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A NEW ITERATION METHOD FOR THE FIXED POINT OF NONEXPANSIVE MAPPINGS

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Abstract

In this paper, a new iteration process is introduced and some strong convergence theorems are obtained for the nonexpansive mapping in Hilbert spaces.

1. Introduction

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, C be a closed convex subset of H. Then a mapping $T : C \to C$ is said to be *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $F : H \to H$ is said to be η-*strong monotone* if there exists a constant $\eta > 0$ such that $\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2$ for any $x, y \in H$. $F : H \to H$ is

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said to be *k-Lipschitzian* if there exists a constant $k > 0$ such that $\| Fx - Fy \| \leq k \| x - y \|$ for any $x, y \in H$.

The interest and importance of construction if fixed points of nonexpansive mappings stem mainly from the fact that it may be applied in many areas, such as imagine recovery and signal processing (see, e.g., [1, 2, 8]). Iterative techniques for approximating fixed points of nonexpansive mappings have been studied by various authors (see, e.g., [1, 3-7, 9-14], etc), using famous Mann iteration method, Ishikawa iteration method, and many other iteration methods such as, viscosity approximation method [5] and CQ method [6].

For reducing the complexity of computation, for a sequence $\{\alpha_n\}$ of real numbers in $[0, 1]$ and an arbitrary point $u \in C$, starting with another arbitrary initial $x_0 \in C$, Halpern [3] defined a sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.1}$$

and got some convergence results. Lions [4] improved his results and proved the strong convergence of $\{x_n\}$ if the sequence $\{\alpha_n\}$ satisfies the following conditions:

(1) $\lim_{n \to \infty} \alpha_n = 0;$

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty;$

(3) $\lim_{n \to \infty} \frac{(\alpha_n - \alpha_{n-1})}{\alpha_n^2} = 0.$

In 1992, Wittmann [12] proved the strong convergence of $\{x_n\}$. His conditions on the parameters $\{\alpha_n\}$ are (1), (2), and

(3)′ $\sum_{n=1}^{\infty} | \alpha_{n+1} - \alpha_n | < \infty.$

In 2002, Xu [13] got another strong convergence theorem of $\{x_n\}$ in the framework of Banach spaces, and conditions on the parameters $\{\alpha_n\}$ are (1), (2), and

(3)* $\lim_{n\to\infty} \frac{(\alpha_n - \alpha_{n-1})}{\alpha_n} = 0.$

For the same aim, Yamada [14] proposed an iteration method as follows: for arbitrary $u_0 \in H$,

$$u_{n+1} = Tu_n - \lambda_{n+1}\mu F(T(u_n)), \ \ n \geq 0, \tag{1.2}$$

where T is a nonexpansive mapping from H to itself, K is the fixed point set of T, F is an η-strong monotone and k-Lipschitzian mapping on K, $\{\lambda_n\}$ is a real sequence in $[0, 1)$, and $0 < \mu < 2\eta/k^2$. Then Yamada got a strong convergence result as $\{\lambda_n\}$ satisfies the following conditions:

(1) $\lim_{n\to\infty} \lambda_n = 0;$

(2) $\sum_{n=1}^{\infty} \lambda_n = \infty;$

(3) $\lim_{n\to\infty} \frac{(\lambda_n - \lambda_{n+1})}{\lambda_{n+1}^2} = 0.$

In 2006, Wang [11] defined a sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^{\lambda_{n+1}}x_n, \ \ n \geq 0, \tag{1.3}$$

where T^{λ} is a mapping from H to itself and defined as follows:

$$T^{\lambda}x = Tx - \lambda\mu F(Tx), \ \ \forall x \in H. \tag{1.4}$$

Then under some suitable conditions, the sequence $\{x_n\}$ is shown to convergence strongly to a fixed point of T and the necessary and sufficient conditions that $\{x_n\}$ converges strongly to a fixed point of T are obtained.

Motivated by the above work, we propose a new explicit iteration scheme with mapping F to approximate the fixed point of nonexpansive mapping in Hilbert space.

