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PROPERTY (T) AND LEFT G-INVARIANT COARSE EMBEDDABILITY OF TOPOLOGICAL GROUPS

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Abstract

Let G be a topological group with a left G-invariant metric d satisfying $\sup_{g,\, h \in G}(d(g,\, h)) = \infty.$ We show that if G admits a left G-invariant coarse embedding into a Hilbert space, then G does not have Property (T). As an application of our main results, we show that if G and H are two non-trivial countable discrete groups, which admit left G-invariant coarse embeddings into a Hilbert space and $\Gamma = G * H$ is their free product, then Γ has not Property (T).

1. Introduction

In [6], Gromov introduced the following concept of coarse embeddings to describe the large scale geometry of groups.

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Definition 1.1. Let X and Y be two metric spaces. We say function $f : X \to Y$ is a *coarse embedding* if there exist two non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ satisfying

(1) $\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y))$ for all $x, y \in X$;

(2) $\lim_{t\to\infty} \rho_1(t) = \infty$.

In 2000, Yu [13] proved the coarse Baum-Connes conjecture for spaces which admitted a uniform embedding into a Hilbert space. As a consequence, for finitely generated groups which admit a uniform embedding into a Hilbert space, they also proved that the strong Novikov conjecture held.

In 2002, Dranishnikov et al. [4] studied the uniform embedding into Hilbert space. In terms of the negative define kernels, they gave a condition which was equivalent to coarse embeddability (a locally finite metric space into a Hilbert space).

In 2004, Nowak [10] gave such characterizations for general metric spaces and obtained that space L_p admits a coarse embedding into a Hilbert space for $0 < p < 2$. Moreover, they showed that coarse embeddings into L_p was guaranteed for a finitely generated group Γ to satisfy the Novikov Conjecture. The relationship among coarse embeddability of general metric spaces was studied more clearly.

In 2000, Yu [13] presented the definition of "Property (A)" which could be regarded as a generalization of amenability for discrete spaces with bounded geometry. Therefore, for a locally compact discrete group with a word length metric, amenability implies Property (A) and Property (A) can imply coarsely embedding into a Hilbert space.

In the mid 60's, Kazhdan [8] introduced the definition of "Property (T)" for the locally compact groups. With this definition, he proved that a large class of lattices was finitely generated. The definition is as follows:

Definition 1.2 [8]**.** A topological group G has *Property* (T) if there exist a compact subset Q and a real number $\varepsilon > 0$ such that, whenever π is a continuous unitary representation of G on a Hilbert space H for which there exists a vector $\xi \in H$ of norm 1 with $\sup_{q \in Q} \| \pi(q)\xi - \xi \| < \varepsilon$, then there exists an invariant vector, namely a vector $\eta \neq 0$ in H such that $\pi(g)\eta = \eta$ for all $g \in G$.

For locally compact groups, amenability and Property (T) are two extreme and opposite properties, "rigid" and "soft", respectively.

There is a natural question:

What is the relationship between Property (*T*) *of a group G and its coarse embeddability*?

In 2008, Kekka et al. [9] characterized Property (T) by using the negative definite functions.

Motivated by these results, the most interesting and considerable issue is the relationship between the negative definite functions and the negative definite kernels of groups. In this paper, a definition of G-invariant coarse embedding is given. The relationship between the G-invariant coarse embedding and Property (T) of the topological groups is studied. Moreover, Property (T) of the free product groups is investigated.

In the next section, some preliminaries are presented. In Section 3, a definition of left G-invariant coarse embedding is given; by using Property (T), the left G-invariant coarse embedding is characterized. The free product of countable discrete groups is studied in Section 4.

2. Preliminaries

In this section, some general facts and concepts are introduced.

Definition 2.1 [9]**.** A continuous real valued kernel Ψ on a topological space X is *conditionally negative type* if

(i) $\Psi(x, x) = 0,\ \Psi(x, y) = \Psi(y, x),$ for all $x, y \in X$,

(ii) $\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \Psi(x_i, x_j) \leq 0$

for any elements $x_1, x_2, ..., x_n$ in X and any real numbers $c_1, c_2, ..., c_n$ with $\sum_{i=1}^{n} c_i = 0$. A continuous real value function ψ on a topological group G is a function of conditionally negative type if the kernel on G defined by $(g, h) \mapsto \psi(h^{-1}g)$ is conditionally negative type.

