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ANALYTICAL STUDY OF A NONLINEAR DIFFUSION EQUATION USING RECONSTRUCTION OF VARIATIONAL ITERATION METHOD

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Abstract

In this paper, authors present the reconstruction of variational iteration method (RVIM) in order to obtain the analytical solution of the diffusion equation with a reaction term. This special kind of equation which is used as a mathematical model in geotechnical engineering, fluid mechanics, heat transfer, plasma physics, plasma waves, thermo-elasticity and chemical physics, occurs nonlinearly. The initial approximation can be freely chosen with possible unknown constants which can be determined by imposing the boundary and initial conditions. Convergence of the solution and effects for the method are discussed. The obtained results reveal that the technique introduced here is very promising and convenient in order to solve nonlinear partial differential equations and nonlinear ordinary differential equations.

1. Introduction

Most scientific problems and phenomena such as diffusion occur nonlinearly and one of the most notable nonlinear equations is diffusion equation. Analyses of diffusion equations in mathematical physics and applied mathematics have been of a considerable interest in the literature. Great contributions have been made to both the theory and applications of the diffusion equations. These kind of equations are increasingly used to model problems in a vast divergence of fields such as dynamical systems, mechanical systems, geotechnical engineering, fluid mechanics, control, chaos, anomalous diffusive and sub diffusive systems, unification of diffusion and wave propagation phenomenon and so on. In a great majority of cases, scientific problems are inherently nonlinear which does not admit analytical solution; hence these equations should be solved using special techniques. In recent decades, semi-exact analytical methods have been playing a great role in salvation of nonlinear ordinary and partial factional differential equations and many methods have been introduced such as homotopy perturbation method by He [1-7], the variational iteration method [8-12] and the decomposition method [13-15]. Some of these methods are too strongly dependent upon the so-called small parameters thus it is worthwhile developing some new analytical techniques independent of small parameters. One of these techniques that has been introduced is RVIM, reconstruction of variational iteration method.

The RVIM offers certain advantages over the known numerical methods. Numerical methods use discretization which gives rise to rounding off errors, hence the accuracy decreases and large computational work, power and time are required. The RVIM method is advantageous since it does not involve discretization of the variables, hence large computational work and round-off errors are avoided.

In this study, RVIM is applied to find an approximate solution of nonlinear differential equations governing diffusion and the result has been compared with the exact solution. This method gives rapid, convergent, successive, and accurate approximations to the formerly mentioned problem. We also aim to confirm that the reconstruction of variational iteration method is powerful, efficient, and promising in handling scientific and engineering problems. The RVIM technique is not only independent of any small parameters but also provides us with a simple way to ensure the convergence of solution series, so that we can always get accurate approximations. The RVIM technique has been successfully applied to many nonlinear problems in science and engineering. All of these verify the great potential and validity of the RVIM technique solving many strong nonlinear problems in science and engineering.

2. Description of the Method

In the following section, an alternative method for finding the optimal value of the Lagrange multiplier by the use of the Laplace transform [16, 17] will be investigated. Assume x, t as two independent variables; consider t as the principal variable and x as the secondary variable. If $u(x, t)$ is a function of two variables x and t, when the Laplace transform is applied with t as a variable, definition of Laplace transform is:

$$\mathbb{L}[u(x, t); s] = \int_0^\infty e^{-st} u(x, t) dt. \tag{1}$$

We have some preliminary notations as

$$\mathbb{L}\left[\frac{\partial u}{\partial t};\ s\right] = \int_0^\infty e^{-st}\frac{\partial u}{\partial t}\,dt = sU(x,\ s) - u(x,\ 0), \tag{2}$$

$$\mathbb{L}\left[\frac{\partial^2 u}{\partial t^2};\ s\right] = s^2U(x,\ s) - su(x,\ 0) - u_t(x,\ 0), \tag{3}$$

where

$$U(x,\ s) = \mathbb{L}[u(x,\ t);\ s]. \tag{4}$$

We often come across functions which are not the transform of some known function, but then, they can possibly be as a product of two functions, each of which is the transform of a known function. Thus, we may be able to write the given function as $U(x,\ s),\ V(x,\ s),$ where $U(s)$ and $V(s)$ are known to the transform of the functions $u(x,\ t),\ v(x,\ t),$ respectively. The convolution of $u(x,\ t)$ and $v(x,\ t)$ is written $u(x,\ t) * v(x,\ t).$ It is defined as the integral of the product of the two functions after one is reversed and shifted.