2. Preliminaries

Let T be a nonexpansive mapping from C into itself, $F : H \to H$ an η-strongly monotone and k-Lipschitzian, $\{\lambda_n\} \subset [0, 1)$, and μ a fixed constant in $(0, 2\eta/k^2)$. Starting with an initial point $x_0 \in H$, the explicit iteration scheme with mapping F is defined as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T^{\lambda_{n+1}} x_n, \ \ n \geq 0, \tag{2.1}$$

where u is an arbitrary point, and T^{λ} is a mapping from H to itself and defined as follows:

$$T^{\lambda} x = Tx - \lambda\mu F(Tx), \ \ \forall x \in H. \tag{2.2}$$

Note. If $\lambda_n = 0$ for any $n \geq 1$, then scheme (2.1) reduces to the famous Halpern iteration scheme (1.1).

If we replace Tx_n in scheme (2.2) with the mean

$$T_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \ \ n \geq 1, \ \ x \in C,$$

then we have

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_n^{\lambda_{n+1}} x_n, \ \ n \geq 0. \tag{2.3}$$

Notice that T is a nonexpansive mapping, so T_n is a nonexpansive mapping too.

We restate the following lemmas which play crucial roles in our proofs.

Lemma 2.1 [13]**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \ \ \forall n \geq 0,$$

where $\{\alpha_n\}, \ \{\beta_n\}, \ \{\gamma_n\}$ *satisfy the conditions*:

(1) $\{\alpha_n\} \subset [0, 1], \ \sum_{n=1}^{\infty}\alpha_n = \infty;$

(2) $\limsup_{n\to\infty} \beta_n \leq 0;$

(3) $\gamma_n \geq 0 (n \geq 0), \ \sum_{n=1}^{\infty}\gamma_n < \infty.$

Then

$$\lim_{n\to\infty} s_n = 0.$$

Lemma 2.2 [14]**.** *Let* $T^{\lambda}x = Tx - \lambda\mu F(Tx)$, *where* $T : H \to H$ *is a nonexpansive mapping and* $F : H \to H$ *is an* η-*strongly monotone and* k-*Lipschitzian mapping*. *If* $0 \leq \lambda_n < 1$ *and* $0 < \mu < 2\eta/k^2$, *then* T^{λ} *is a contraction and satisfies*

$$\| T^{\lambda}x - T^{\lambda}y \| \leq (1-\lambda\tau)\| x - y \|, \forall x, \ y \in H,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

Lemma 2.3 [9]**.** *Let X be a uniformly smooth Banach space*, *C be a closed convex subset of X and T be a nonexpansive mapping such that* $F(T) \neq \phi$. *Let* $\{z_t\} \in C$ *be defined as follows*:

$$z_t = tu + (1-t)Tz_t, \tag{2.4}$$

where $t \in (0, 1)$. *Then the strong* $\lim_{t\to 0} z_t$ *exists and is a fixed point of T.*

Note. If X is a Hilbert space, then the result owes to Browder [1].

Lemma 2.4 [8]**.** *Let E be a real Banach space*. *Then for arbitrary* $x, \ y \in E$ *and* $J : E \to 2^{E^*}$ *is a normalized duality mapping*, *the following inequality holds*:

$$\| x + y \|^2 \leq \| x \|^2 + 2\langle y, \ j(x + y)\rangle.$$

Lemma 2.5 [15]. *If E is a uniform Banach space and $\tilde{C}$ is a bounded subset of C, then*

$$\limsup_{n\to\infty,\, x\in\tilde{C}} \| T(T_n(x)) - T_n(x) \| = 0.$$

3. Main Results

Lemma 3.1. *Let H be a Hilbert space, C be a closed convex subset of H, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \phi$, and $F : H \to H$ be an η-strongly monotone and k-Lipschitzian mapping. For any given $x_0 \in C$, $\{x_n\}$ is generated by* (2.1). *If the sequences $\{\alpha_n\}, \{\lambda_n\} \subset [0, 1)$ satisfy the conditions*:

(1) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \lim_{n\to\infty} \frac{(\alpha_n - \alpha_{n-1})}{\alpha_n} = 0;$

(2) $\sum_{n=1}^{\infty} \lambda_n < \infty;$

(3) $0 < \mu < 2\eta/k^2,$

then

(1) $\{x_n\}$ *is bounded, so are* $\{Tx_n\}$ *and* $\{Fx_n\}$;