Example 2.2. Let G be a topological group, and let α be an affine isometric action of G on a real Hilbert space H, from example C.2.2 ii in [9], for any $\xi \in H$, the function

$$\psi : G \to R, \quad g \to \| \alpha(g)\xi - \xi \|^2$$

is a function of conditionally negative type.

Definition 2.3. Let X be a metric space. Let G be a topological group acting continuously on the metric space X. A continuous real valued function f on a topological space $X \times X$ is *left group-invariant* if $\Psi(x, x) = \Psi(gx, gy)$, for all $g \in G$, $y \in X$. Throughout this article, "left group-invariant" will be denoted by "left G-invariant". A continuous real valued function Ψ on a topological group $G \times G$ is *left G-invariant* if $\Psi(x, x) = \Psi(gx, gy)$, for all $g, x, y \in G$. A metric d on a topological group G is left G-invariant if $d(x, x) = d(gx, gy)$, for all $g, x, y \in G$.

Theorem 2.4. *Let Ψ be a kernel of conditionally negative type on a topological group G which is left G-invariant if only if there exists a function of conditionally negative type ψ, such that $\Psi(h, g) = \psi(g^{-1}h)$.*

Proof. If $\psi(g^{-1}h) = \Psi(h, g)$, then $\Psi(ah, ag) = \psi((ag)^{-1}(ah)) = \psi(g^{-1}a^{-1}ah) = \psi(g^{-1}h) = \Psi(h, g)$.

If Ψ is left G-invariant, then $\Psi(h, g) = \Psi(g^{-1}h, g^{-1}g) = \Psi(g^{-1}h, e)$. Denote $\psi(g^{-1}h) \equiv \Psi(g^{-1}h, e)$, then we can obtain that ψ is a function of conditionally negative type. $\square$

The key to give the relationship between the maps into Hilbert spaces and the negative kernels is the following theorem:

Theorem 2.5 [12]**.** *A continuous real valued kernel* Ψ *in a topological space X is of conditionally negative type if and only if there exist a Hilbert space H and a map* $f : X \to H$ *such that*

$$\Psi(x, y) = \| f(x) - f(y) \|^2, \quad \textit{for all } x, y \in X.$$

From Definition 2.3, Theorems 2.4 and 2.5, we have:

Theorem 2.6. *A continuous real valued kernel* Ψ *in a topological group G is a function of conditionally negative type if and only if there exist a Hilbert space H, a left G-invariant map* $f : G \to H$ *and a map* ψ *such that*

$$\Psi(h, g) = \psi(g^{-1}h) = \| f(h) - f(g) \|^2, \quad \textit{for all } h, g \in G.$$

Definition 2.7. A topological group *G* has *Property* (*FH*) if every affine isometric action of *G* in a real Hilbert space has a fixed point.

The following theorem describes the relationship between the bounded function of conditionally negative type and Property (FH).

Theorem 2.8 [9]**.** *Let G be a topological group*. *Then the following two statements are equivalent*:

(1) *G has Property* (*FH*),

(2) *every function of conditionally negative type on G is bounded*.

The following theorem shows that Property (T) and Property (FH) are closely related. Part (1) is due to Delorme [3]. Part (2) is due to Guichardet [7].

Theorem 2.9 ([3], [7])**.** *Let G be a topological group*.

(1) *If G has Property* (*T*), *then G has Property* (*FH*).

(2) *If G is a* σ*-compact locally compact group and G has Property* (*FH*), *then G has Property* (*T*).

3. Left G-invariant Coarse Embeddings of Topological Groups and Property (T)

In this section, we give a definition of left G-invariant coarse embedding, and then characterize the left G-invariant coarse embedding by using Property (T).

Definition 3.1. Let X and Y be two metric spaces. Let G be a topological group acting continuously on the metric space X. A continuous function $f : X \to Y$ is called a *left G-invariant coarse embedding* if

(1) $f : X \to Y$ is a coarse embedding,

(2) f is a left G-invariant function on X,

(3) d is a left G-invariant metric on X.

The following theorem will show the connection between the left G-invariant coarse embeddability of topological groups and Property (T).

Theorem 3.2. *If a topological group G with a left G-invariant metric d admits a G-invariant coarse embedding into a Hilbert space and* $\sup_{g, h \in G}(d(g, h)) = \infty$, *then G has not Property* (T).

