q^*$, $H_q^*\zeta_2$, $\overline{I_q}$, $\overline{I_q\zeta_2}$, $\overline{I_q^*}$ *and* $\overline{I_q^*\zeta_2}$ *are those given in* (1.2) *and* (1.3).

**Proof of Theorem I.** For $q = 0$, the theorem is shown in [11] (cf. [3], [1]). For $q = 1, 2$, Proposition 3.4 says that $E_{13}^{s,t}(Q^{\wedge q} \wedge V(1)) = 0$ for $s > 12$, and so $E_{13}^{s,t}(Q^{\wedge q} \wedge V(1)) = E_\infty^{s,t}(Q^{\wedge q} \wedge V(1))$ since the differential $d_r = 0$ for $r \geq 13$. The homotopy group $\pi_k(Q^{\wedge q} \wedge V(1))$ for each $k \in \mathbf{Z}$ is a $\mathbf{Z}/3$-vector space since $Q^{\wedge q} \wedge V(1)$ is a $V(0)$-module spectrum. Therefore, there is no extension problem and we obtain the theorem from Proposition 3.4.

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