(2) $\lim_{n\to\infty} \| x_n - Tx_n \| = 0.$

Proof. (1) For any $q \in F(T)$, it follows from (2.1) that

$$\| x_{n+1} - q \| = \| \alpha_n(u - q) + (1 - \alpha_n)(T^{\lambda_{n+1}} x_n - q) \|$$

$$\leq \alpha_n \| u - q \| + (1 - \alpha_n) \| T^{\lambda_{n+1}} x_n - q \|,$$

where

$$\begin{aligned}\| T^{\lambda_{n+1}} x_n - q \| &= \| T^{\lambda_{n+1}} x_n - T^{\lambda_{n+1}} q + T^{\lambda_{n+1}} q - q \| \\ &\leq \| T^{\lambda_{n+1}} x_n - T^{\lambda_{n+1}} q \| + \| T^{\lambda_{n+1}} q - q \| \\ &\leq (1 - \lambda_{n+1}\tau) \| x_n - q \| + \lambda_{n+1}\mu \| F(q) \|.\end{aligned}$$

Thus

$$\begin{aligned}\| x_{n+1} - q \| &\leq \alpha_n \| u - q \| + (1 - \alpha_n)(1 - \lambda_{n+1}\tau) \| x_n - q \| \\ &\quad + (1 - \alpha_n)\lambda_{n+1}\mu \| F(q) \| \\ &\leq \alpha_n \| u - q \| + (1 - \alpha_n) \| x_n - q \| + \lambda_{n+1}\mu \| F(q) \|.\end{aligned}$$

Then by induction, we get

$$\| x_n - q \| \leq \max\{\| u - q \|, \| x_0 - q \|\} + \mu \| F(q) \| \sum_{n=1}^{\infty} \lambda_n.$$

Since $\sum_{n=1}^{\infty} \lambda_n < \infty$, so $\{x_n\}$ is bounded, $\{Tx_n\}$ and $\{F(Tx_n)\}$ are bounded, too.

(2) It follows from (2.1) and (2.2) that

$$\| x_{n+1} - T^{\lambda_{n+1}} x_n \| = \alpha_n \| u - T^{\lambda_{n+1}} x_n \| \to 0 \ (n \to \infty), \tag{3.1}$$

and

$$\begin{aligned}\| x_{n+1} - x_n \| &= \| \alpha_n u + (1 - \alpha_n) T^{\lambda_{n+1}} x_n - \alpha_{n-1} u - (1 - \alpha_{n-1}) T^{\lambda_n} x_{n-1} \| \\ &= \| (\alpha_n - \alpha_{n-1})(u - T^{\lambda_{n+1}} x_{n-1}) + (1 - \alpha_n)(T^{\lambda_{n+1}} x_n - T^{\lambda_n} x_{n-1}) \| \\ &\leq (\alpha_n - \alpha_{n-1}) \| u - T^{\lambda_n} x_{n-1} \| + (1 - \alpha_n) \| T^{\lambda_{n+1}} x_n - T^{\lambda_n} x_{n-1} \| \\ &\leq | \alpha_n - \alpha_{n-1} | \| u - T^{\lambda_n} x_{n-1} \| + (1 - \alpha_n) \| x_n - x_{n-1} \| \\ &\quad + (1 - \alpha_n)\mu \| \lambda_{n+1} F(Tx_n) - \lambda_n F(Tx_{n-1}) \| \\ &= (1 - \alpha_n) \| x_n - x_{n-1} \| + | \alpha_n - \alpha_{n+1} | M + \gamma_n,\end{aligned}$$

where $M := \sup_{n\geq 1} \| u - T^{\lambda_n} x_{n-1} \| < \infty$, $\beta_n := \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} M \to 0$,

$$\gamma_n := (1 - \alpha_n)\mu \| \lambda_{n+1} F(Tx_n) - \lambda_n F(Tx_{n-1}) \|$$

satisfies $\gamma_n \geq 0 (n \geq 1)$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. By Lemma 2.1, we have $\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0$.

Thus,

$$\begin{aligned} \| x_n - Tx_n \| &= \| x_n - T^{\lambda_{n+1}} x_n + \lambda_{n+1} \mu F(Tx_n) \| \\ &= \| x_n - x_{n+1} + x_{n+1} - T^{\lambda_{n+1}} x_n - \lambda_{n+1} \mu F(Tx_n) \| \\ &\leq \| x_{n+1} - x_n \| + \| x_{n+1} - T^{\lambda_{n+1}} x_n \| + \lambda_{n+1} \mu \| F(Tx_n) \|. \end{aligned}$$

In addition, $\sum_{n=1}^{\infty} \lambda_n < \infty \Rightarrow \lim_{n\to\infty} \lambda_n = 0$. Thus, $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$. The proof is completed.

