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ON THE COMPOSITION OF THE DISTRIBUTIONS $x_-^{-s} \ln^m x_-$ AND x_+^r

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Abstract

The neutrix composition $F(f(x))$ of a distribution $F(x)$ and a locally summable function $f(x)$ is said to exist and be equal to the distribution $h(x)$, if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. It is proved that if $G_{s,m}(x)$ denotes the distribution $(\ln^{m+1} x_-)^{(s)}$, then the neutrix composition $G_{s,m}(x_+^r)$ exists and

$$G_{s,m}(x_+^r) = \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{(-1)^{rs+s-1} c_{s,i} B_{0,m-i+1}(s, 1)}{r(rs-1)!} \delta^{(rs-1)}(x),$$

for $r, s, m = 1, 2, \ldots,$ where

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$$c_{s,i} = \int_0^1 v^s \ln^i v \rho^{(s)}(v)\, dv$$

for $i = 0, 1, 2, \ldots, s,$ and B denotes the Beta function.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions φ with compact support and let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$. We let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$ and let $\mathcal{D}'[a, b]$ be the space of distributions defined on $\mathcal{D}[a, b]$.

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots,$ it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is a distribution in $\mathcal{D}'$ and $F_n(x) = \langle F(x-t), \delta_n(x) \rangle$, then $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to $F(x)$.

Now let $f(x)$ be an infinitely differentiable function having a single simple root at the point $x = x_0$. Gel'fand and Shilov defined the distribution $\delta^{(r)}(f(x))$ by the equation

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{|f'(x)|}\frac{d}{dx}\right]^r \delta(x - x_0),$$

for $r = 0, 1, 2, \ldots,$ see [8].

In order to give a more general definition for the composition of distributions, the following definition for the neutrix composition of distributions was given in [2] and was originally called the composition of distributions.

Definition 1. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$N-\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and N is the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \ln^r n : \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$.

Note that taking the neutrix limit of a function $f(n)$, is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$.

We need the following lemma, which can be easily proved by induction:

Lemma 1.

$$\int_0^1 t^i \rho^{(s)}(t)dt = \begin{cases} 0, & 0 \le i < s, \\ \dfrac{(-1)^s s!}{2}, & i = s \end{cases}$$

for $s = 0, 1, 2, \ldots$.

The following theorems were proved in [2], [4], [5], [6] and [7], respectively.

Theorem 1. *If* $F_\lambda(x)$ *denotes the distribution* x_-^λ, *then the neutrix composition* $F_\lambda(x_+^{-r/\lambda})$ *exists and*

$$F_\lambda(x_+^{-r/\lambda}) = \frac{(-1)^{r-1}\pi\lambda\,\mathrm{cosec}(\pi\lambda)}{2r(r-1)!}\delta^{(r-1)}(x)$$

for $\lambda < 0$, $\lambda \neq -1, -2, \ldots$ *and* $r = 1, 2, \ldots$.

Theorem 2. *If* $F_s(x)$ *denotes the distribution* x_-^{-s}, *then the neutrix composition* $F_s(x_+^r)$ *exists and*

$$F_s(x_+^r) = \frac{(-1)^{rs+s}c(\rho)}{r(rs-1)!}\delta^{(rs-1)}(x) \tag{1}$$

for $r, s = 1, 2, \ldots$, *where* $c(\rho) = \int_0^1 \ln t\rho(t)\,dt$.

Note that in this theorem, the distribution x_-^{-s} is defined by

$$x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!},$$

for $s = 1, 2, \ldots$.

Theorem 3. *If* $F_{\lambda,m}(x)$ *denotes the distribution* $x_+^\lambda \ln^m x_+$, *then the neutrix composition* $F_{\lambda,m}(x_+^\mu)$ *exists and*

$$F_{\lambda,m}(x_+^\mu) = \mu^m x_+^{\lambda\mu} \ln^m x_+$$

for $\lambda < 0$, $\mu > 0$ *and* $\lambda, \lambda\mu \neq -1, -2, \ldots$.

Theorem 4. *If* $F_{s,m}(x)$ *denotes the distribution* $x_+^{-s} \ln^m x_+$, *then the neutrix composition* $F_{s,m}(x_+^\mu)$ *exists and*

$$F_{s,m}(x_+^{\mu}) = \mu^m x_+^{-s\mu} \ln^m x_+$$

for $m = 0, 1, 2, \ldots, s = 1, 2, \ldots, \mu > 0$ *and* $s\mu \neq 1, 2, \ldots$.

