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A STUDY ON RIGHT SUBSTRUCTURES IN NEAR-RINGS

Yong Uk Cho

Department of Mathematics
College of Education
Silla University
Pusan 617-736, Korea
e-mail: yucho@silla.ac.kr

Abstract

In this paper, we denote R a near-ring. We initiate a study of substructures in R, and relationship between them.

Next, we investigate some isomorphic properties of near-rings and then some characterizations of right ideal structures in near-rings.

1. Introduction

A near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $a(b + c) = ab + ac$ for all a, b, c in R. We note that obviously, $a0 = 0$ and $a(-b) = -ab$ for all a, b in R, but in general, $0a \neq 0$ and $(-a)b \neq -ab$.

If R has a unity 1, then R is called *unitary*. An element d in R is called *distributive* if $(a + b)d = ad + bd$ for all a and b in R.

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An *ideal* of R is a subset I of R such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $aI \subset I$ for all $a \in R$, (iii) $(I + a)b - ab \subset I$ for all a, $b \in R$. If I satisfies (i) and (ii), then it is called a *left ideal* of R. If I satisfies (i) and (iii), then it is called a *right ideal* of R [4].

On the other hand, an *R-subgroup* of R is a subset H of R such that (i) $(H, +)$ is a subgroup of $(R, +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii), then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii), then it is called a *right R-subgroup* of R. In case, $(H, +)$ is normal in above, we say that *normal R-subgroup*, *normal left R-subgroup* and *normal right R-subgroup* instead of R-subgroup, left R-subgroup and right R-subgroup, respectively. Note that the normal left R-subgroups of R are equivalent to the left ideals of R.

We consider the following substructures of near-rings: Given a near-ring R, $R_0 = \{a \in R \mid 0a = 0\}$ which is called the *zero symmetric part* of R,

$$R_c = \{a \in R \mid 0a = a\} = \{a \in R \mid ra = a, \text{ for all } r \in R\} = \{0a \in R \mid a \in R\}$$

which is called the *constant part* of R, and $R_d = \{a \in R \mid a \text{ is distributive}\}$ which is called the *distributive part* of R.

A non-empty subset S of a near-ring R is said to be a *subnear-ring* of R, if S is a near-ring under the operations of R, equivalently, for all a, b in S, $a - b \in S$ and $ab \in S$. Sometimes, we denote it by $S < R$.

We note that R_0 and R_c are subnear-rings of R, R_d is a subsemigroup of $(R, \cdot)$, but is not a subnear-ring of R. A near-ring R with the extra axiom $0a = 0$ for all $a \in R$, that is $R = R_0$, is said to be *zero symmetric*, also, in case $R = R_c$, R is called a *constant* near-ring, and in case $R = R_d$, R is called a *distributive* near-ring.

Moreover, we note that R_0 is a right ideal of R, but not generally ideal of R, also R_c is an R-subgroup of R, but in general neither a right nor a left ideal of R.

Let $(G, +)$ be a group (not necessarily abelian). We may obtain some examples of near-rings as follows:

First, if we define multiplication on G as $xy = y$ for all x, y in G, then $(G, +, \cdot)$ is a near-ring, because $(xy)z = z = x(yz)$ and $x(y + z) = y + z = xy + xz$, for all x, y, z in G, but in general, $0x = 0$ and $(x + y)z = xz + yz$ are not true. These kinds of near-rings are constant near-rings.

For the remainder basic concepts and results on near-rings, we refer to Pilz [4].

2. Characterizations of Right Ideal Structures in Near-rings

Let R and S be two near-rings. Then a mapping f from R to S is called a *near-ring homomorphism* [4] if (i) $(a + b)f = af + bf$, (ii) $(ab)f = afbf$, for all $a, b \in R$. Obviously, $Rf < S$ and $Tf^{-1} = \{a \in R \mid af \in T\} < R$ for any $T < S$. As in ring theory, Rf is called the *image* of f which is denoted by *Imf*, also, $\{0\}f^{-1} = \{a \in R \mid af = 0\}$ is called the *kernel* of f which is denoted by $Ker\ f$.

We can replace homomorphism by monomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as in ring theory [1].

From now on, we will consider the isomorphism theorem in near-rings (or, R-groups) which is only mentioned already in [4], we can reprove it more concretely as follows.

Let $f : R \to S$ be a near-ring homomorphism. Then certainly, $f : R^+ \to S^+$ be a group homomorphism, where $R^+ = (R, +)$, and so as group

$$R^+/Ker\ f \cong R^+ f.$$

Putting $K := Ker\ f$, $(K, +)$ is a normal subgroup of $(R, +)$ and $R/K = \{a + K \mid a \in R\}$. The addition in R defines an addition in R/K by

$$(a + K) + (b + K) = (a + b) + K.$$

This addition is well defined in group theory.

Would it make

$$(a + K)(b + K) = ab + K$$

a well defined binary operation? It is affirmative in the following statement:

Lemma 2.1. *Let K be the kernel of a near-ring homomorphism* $f : R \to S$. *Then* $(R/K, +, \cdot) \cong Imf$.

Proof. If $(a + K)(b + K) = ab + K$ is a well defined binary operation, then easily, $(R/K, +, \cdot)$ is a near-ring.

Suppose that $a + K = a' + K$ and $b + K = b' + K$. Then there exist $x, y \in K$ such that $a = a' + x$ and $b = b' + y$. We need to show that $ab + K = a'b' + K$ or equivalently, $ab - a'b' \in K$.