Theorem 3.2. *Let H be a Hilbert space, C be a closed convex subset of H, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \phi$, and $F : H \to H$ be an η-strongly monotone and k-Lipschitzian mapping. For any given $x_0 \in C$, $\{x_n\}$ is generated by* (2.1). *If the sequences $\{\alpha_n\}, \{\lambda_n\} \subset [0, 1)$ satisfy the conditions*:

(1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \frac{(\alpha_n - \alpha_{n-1})}{\alpha_n} = 0$;

(2) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(3) $0 < \mu < 2\eta/k^2$,

then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By equation (2.3), we can write

$$z_t - x_n = t(u - x_n) + (1-t)(Tz_t - x_n).$$

It follows from Lemma 2.4 that

$$\begin{aligned}\| z_t - x_n \|^2 &\leq (1-t)^2 \| Tz_t - x_n \|^2 + 2t\langle u - x_n,\ j(z_t - x_n)\rangle \\ &\leq (1-t)^2(\| Tz_t - Tx_n \| + \| Tx_n - x_n \|)^2 \\ &\quad + 2t(\| z_t - x_n \| + \langle u - z_t,\ j(z_t - x_n)\rangle) \\ &\leq (1+t^2)\| z_t - x_n \| + \| Tx_n - x_n \|(2\| z_t - x_n \| + \| Tx_n - x_n \|) \\ &\quad + 2t\langle u - z_t,\ j(z_t - x_n)\rangle.\end{aligned}$$

Hence

$$\begin{aligned}&\langle u - z_t,\ j(x_n - z_t)\rangle \\ &\leq \frac{t}{2}\| z_t - x_n \|^2 + \frac{\| Tx_n - x_n \|}{2t}(2\| z_t - x_n \| + \| Tx_n - x_n \|).\end{aligned}$$

Taking $\limsup$ as $n \to \infty$ yields

$$\limsup_{n\to\infty}\langle u - z_t,\ j(x_n - z_t)\rangle \leq \limsup_{n\to\infty}\frac{t}{2}\| z_t - x_n \|^2.$$

Letting $t \to 0$, noting the fact that $z_t \to z$ in norm and the fact that the duality map j is norm-to-norm uniformly continuous on bounded sets on X, we get

$$\limsup_{n\to\infty}\langle u - z,\ j(x_n - z)\rangle \leq 0. \tag{3.2}$$

In addition, from (2.1) and (2.2), we can write

$$x_{n+1} - z = \alpha_n(u - z) + (1-\alpha_n)(T^{\lambda_{n+1}}x_n - z).$$

By Lemma 2.4, we have

$$\begin{aligned}&\| x_{n+1} - z \|^2 \\ &\leq (1-\alpha_n)^2\| T^{\lambda_{n+1}}x_n - z \|^2 + 2\alpha_n\langle u - z,\ j(x_{n+1} - z)\rangle\end{aligned}$$

$$\leq (1-\alpha_n)(\| T^{\lambda_{n+1}} x_n - T^{\lambda_{n+1}} z \| + \| T^{\lambda_{n+1}} z - z \|)^2$$

$$+ 2\alpha_n \langle u - z,\ j(x_{n+1} - z)\rangle$$

$$\leq (1-\alpha_n)(1-\lambda_{n+1}\tau)^2 \| x_n - z \|^2 + 2\alpha_n \langle u - z,\ j(x_{n+1} - z)\rangle$$

$$+ (1-\alpha_n)\| T^{\lambda_{n+1}} z - z \| (2\| T^{\lambda_{n+1}} x_n - T^{\lambda_{n+1}} z \| + \| T^{\lambda_{n+1}} z - z \|)$$

$$\leq (1-\alpha_n)\| x_n - z \|^2 + 2\alpha_n \langle u - z,\ j(x_{n+1} - z)\rangle$$

$$+ (1-\alpha_n)\| T^{\lambda_{n+1}} z - z \| (2\| T^{\lambda_{n+1}} x_n - T^{\lambda_{n+1}} z \| + \| T^{\lambda_{n+1}} z - z \|),$$

where $\beta := 2\langle u - z,\ j(x_{n+1} - z)\rangle$ satisfies $\limsup_{n\to\infty} \beta_n \leq 0$ by (3.2), and

$$\gamma_n := \| T^{\lambda_{n+1}} z - z \| (2\| T^{\lambda_{n+1}} x_n - T^{\lambda_{n+1}} z \| + \| T^{\lambda_{n+1}} z - z \|)$$

satisfies $\gamma_n \geq 0,\ \sum_{n=1}^{\infty} \gamma_n < \infty.$ Apply Lemma 2.1, we see that $\lim_{n\to\infty} \| x_n - z \| = 0,$ that is, $x_n \to z.$ The proof is completed.