Theorem 5. *If* $F_{\lambda,s}(x)$ *denotes the distribution* $x_-^{\lambda} \ln^s x_-$, *then the neutrix composition* $F_{\lambda,s}(x_+^{-r/\lambda})$ *exists and*

$$F_{\lambda,s}(x_+^{-r/\lambda}) =$$

$$\frac{(-1)^r \lambda}{r!} \sum_{i=0}^{s} \sum_{j=0}^{i} \binom{i}{j} \binom{s}{i} \frac{c_{m,i} B_{s-i,0}(\lambda+1, m) B_{0,i-j}(-\lambda, \lambda+m+1)}{(m-1)!} \delta^{(r-1)}(x),$$

for $\lambda < 0, \lambda \neq -1, -2, \ldots, r = 1, 2, \ldots$ *and* $s = 0, 1, 2, \ldots,$ *where*

$$c_{m,i} = \int_0^1 v^m \ln^i v \rho^{(m)}(v) dv$$

for $i = 0, 1, 2, \ldots, s$ *and* $-m-1 < \lambda < -m,$ *for* $m = 1, 2, \ldots$.

In the particular case $m = 0$, *we have*

$$F_{\lambda,s}(x_+^{-r/\lambda}) = \frac{(-1)^r \lambda}{r!} \sum_{j=0}^{s} \binom{s}{j} c_{0,j} B_{0,s-j}(-\lambda, \lambda+1) \delta^{(r-1)}(x).$$

In the following, the distribution $x_-^{-1} \ln^m x_-$ is defined by

$$x_-^{-1} \ln^m x_- = -\frac{(\ln^{m+1} x_-)'}{m+1}$$

for $m = 1, 2, \ldots$ and the distribution $x_-^{-s-1} \ln^m x_-$ is defined inductively by the equation

$$x_-^{-s-1} \ln^m x_- = \frac{m x_-^{-s-1} \ln^{m-1} x_- + (x_-^{-s} \ln^m x_-)'}{s}$$

for $s, m = 1, 2, \ldots$. Note that this is not the same as Gel'fand and Shilov's definition, see [8].

Putting $G_{s,m}(x) = (\ln^{m+1} x_-)^{(s)}$, for $s, m = 1, 2, \ldots,$ we see that $G_{s,m}(x)$ is of the form

$$G_{s,m}(x) = \sum_{i=0}^{m} a_{s,m,i} x_-^{-s} \ln^i x_-,$$

for $s, m = 1, 2, \ldots,$ where $a_{s,m,i} = 0$ if $i \le m - s$.

In particular,

$$G_{1,m}(x) = -(m+1)x_-^{-1} \ln^m x_- \tag{2}$$

for $m = 1, 2, \ldots$ and

$$G_{s,1}(x) = 2(s-1)!\phi(s-1)x_-^{-s} - 2(s-1)!x_-^{-s} \ln x_-, \tag{3}$$

for $s = 1, 2, \ldots,$ where

$$\phi(s) = \begin{cases} \sum_{i=1}^{s} i^{-1}, & s \ge 1, \\ 0, & s = 0. \end{cases}$$

We first of all prove

Theorem 6. *The neutrix composition $G_{s,m}(x_+^r)$ exists and*

$$G_{s,m}(x_+^r) = \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{(-1)^{rs+s-1} c_{s,i} B_{0,m-i+1}(s,1)}{r(rs-1)!} \delta^{(rs-1)}(x), \tag{4}$$

for $r, s, m = 1, 2, \ldots,$ where

$$c_{s,i} = \int_0^1 v^s \ln^i v \rho^{(s)}(v)\,dv$$

for $i = 0, 1, 2, \ldots, s$.

In particular

$$G_{1,m}(x_+^r) = \frac{(-1)^r (c_{1,m+1} - mc_{1,m})}{r!} \delta^{(r-1)}(x) \tag{5}$$

for $r, m = 1, 2, \ldots$.

Proof. Putting $G_{n,s,m}(x) = G_{s,m}(x) * \delta_n(x)$, we have

$$G_{n,s,m}(x) = \langle G_{s,m}(x-t), \delta_n(t)\rangle$$

$$= (-1)^s \langle \ln^{m+1}(x-t)_-, \delta_n^{(s)}(t)\rangle$$

$$= \begin{cases} (-1)^s \int_x^{1/n} \ln^{m+1}(t-x)\delta_n^{(s)}(t)\,dt, & -1/n < x < 1/n, \\ (-1)^s \int_{-1/n}^{1/n} \ln^{m+1}(t-x)\delta_n^{(s)}(t)\,dt, & x < -1/n, \\ 0, & x > 1/n \end{cases}$$

and it follows that

$$G_{n,s,m}(x_+^r) = \begin{cases} (-1)^s \int_{x^r}^{1/n} \ln^{m+1}(t-x^r)\delta_n^{(s)}(t)\,dt, & 0 < x^r < 1/n, \\ (-1)^s \int_0^{1/n} \ln^{m+1} t\delta_n^{(s)}(t)\,dt, & x < 0, \\ 0, & x^r > 1/n. \end{cases}$$

