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THE EXACT TRAVELING WAVE SOLUTIONS TO THE $K(1, 2)$ EQUATION

Chun-Yan Wang

Department of Mathematics
Northeast Petroleum University
Daqing 163318, P. R. China
e-mail: chunyanmyra@163.com

Abstract

Using the complete discrimination system for polynomial, we give the classification of single traveling wave solutions to the $K(m, n)$ equation for $m = 1$ and $n = 2$.

1. Introduction

The classifications of single traveling wave solutions to some nonlinear differential equations have been obtained extensively by the complete discrimination system for polynomial method proposed by Liu [1-3]. In the present paper, we consider the $K(m, n)$ equation [4, 5], which reads as

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \tag{1}$$

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where *m*, *n* are parameters. We will give the classification of single traveling wave solutions to the $K(m, n)$ equation for $m = 1$ and $n = 2$.

2. Classification of Traveling Wave Solutions

In order to obtain the traveling wave solutions, we take a wave transformation $u = u(\xi)$ and $\xi = x - kt$, the $K(m, n)$ equation is reduced to the following ODE:

$$-ku' + a(u^m)' + (u^n)''' = 0. \tag{2}$$

Integrating equation (2) once and let $w = u^n$, we have:

$$-kw^{\frac{1}{n}} + aw^{\frac{m}{n}} + w'' + C_0 = 0, \tag{3}$$

where C_0 is an integral constant. Multiplying the both sides of equation (3) by w' and integrating it once, we can obtain

$$(w')^2 = 2k\frac{n}{n+1}w^{\frac{n+1}{n}} - 2a\frac{n}{m+1}w^{\frac{m+n}{n}} - 2C_0w - 2C_1, \tag{4}$$

where C_1 is an integral constant. Owing to $w = u^n$, the solutions of *u* can be given from

$$\pm(\xi - \xi_0) = \int \frac{du}{\sqrt{\frac{2k}{n(n+1)}u^{3-n} - \frac{2a}{n(n+m)}u^{m-n+2} - \frac{2C_0}{n^2}u^{2-n} + C_1u^{2-2n}}}. \tag{5}$$

The classifications of single traveling wave solutions to the $K(m, n)$ equation for some values of *m* and *n* by using the discrimination system are the following:

In the present paper, we consider $m = 1$ and $n = 2$.

Case 1. $C_1 = 0$. The corresponding solutions are

$$u = \frac{k-a}{12}(\xi - \xi_0)^2 + \frac{3C_0}{2(k-a)}. \tag{6}$$

Case 2. $C_1 \neq 0$. Performing the transformation $t = \left(\frac{k-a}{3}\right)^{\frac{1}{3}} u$, the general solution can be obtained from the following quadrature:

$$\pm a_3^{\frac{2}{3}}(\xi - \xi_0) = \int \frac{tdt}{\sqrt{t^3 + d_2 t^2 + d_0}}, \tag{7}$$

where $d_2 = -\frac{C_0}{2}\left(\frac{k-a}{3}\right)^{-\frac{2}{3}}$, and C_0, C_1 are arbitrary constants. Denote $F(t) = t^3 + d_2 t^2 + d_0$, and its complete discrimination systems are

$$\Delta = -27\left(\frac{2d_2^3}{27} + d_0\right)^2 + 4\left(\frac{d_2^2}{3}\right)^3, \quad D = -\frac{d_2^2}{3}. \tag{8}$$

According to the complete discrimination system (8) for polynomial $F(t)$, there exist four cases to be discussed:

Case 2.1. $\Delta = 0$, $D < 0$. Then we have $F(t) = (t - \alpha)^2(t - \beta)$, where α, β are real numbers.

When $\beta > \alpha$,

$$\pm \frac{1}{2}\left(\frac{k-a}{3}\right)^{\frac{2}{3}}(\xi - \xi_0) = \sqrt{\left(\frac{k-a}{3}\right)^{\frac{1}{3}} u - \beta} - 2\sqrt{\frac{\beta}{3}} \arctan \frac{\sqrt{\left(\frac{k-a}{3}\right)^{\frac{1}{3}} u - \beta}}{\sqrt{3\beta}}. \tag{9}$$

When $\beta < \alpha$,

$$\pm \frac{1}{2}\left(\frac{k-a}{3}\right)^{\frac{2}{3}}(\xi - \xi_0) = \sqrt{\left(\frac{k-a}{3}\right)^{\frac{1}{3}} u - \beta} + \sqrt{-\frac{\beta}{3}} \ln\left|\frac{\sqrt{-3\beta} + \sqrt{\left(\frac{k-a}{3}\right)^{\frac{1}{3}} u - \beta}}{\sqrt{-3\beta} - \sqrt{\left(\frac{k-a}{3}\right)^{\frac{1}{3}} u - \beta}}\right|. \tag{10}$$

Case 2.2. $\Delta = 0$, $D = 0$. Then we have $F(t) = (t - \alpha)^3$. The corresponding solutions are:

$$u = 2\left(\frac{k-a}{3}\right)^{-\frac{1}{3}}\alpha + \frac{k-a}{24}(\xi - \xi_0)^2 \pm \frac{1}{8}\left(\frac{k-a}{3}\right)^{\frac{1}{3}}(\xi - \xi_0)$$

$$\times\sqrt{\left(\frac{k-a}{3}\right)^{\frac{4}{3}}(\xi - \xi_0)^2 + 16\alpha}. \tag{11}$$

Case 2.3. $\Delta > 0$, $D < 0$. Then we have $F(t) = (t - \alpha)(t - \beta)(t - \gamma)$.

Case 2.4. $\Delta < 0$. Then we have $F(t) = (t - \alpha)(t^2 + pt + q)$.

For Cases 2.3 and 2.4, the corresponding solutions can be expressed by hyper-elliptic functions or hyper-elliptic integral. We omit them for simplicity.

References

[1] C. S. Liu, The classification of travelling wave solutions and superposition of multi-solutions to Camassa-Holm equation with dispersion, Chin. Phys. 16 (2007), 1832-1837.

[2] C. S. Liu, Representations and classification of traveling wave solutions to sinh-Gordon equation, Commun. Theor. Phys. 49 (2008), 153-158.

[3] C. S. Liu, Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations, Comput. Phys. Commun. 181 (2010), 317-324.

[4] P. Rosenau and J. M. Hyman, Compactons: solitons with finite wavelength, Phys. Rev. Lett. 70 (1993), 564-567.

[5] P. Rosenau, Nonlinear dispersion and compact structures, Phys. Rev. Lett. 73 (1994), 1737-1741.