We will use the following lemma to prove Theorem 3.2:

Lemma 3.3. *A topological group G with a left G-invariant metric d admits a G-invariant coarse embedding into a Hilbert space, if and only if, there exist a function of conditionally negative type* $\psi : G \to H$ *and two non-decreasing functions* $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$, *satisfying*

(1) $\rho_1(d_G(h, g)) \leq \psi(g^{-1}h) \leq \rho_2(d_G(h, g))$ *for all* $h, g \in G$,

(2) $\lim_{t \to \infty} \rho_1(t) = \infty$.

Proof. $\Rightarrow$ If there exist non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ satisfying

(1) $\rho_1(d_G(h, g)) \leq \| f(h) - f(g) \| \leq \rho_2(d_G(h, g))$ for all $h, g \in G$,

(2) $\lim_{t \to \infty} \rho_1(t) = \infty$,

define $N(h, g) = \| f(h) - f(g) \|^2$, then N is a kernel of conditionally negative type. In fact, we have:

(i) $N(h, h) = 0$;

(ii) When $\Sigma C_i = 0$ and $g_1, g_2, \ldots, g_n \in G$, then

$$\sum_{i,\ j=1}^{n} N(g_i, g_j) c_i c_j$$

$$= \sum_{i,\ j=1}^{n} (\| f(g_i) \|^2 + \| f(g_j) \|^2 - 2\langle f(g_i), f(g_j) \rangle) c_i c_j$$

$$= \left(\sum_{j=1}^{n} c_j \right) \left(\sum_{i=1}^{n} \| f(g_i) \|^2 c_i \right) + \left(\sum_{i=1}^{n} c_i \right) \left(\sum_{j=1}^{n} \| f(g_j) \|^2 c_j \right)$$

$$- 2 \left(\sum_{i=1}^{n} c_i \| f(g_i) \|^2 \right)$$

$$= -2 \left\| \sum_{i=1}^{n} (c_i) f(g_i) \right\|^2 \le 0;$$

(iii) Notice that *f* is a left G-invariant function on *G*, hence,

$$N(h, g) = \| f(h) - f(g) \|^2 = \| f(g^{-1}h) - f(e) \| = N(g^{-1}h, e),$$

where e is unit element. Denote $\sqrt{N(g^{-1}h, e)} = \psi(g^{-1}h)$, then ψ is a function of conditionally negative type on *G*. Hence there exist a function of conditionally negative type ψ on G and two non-decreasing functions $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$, satisfying

(1) $\rho_1(d_G(h, g)) \le \psi(g^{-1}h) \le \rho_2(d_G(h, g))$ for all $h, g \in G$,

(2) $\lim_{t \to \infty} \rho_1(t) = \infty$.

$\Leftarrow$ Suppose there exists a function of conditionally negative type $\psi : G \to H$ satisfying

(1) $\rho_1(d_G(h, g)) \le \psi(g^{-1}h) \le \rho_2(d_G(h, g))$ for all $h, g \in G$,

(2) $\lim_{t\to\infty} \rho_1(t) = \infty$.

From Theorem 2.6, there exist a Hilbert space H, a left G-invariant map $f : G \to H$ and a map ψ such that $\psi(g^{-1}h) = \| f(h) - f(g)\|^2$, for all $h, g \in G$. That is

$$\sqrt{\rho_1(d_G(h, g))} \le \| f(h) - f(g)\| \le \sqrt{\rho_2(d_G(h, g))}. \qquad \square$$

Proof of Theorem 3.2. G admits a G-invariant coarse embedding into a Hilbert space. From Lemma 3.3, there exist a function of conditionally negative type $\psi : G \to H$ and two non-decreasing functions $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$, satisfying

(1) $\rho_1(d_G(h, g)) \le \psi(h^{-1}g) \le \rho_2(d_G(h, g))$ for all $h, g \in G$,

(2) $\lim_{t\to\infty} \rho_1(t) = \infty$.

Now suppose G has Property (T), from Theorem 2.9, G has Property (FH). From Theorem 2.8, every function of conditionally negative type on G is bounded.

Due to $\sup_{g, h\in G}(d(g, h)) = \infty$ and (2) in Lemma 3.3, the function of conditionally negative type ψ is an unbounded function on G. This is a contradiction. Thus we complete the proof. $\square$

By using Theorem 3.2, we can give an example which has coarse embeddability but not G-invariant coarsely embeddability.

Example 3.4. The groups $SL_3(Z)$ are coarsely embeddable in a Hilbert space but not G-invariant coarsely embeddable in any Hilbert space.