Now let φ be an arbitrary function in $\mathcal{D}$. We may suppose that $\varphi(x)$ is in $\mathcal{D}[a, b]$, where $a < 0 < b$. By Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{rs-1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{\varphi^{(rs)}(\xi x)}{(rs)!} x^{rs},$$

where $0 < \xi < 1$. We therefore need to evaluate

$$N-\lim_{n\to\infty} \langle G_{n,s,m}(x_+^r), \varphi(x)\rangle = N-\lim_{n\to\infty} \sum_{k=0}^{rs-1} \frac{\varphi^{(k)}(0)}{k!} \int_a^b x^k G_{n,s,m}(x_+^r)\,dx$$

$$+ N-\lim_{n\to\infty} \int_a^b \frac{x^{rs}\varphi^{(rs)}(\xi x)}{(rs)!} G_{n,s,m}(x_+^r)\,dx. \quad (6)$$

If now $n^{-1/r} < b$, we have

$$\int_a^b x^k G_{n,s,m}(x_+^r)\,dx = (-1)^s \int_0^{n^{-1/r}} x^k \int_{x^r}^{1/n} \ln^{m+1}(t - x^r)\,\delta_n^{(s)}(t)\,dt\,dx$$

$$+ (-1)^s \int_a^0 x^k \int_0^{1/n} \ln^{m+1} t\,\delta_n^{(s)}(t)\,dt\,dx$$

$$= I_k(n) + J_k(n). \tag{7}$$

Putting $nt = v$, we have

$$J_k(n) = (-1)^s n^s \int_a^0 x^k \int_0^1 (\ln v - \ln n)^{m+1} \rho^{(s)}(v)\,dv\,dx$$

and it follows that

$$\underset{n\to\infty}{N-\lim}\, J_k(n) = 0, \tag{8}$$

for $k = 0, 1, 2, \ldots$.

Further, putting $nx^r = u$, we have

$$I_k(n) = \frac{(-1)^s n^{s-(k+1)/r}}{r} \int_0^1 u^{(k+1)/r-1} \int_u^1 \ln^{m+1}\left(\frac{v}{n} - \frac{u}{n}\right) \rho^{(s)}(v)\,dv\,du$$

$$= \frac{(-1)^s n^{s-(k+1)/r}}{r} \int_0^1 \rho^{(s)}(v) \int_0^v u^{(k+1)/r-1} \ln^{m+1}\left(\frac{v}{n} - \frac{u}{n}\right) du\,dv. \tag{9}$$

It follows that

$$\underset{n\to\infty}{N-\lim}\, I_k(n) = 0 \tag{10}$$

for $k = 0, 1, 2, \ldots, rs - 2$.

When $k = rs - 1$, we have from equation (9),

$$\underset{n\to\infty}{N-\lim}\, I_{rs-1} = \frac{(-1)^s}{r} \int_0^1 \rho^{(s)}(v) \int_0^v u^{s-1} \ln^{m+1}(v - u)\,du\,dv. \tag{11}$$

Putting $u = vw,$ we have

$$\int_0^v u^{s-1} \ln^{m+1}(v-u)du = v^s \int_0^1 w^{s-1}[\ln v + \ln(1-w)]^{m+1} dw$$

$$= v^s \sum_{i=0}^{m+1} \binom{m+1}{i} \ln^i v \int_0^1 w^{s-1} \ln^{m-i+1}(1-w)dw$$

$$= v^s \sum_{i=0}^{m+1} \binom{m+1}{i} B_{0,\, m-i+1}(s,\, 1) \ln^i v, \tag{12}$$

where $B(\lambda,\, \mu)$ denotes the Beta function and

$$B_{i,\, j}(\lambda,\, \mu) = \frac{\partial^{i+j}}{\partial^i \lambda \partial^j \mu} B(\lambda,\, \mu)$$

for $i,\; j = 0, 1, 2, \ldots.$ It now follows from equations (11) and (12) that

$$N - \lim_{n\to\infty} I_{rs-1} = \frac{(-1)^s}{r} \sum_{i=0}^{m+1} \binom{m+1}{i} B_{0,\, m-i+1}(s,\, 1) \int_0^1 v^s \ln^i v\rho^{(s)}(v)dv$$

$$= \frac{(-1)^s}{r} \sum_{i=0}^{m+1} \binom{m+1}{i} c_{s,\, i} B_{0,\, m-i+1}(s,\, 1). \tag{13}$$