Now, $ab = (a' + x)(b' + y) = (a' + x)b' + (a' + x)y$. Since $(a' + x)y$ is in K, putting $(a' + x)y = k$ in K, $ab = (a' + x)b' + k$ and $ab - a'b' = (a' + x)b' + k - a'b' = (a' + x)b' - a'b' + k' \in K$, for some $k' \in K$. Hence, multiplication is well defined.

As groups, $(R/K, +) \cong (Rf, +)$, where a mapping $F : R/K \to Rf$ which is defined by $(a + K)F = af$ is the group isomorphism. Now, we have

$$((a + K)(b + K))F = (ab + K)F = abf = afbf = (a + K)F(b + K)F.$$

Consequently, F is a near-ring isomorphism. □

We can obtain the following fundamental theorem in near-ring homomorphism as in ring theory:

Proposition 2.2. *Let* $f : R \to S$ *be a near-ring epimorphism with the kernel K of f, and let* $\pi : R \to R/K$ *defined by* $a\pi = a + K$ *be the natural epimorphism. Then the isomorphism* $F : R/K \to S$ *which is defined by* $(a + K)F = af$ *is unique such that* $\pi F = f$.

Proof. By Lemma 2.1, there exists a near-ring isomorphism $f : R \to S$.

Next, to show that $\pi F = f$, let $a \in R$, and we get $a(\pi F) = (a\pi)F = (a + K)F = af$. Hence, $\pi F = f$.

Finally, to show that the "uniqueness", if $F' : R/K \to S$ is a near-ring isomorphism such that $\pi F' = f$, then for all $a + K \in R/K$, we have

$$(a + K)F' = (a\pi)F' = a(\pi F') = af = (a\pi)F = (a + K)F. \qquad \square$$

Analogously, we can prove the isomorphism theorem and fundamental theorem for *R*-groups.

The following are some characterizations of ideal structures of near-rings, in particular right ideal structures, which are obtained using the fact of the proof in Lemma 2.1.

Proposition 2.3. *Let $(R, +, \cdot)$ be a near-ring. Suppose that $(K, +)$ is a normal subgroup of $(R, +)$ and K is a left R-subgroup of R. Then the following conditions are equivalent*:

(1) *K is the kernel of a near-ring homomorphism.*

(2) $(x + a)b - ab \subset K$ *for all* $x \in K$ *and* $a, b \in R$.

(3) $(a + x)b - ab \subset K$ *for all* $x \in K$ *and* $a, b \in R$.

(4) $-ab + (a + x)b \subset K$ *for all* $x \in K$ *and* $a, b \in R$.

(5) $-ab + (x + a)b \subset K$ *for all* $x \in K$ *and* $a, b \in R$.

(6) *K is a right ideal of R.*

Proof. (1) $\Rightarrow$ (2) Suppose K is the kernel of a near-ring homomorphism $f : R \to S$, that is, $K := Ker\, f = \{a \in R \mid af = 0\}$. Then for all $x \in K$ and $a, b \in R$, $((x + a)b - ab) f = (xf + af)bf - afbf = 0$ since $xf = 0$. Hence $(x + a)b - ab \subset K$ for all $x \in K$ and $a, b \in R$.

(2) $\Rightarrow$ (1) Assume the condition that $(x + a)b - ab \subset K$ for all $x \in K$

and $a, b \in R$. Since $(K, +)$ is a normal subgroup of $(R, +)$, there exists a quotient group $(R/K, +)$ and the natural group epimorphism $\pi : R \to R/K$ defined by $a\pi = a + K$. Now $K = Ker\,\pi$ as a group homomorphism. We need only show that $(ab)\pi = a\pi b\pi$, that is, $ab + K = (a + K)(b + K)$. To do this, we must show that

$$(a + K)(b + K) = ab + K$$

is a well defined binary operation.

We take that $a + K = a' + K, b + K = b' + K$. So there exist $x, y \in K$ such that $a = a' + x, b = b' + y$. Hence

$$ab = (a' + x)(b' + y) = (a' + x)b' + (a' + x)y$$

$$= (a' + x)b' - a'b' + a'b' + (a' + x)y \in K + a'b',$$

since K is a left R-subgroup of R, $(a' + x)y \in K$, also, by assumption, $(a' + x)b' - a'b' \in K$. Hence we see that $ab - a'b' \in K$, equivalently, $ab + K = a'b' + K$. Consequently, $(a + K)(b + K) = ab + K$ is a well defined binary operation.

(2) $\Leftrightarrow$ (3) Let $x \in K$ and $a, b \in R$. Then from $(K, +)$ is a normal subgroup of $(R, +)$, $x + a \in K + a = a + K$, so that there exist $x' \in K$ such that $x + a = a + x'$. Analogously, there exist $x'' \in K$ such that $a + x = x'' + a$.

(3) $\Leftrightarrow$ (4) Let $x \in K$ and $a, b \in R$. Then $(a + x)b - ab \subset K \Leftrightarrow (a + x)b + K = ab + K \Leftrightarrow -ab + (a + x)b \subset K$, because of $-ab + (a + x)b \subset K = -[-(a + x)b + ab] = -[(a + x)(-b) - a(-b) \in K]$.

(2) $\Leftrightarrow$ (5) can be proved as similar method of the proof of (2) $\Leftrightarrow$ (3).

(1) $\Leftrightarrow$ (6) is obviously proved from the definition of right ideal structure.

□

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