In fact, $SL_3(Z)$ has Property (T). It is well known that $SL_3(Z)$ is coarsely embeddable in a Hilbert space. When d is the word metric of $SL_3(Z)$, then $\sup_{g, h \in SL_3}(d(g, h)) = \infty$. From Theorem 3.2, $SL_3(Z)$ is not G-invariant coarsely embeddable.

4. Free Product of Countable Discrete Groups

We recall some elementary facts about length functions, metrics on countable discrete groups and free products, details can be found in [2].

An integer valued function l on a group G is a length function if

(1) $l(g) = l(g^{-1}) \geq 0$ for all $g \in G$,

(2) $l(e) = 0$,

(3) $l(gh) \leq l(g) + l(h)$, for all $g, h \in G$, where e is unit element in G.

A length function l is non-degenerate if $l(g) = 0$ implies that $g = e$.

For any non-degenerate length function l on G, we define d on G by $d(g, h) = l(gh^{-1})$, then d is a left G-invariant metric on G.

Let G and H be two countable discrete groups and let $\Gamma = G * H$ be their free product. Every element $g \in \Gamma$ is uniquely expressed in normal form as a reduced word $g = x_1 \cdots x_p$, where $x_i \in G \bigcup H$, $x_i \neq e$ and if $x_i \in G$ (or H), then $x_{i+1} \in H$ (or G), as appropriate.

Let l_G and l_H be two non degenerate integer valued length functions on G and H, respectively. Define an integer valued function l_Γ on Γ by

$$l_\Gamma(g) = \sum_1^n l_G(a_i) + \sum_1^n l_H(b_i),$$

where $g = a_1 b_1 \cdots a_n b_n$ is a product without cancellation and $a_i \in G$, $b_i \in H$. It is obvious that l_Γ is a length function on Γ. If $g, g' \in \Gamma$, we denote g and g' as products without cancellation, $g = hx x_1 \cdots x_n$, $g' =$

$hx'x'_1 \cdots x_m$, where h is the common part of g and g', $x \neq x' \in G$ (or H) and $x_1, \ldots, x_n$ are alternately elements of G and H (or H and G) and similarly for $x'_1, \ldots, x'_m$, and with the convention that empty sums are zero, a metric d_Γ can be defined by

$$d_\Gamma(g, g') = \sum_1^n l_{G,H}(x_i) + d_{G,H}(x, x') + \sum_1^m l_{G,H}(x'_j),$$

where we have written $l_{G,H}$ to mean l_G or l_H as appropriate, and similarly for $d_{G,H}$. It is obvious that d_Γ is a left Γ-invariant metric.

The following result is important to give the relationship between the free product and the coarse embedding.

Theorem 4.1. *Let G and H be two non-trivial countable discrete groups. Let* $\Gamma = G * H$ *be their free product. If both G and H admit the G-invariant coarse embeddings into a Hilbert space, then so does* Γ.

Proof. From the proof of Theorem in [2], we know that the coarse map F of Γ is left G-invariant and the metric d of Γ is also left G-invariant, hence F is a left G-invariant coarse embedding. □

From Property (T) of free product of the countable discrete groups and Theorem 3.2, we obtain the following theorem.

Theorem 4.2. *Let G and H be two non-trivial countable discrete groups and let* $\Gamma = G * H$ *be their free product. If both G and H admit coarse embeddings into a Hilbert space, then* Γ *has not Property* (*T*).

Proof. Suppose Γ has Property (T). From the definition of l_Γ and d_Γ, d_Γ is a left G-invariant metric. From Theorem 4.1, Γ admits a G-invariant coarse embedding into a Hilbert space. It is obvious that $\sup_{g, h \in \Gamma} d_\Gamma(g, h) = \infty$. From Theorem 3.2, we know that Γ cannot have Property (T), which is a contradiction. □

Corollary 4.3. *Let G and H be two non-trivial countable discrete groups with finite asymptotic dimension*, *then* $\Gamma = G * H$ *has not Property* (*T*).

Proof. Since both G and H have finite asymptotic dimension, by Theorem 43 in [1], G and H admit a coarse embedding into a Hilbert space. From Theorem 4.2, $\Gamma = G * H$ has not Property (T). □

Remark 4.4. Asymptotic dimension of groups can be found in [1], [5]. Corollary 4.3 is true for two non-trivial exact countable discrete groups which are not two non-trivial topological metric groups with finite asymptotic dimension, for detail see [11].

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