Convolution Theorem. *If* $U(x,\ s),\ V(x,\ s)$ *are the Laplace transform of* $u(x,\ t),\ v(x,\ t),$ *when the Laplace transform is applied to* t *as a variable, respectively, then* $U(x,\ s)\cdot V(x,\ s)$ *is the Laplace transform of* $\int_0^t u(x,\ t-\varepsilon)v(x,\ \varepsilon)d\varepsilon,$

$$\mathbb{L}^{-1}[U(x,\ s)\cdot V(x,\ s)] = \int_0^t u(x,\ t-\varepsilon)v(x,\ \varepsilon)d\varepsilon. \tag{5}$$

To facilitate our discussion of reconstruction of variational iteration method, introducing the new linear or nonlinear function $h(u(t,\ x)) = f(t,\ x) - N(u(t,\ x))$ and considering the new equation, rewrite $h(u(t,\ x)) = f(t,\ x) - N(u(t,\ x))$ as

$$L(u(t,\ x)) = h(t,\ x,\ u). \tag{6}$$

Now, for implementation the correctional function of VIM based on new idea of Laplace transform, applying Laplace transform to both sides of the above equation so that we introduce artificial initial conditions to zero for main problem, then left hand side of equation after transformation is featured as

$$\mathbb{L}[L\{u(x, t)\}] = U(x, s)P(s), \tag{7}$$

where $P(s)$ is polynomial with the degree of the highest order derivative of the selected linear operator

$$\mathbb{L}[L\{u(x, t)\}] = U(x, s)P(s) = \mathbb{L}[h\{(x, t, u)\}]. \tag{8}$$

Then

$$U(x, s) = \frac{\mathbb{L}[h\{(x, t, u)\}]}{P(s)}. \tag{9}$$

Suppose that $D(s) = \dfrac{1}{P(s)}$, and $\mathbb{L}[h\{(x, t, u)\}] = H(x, s)$. Therefore, using the convolution theorem, we have

$$U(x, s) = D(s) \cdot H(x, s) = \mathbb{L}\{(d(t) * h(x, t, u))\}. \tag{10}$$

Taking the inverse Laplace transform on both sides of equation

$$u(x, t) = \int_0^t d(t - \varepsilon)h(x, \varepsilon, u)d\varepsilon. \tag{11}$$

Thus, the following reconstructed method of variational iteration formula can be obtained

$$u_{n+1}(x, t) = u_0(x, t) + \int_0^t d(t - \varepsilon)h(x, \varepsilon, u_n)d\varepsilon. \tag{12}$$

And $u_0(x, t)$ is initial solution with or without unknown parameters. In absence of unknown parameters, $u_0(x, t)$ should satisfy initial/boundary conditions.

3. Application of the Proposed Method

The mentioned method (RVIM) is able to solve a wide range of linear and nonlinear equations. In this paper, we illustrated basic concepts of reconstructed of variational iteration method [16] as seen in following we concentrated on solution of diffusion equation that is placed in nonlinear equations classes:

$$\frac{\partial}{\partial t}u(x, t) = \frac{\partial^2}{\partial x^2}u(x, t) + u^2(x, t) - u^3(x, t), \quad 0 < x < 10, \quad t > 0. \tag{13}$$

Consider equation (13) as:

$$u_t = u_{xx} + u^2 - u^3 \tag{14}$$

with the initial and boundary conditions taken from the exact solution given by:

$$u(x, t) = \frac{1}{1 + e^{\varphi(x - \varphi t)}}. \tag{15}$$

And we consider $\varphi = \frac{\sqrt{2}}{2}$.