When $k = rs,$ we have from equation (9) that

$$|\, I_{rs}(n) \,| = O(n^{-1/n} \ln^{m+1} n)$$

and so

$$\lim_{n\to\infty} \int_0^b x^{rs} \varphi^{(rs)}(\xi x) G_{n,\, s,\, m}(x_+^r)dx = 0. \tag{14}$$

Further, putting $nt = v$ again, we have

$$\int_a^0 x^{rs} \varphi^{(rs)}(\xi x)(x) G_{n,\, s,\, m}(x_+^r)dx$$

$$= (-1)^s \int_a^0 x^{rs} \varphi^{(rs)}(\xi x)(x) \int_0^{1/n} \ln^{m+1} t \delta_n^{(s)}(t)\, dt dx$$

$$= (-1)^s n^s \int_a^0 x^{rs} \varphi^{(rs)}(\xi x)(x) \int_0^1 (\ln v - \ln n)^{m+1} \rho^{(s)}(v)\, dv dx.$$

Thus

$$N-\lim_{n\to\infty} \int_a^0 x^{rs} \varphi^{(rs)}(\xi x)(x) G_{n,s,m}(x_+^r)\, dx = 0 \tag{15}$$

and so it follows from equations (14) and (15) that

$$N-\lim_{n\to\infty} \int_a^b x^{rs} \varphi^{(rs)}(\xi x)(x) G_{n,s,m}(x_+^r)\, dx = 0. \tag{16}$$

Now using equations (6) to (8), (10), (11), (13) and (16), we have

$$N-\lim_{n\to\infty} \langle G_{n,s,m}(x_+^r), \varphi(x)\rangle$$

$$= \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{(-1)^s c_{s,i} B_{0,m-i+1}(s,1) \varphi^{(rs-1)}(0)}{r(rs-1)!}$$

$$= \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{(-1)^{rs+s-1} c_{s,i} B_{0,m-i+1}(s,1)}{r(rs-1)!} \langle \delta^{(rs-1)}(x), \varphi(x)\rangle$$

for arbitrary $\varphi(x)$ in $\mathcal{D}$ and equation (4) follows.

Equation (5) follows on noting that $c_{1,i} = 0$ for $i = 0, 1, 2, ..., m-1$ and

$$B_{0,0}(1, 1) = 1 = -B_{0,1}(1, 1).$$

We now prove the following generalization of equation (1).

Theorem 7. *If* $F_{s,m}(x)$ *denotes the distribution* $x_-^{-s} \ln^m x_-$, *then the neutrix composition* $F_{s,m}(x_+^r)$ *exists for* $r, s, m = 1, 2, \ldots$.

In particular,

$$F_{1,m}(x_+^r) = \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{(-1)^{r-1} c_{1,i} B_{0,m-i+1}(1, 1)}{(m+1)r!} \delta^{(r-1)}(x) \tag{17}$$

for $r, m = 1, 2, \ldots$ *and*

$$F_{s+1,1}(x_+^r) = \sum_{i=0}^{2} \binom{2}{i} \frac{(-1)^{rs+s} c_{s,i} B_{0,2-i}(s, 1)}{2r(rs-1)!(s-1)!} \delta^{(rs-1)}(x)$$

$$+ \frac{(-1)^{rs+s} \phi(s-1) c(\rho)}{r(rs-1)!} \delta^{(rs-1)}(x). \tag{18}$$

Proof. We note first of all that from equation (2), we have

$$F_{1,m}(x_+^r) = -(m+1)^{-1} G_{1,m}(x_+)$$

and so equation (17) follows immediately.

Next, it follows from equations (1), (3) and (4) that

$$G_{s,1}(x_+^r) = 2(s-1)!\phi(s-1)F_s(x_+^r) - 2(s-1)!F_{s,1}(x_+^r)$$

$$= \frac{2(-1)^{rs+s}(s-1)!\phi(s-1)c(\rho)}{r(rs-1)!} \delta^{(rs-1)}(x) - 2(s-1)!F_{s,1}(x_+^r)$$

$$= \sum_{i=0}^{2} \binom{2}{i} \frac{(-1)^{rs+s-1} c_{s,i} B_{0,2-i}(s, 1)}{r(rs-1)!} \delta^{(rs-1)}(x)$$

and equation (18) follows. This completes the proof of the theorem.

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