At first, we rewrite equation (14) based on t as selected linear operator as:

$$\mathbb{L}\{u(x, t)\} = u_t = \overbrace{u_{xx} + u^2 - u^3}^{h(x,t,u)}. \tag{16}$$

Now Laplace transform is implemented with respect to independent variable t on both sides of equation (16) and by using the new artificial initial condition (which all of them are zero), we have

$$s \bigcup (x, t) = \mathbb{L}\{h(x, t, u)\}, \tag{17}$$

$$\bigcup (x, t) = \frac{\mathbb{L}\{h(x, t, u)\}}{s}, \tag{18}$$

and whereas Laplace inverse transform of $1/s$ is as follows:

$$\mathbb{L}^{-1}\left[\frac{1}{s}\right] = 1. \tag{19}$$

Therefore, by using the Laplace inverse transform and convolution theorem it is concluded that

$$u(x, t) = \int_0^t h(x, \varepsilon, u)\, d\varepsilon. \tag{20}$$

Hence, the following iterative formula for the approximate solution, subject to the initial condition (15) can be obtained:

$$u_{n+1}(x, t) = u_0(x, t) + \int_0^t u_{n_{xx}}(x, \varepsilon) + u_n^2(x, \varepsilon) - u_n^3(x, \varepsilon)\, d\varepsilon. \tag{21}$$

Considering the auxiliary linear operator has been chosen as equation (16), we start an arbitrary initial approximation that satisfied initial condition:

$$u(x, \varnothing) = \frac{1}{1 + e^{\frac{\sqrt{2}}{2}x}}, \tag{22}$$

$$u_1(x, t) = u_0(x, t) + \int_0^t u_{0_{xx}}(x, \varepsilon) + u_0^2(x, \varepsilon) - u_0^3(x, \varepsilon)\, d\varepsilon. \tag{23}$$

Therefore, the following first-order approximate solution is derived:

$$u_1(x, t) = \frac{1}{1 + e^{\frac{\sqrt{2}}{2}x}} + \frac{\left(e^{\frac{\sqrt{2}}{2}x}\right)^2 t}{\left(1 + e^{\frac{\sqrt{2}}{2}x}\right)^3} - \frac{\left(\frac{1}{2} e^{\frac{\sqrt{2}}{2}x}\right) t}{\left(1 + e^{\frac{\sqrt{2}}{2}x}\right)^2}$$

$$+ \frac{t}{\left(1 + e^{\frac{\sqrt{2}}{2}x}\right)^2} - \frac{t}{\left(1 + e^{\frac{\sqrt{2}}{2}x}\right)^3}. \tag{24}$$

By the iteration formula (21), we can calculate other values of u (such as $u_2, u_3, \ldots$).

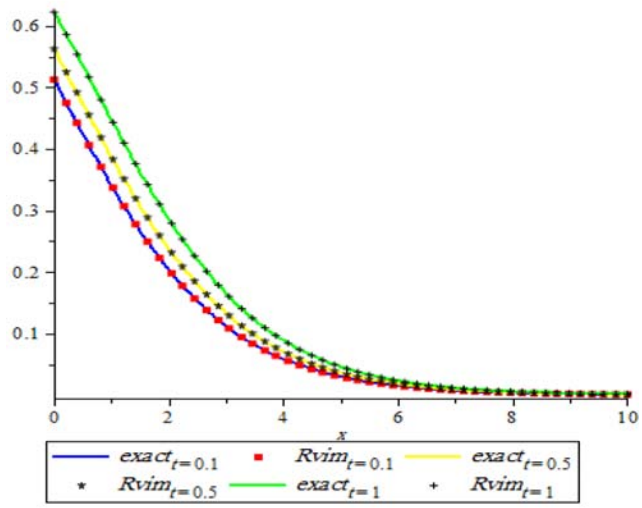

Figure 1. The comparison between the exact and RVIM solutions.

In Figure 1, the comparison between the exact and RVIM solutions is expressed as it is shown there is a good accordance between RVIM solutions and the exact ones.

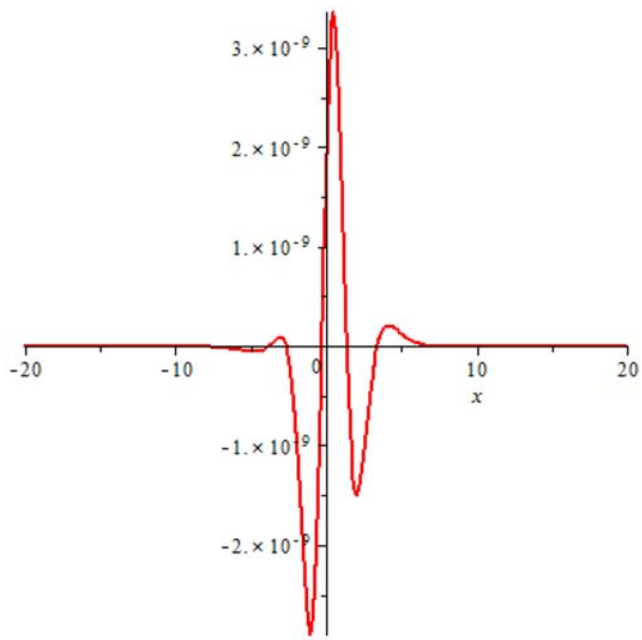

Figure 2. Error of RVIM for diffusion equation.

Table 1. The comparison between the exact and RVIM solutions

	$T = 0.1$		$T = 0.5$		$T = 1$	
x	Exact	RVIM	Exact	RVIM	Exact	RVIM
0	0.512497396	0.512497398	0.562176500	0.562182334	0.622459331	0.622647373
1	0.341389832	0.341389833	0.387672399	0.387678638	0.448407586	0.448636783
2	0.203556341	0.203556339	0.237902502	0.237897690	0.286138386	0.285978021
5	0.029726649	0.029726649	0.036070809	0.036071190	0.045846141	0.045858787
10	0.000892075	0.000892075	0.001089367	0.001089361	0.001398343	0.001398136
20	7.58338195E-7	7.58338194E-7	9.26236208E-7	9.26230085E-7	1.18931052E-6	1.18910584E-6
100	7.58338195E-7	7.58338194E-7	9.26236208E-7	9.26230085E-7	1.18931052E-6	1.18910584E-6

Table 2. Errors of RVIM for different values of t

x	$T = 0.1$	$T = 0.5$	$T = 1$
0	1.8092E-9	5.8334E-6	1.8804E-4
1	1.8126E-9	6.2383E-6	2.2919E-4
2	1.4902E-9	4.8115E-6	1.6036E-4
5	1.1852E-10	3.8086E-7	1.2646E-5
10	1.9108E-12	6.1812E-9	2.0675E-7
20	1.8940E-15	6.1232E-12	2.0467E-10
100	5.1290E-40	1.6581E-36	5.5425E-35

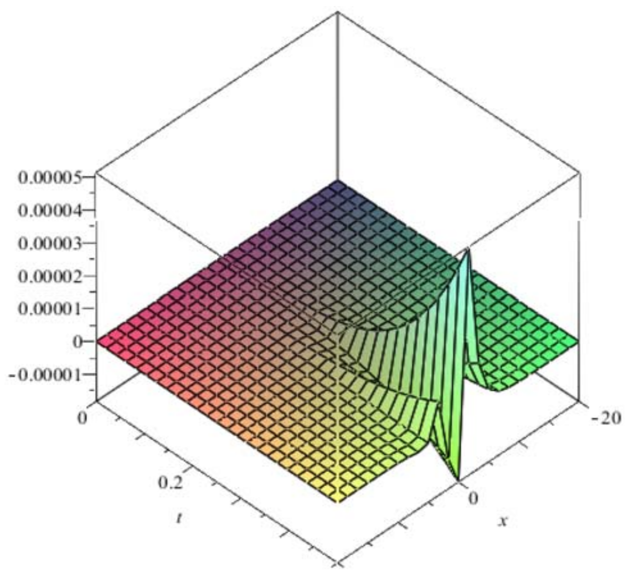

Figure 3. Error of RVIM for diffusion equation.

In Table 1, the values of both exact solution and RVIM method for different values of t and x are expressed. The error of u has strong relevance to two independent parameters, x and t. x range is chosen between 0 and 100 and t is between 0.1 to 1. Error depends on various amounts of x and t.

In Table 2, the absolute error between the exact and RVIM solutions is shown, which it illustrates RVIM is a method including the high accuracy and reliability.

4. Conclusion

In this paper, reconstruction of variational iteration method is applied to nonlinear diffusion equation which is useful in many fields such as: geotechnical engineering, fluid mechanics, anomalous diffusive and sub diffusive systems, etc. The accuracy of the method is acceptable and the resulted solution is close to the numerical solution. In spite of simplicity and requiring less computation, RVIM has a rapid convergence, high accuracy and efficiency and presents acceptable results. RVIM also requires less number of parameters which can be count as another advantage for it. The reliability of the method and the reduction in the size of the computational domain gives a wide applicability to it. Besides, the recent appearance of nonlinear differential equations as models in some fields of engineering sciences makes it necessary to investigate methods (analytical and numerical) to solve such equations and we hope the work we have done to be a step in this direction. We sincerely hope this method can be applied to a wider range of problems.

References

[1] A. Sadighi and D. D. Ganji, Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation and variational iteration methods, Int. J. Nonlinear Sci. Numer. Simul. 8(3) (2007), 435-444.

[2] D. D. Ganji and A. Sadighi, Application of He’s homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations, Int. J. Nonlinear Sci. Numer. Simul. 7(4) (2006), 411-418.

[3] M. Rafei and D. D. Ganji, Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method, Int. J. Nonlinear Sci. Numer. Simul. 7(3) (2006), 321-328.

[4] A. Sadighi and D. D. Ganji, Exact solutions of Laplace equations by homotopy-perturbation and variational iteration methods, Phys. Lett. A 367(1-2) (2007), 83-87.

[5] E. Hesameddini and H. Latifizadeh, Reconstruction of variational iteration algorithms using the Laplace transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009), 1377-1382.

[6] A. A. Imani, D. D. Ganji, Houman B. Rokni, H. Latifizadeh, Esmail Hesameddini and M. Hadi Rafiee, Approximate traveling wave solution for shallow water wave equation, Appl. Math. Model. 36(4) (2012), 1550-1557.

[7] K. Vajravelu and B. V. R. Kumar, Analytic and numerical solutions of coupled nonlinear system arising in three-dimensional rotating flow, Int. J. Non-Linear Mech. 39 (2004), 13-24.

[8] S. S. Ganji, D. D. Ganji and S. Karimpour, He's energy balance and He's variational methods for nonlinear oscillations in engineering, Int. J. Modern Phys. B 23(3) (2009), 461-471.

[9] J.-H. He, Variational iteration method – a kind of non-linear analytical technique: some examples, Int. J. Non-Linear Mech. 34(4) (1999), 699-708.

[10] Hafez Tari, D. D. Ganji and H. Babazadeh, The application of He's variational iteration method to nonlinear equations arising in heat transfer, Phys. Lett. A 363(3) (2007), 213-217.

[11] M. Rafei, D. D. Ganji, H. Daniali and H. Pashaei, The variational iteration method for nonlinear oscillators with discontinuities, J. Sound Vibration 305(4-5) (2007), 614-620.

[12] D. D. Ganji, G. A. Afrouzi and R. A. Talarposhti, Application of He's variational iteration method for solving the reaction-diffusion equation with ecological parameters, Comput. Math. Appl. 54 (2007), 1010-1017.

[13] G. Adomian, Solving Frontier Problems of Physics: The Composition Method, Kluwer, Boston, 1994.

[14] G. Adomian, Solution of physical problems by decomposition, Comput. Math. Appl. 27 (1994), 145-154.

[15] J. I. Ramos, Piecewise-adaptive decomposition methods, Chaos Solitons Fractals 40(4) (2009), 1623-1636.

[16] E. Hesameddini and H. Latifizadeh, Reconstruction of variational iteration algorithms using the Laplace transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009), 1377-1382.

[17] D. D. Ganji, Houman B. Rokni, M. Hadi Rafiee, A. A. Imani, M. Esfandyaripour and M. Sheikholeslami, Reconstruction of variational iteration method for boundary value problems in structural engineering and fluid mechanics, Int. J. Nonlinear Dynamics in Engineering and Sciences 3(1) (2011), 1-10.