Theorem 3.3. *Let H be a Hilbert space, C be a closed convex subset of H, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \phi$, and $F : H \to H$ be an η-strongly monotone and k-Lipschitzian mapping. For any given $x_0 \in C$, $\{x_n\}$ is generated by* (2.3). *If the sequences $\{\alpha_n\}, \{\lambda_n\} \subset [0, 1)$ satisfy the conditions*:

(1) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty;$

(2) $\sum_{n=1}^{\infty} \lambda_n < \infty;$

(3) $0 < \mu < 2\eta/k^2,$

then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. For any $q \in F(T)$, it follows from (2.3) that

$$\| x_{n+1} - q \| = \| \alpha_n (u - q) + (1 - \alpha_n)(T_n^{\lambda_{n+1}} x_n - q) \|$$

$$\leq \alpha_n \| u - q \| + (1 - \alpha_n) \| T_n^{\lambda_{n+1}} x_n - q \|,$$

where

$$\| T_n^{\lambda_{n+1}} x_n - q \| = \| T_n^{\lambda_{n+1}} x_n - T_n^{\lambda_{n+1}} q + T_n^{\lambda_{n+1}} q - q \|$$

$$\leq \| T_n^{\lambda_{n+1}} x_n - T_n^{\lambda_{n+1}} q \| + \| T_n^{\lambda_{n+1}} q - q \|$$

$$\leq (1 - \lambda_{n+1}\tau) \| x_n - q \| + \lambda_{n+1}\mu \| F(q) \|.$$

Thus

$$\| x_{n+1} - q \| \leq \alpha_n \| u - q \| + (1 - \alpha_n)(1 - \lambda_{n+1}\tau) \| x_n - q \|$$

$$+ (1 - \alpha_n)\lambda_{n+1}\mu \| F(q) \|$$

$$\leq \alpha_n \| u - q \| + (1 - \alpha_n) \| x_n - q \| + \lambda_{n+1}\mu \| F(q) \|.$$

Then by induction, we get

$$\| x_n - q \| \leq \max\{\| u - q \|, \| x_0 - q \|\} + \mu \| F(q) \| \sum_{n=1}^{\infty} \lambda_n.$$

Since $\sum_{n=1}^{\infty} \lambda_n < \infty$, so $\{x_n\}$ is bounded, $\{T_n x_n\}$ and $\{F(T_n x_n)\}$ are bounded, too.

It follows from (2.3) and Lemma 2.4 that

$$\| z_t - T_n^{\lambda_{n+1}} x_n \|^2$$

$$= \| (1 - t)(Tz_t - T_n^{\lambda_{n+1}} x_n) + t(u - T_n^{\lambda_{n+1}} x_n) \|^2$$

$$\leq (1 - t)^2 \| Tz_t - T(T_n^{\lambda_{n+1}} x_n) + T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n \|^2$$

$$+ 2t \langle u - T_n^{\lambda_{n+1}} x_n, z_t - T_n^{\lambda_{n+1}} x_n \rangle$$

$$\leq (1-t)^2(\| Tz_t - T(T_n^{\lambda_{n+1}} x_n) \| + \| T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n \|)^2$$

$$+ 2t\langle u - z_t + z_t - T_n^{\lambda_{n+1}} x_n,\ z_t - T_n^{\lambda_{n+1}} x_n \rangle$$

$$\leq (1+t^2)\| z_t - T_n^{\lambda_{n+1}} x_n \|^2 + 2t\langle u - z_t,\ z_t - T_n^{\lambda_{n+1}} x_n \rangle$$

$$+ \| T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n \|$$

$$\cdot (2\| z_t - T_n^{\lambda_{n+1}} x_n \| + \| T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n \|).$$

Hence

$$\begin{aligned}\langle u - z_t, T_n^{\lambda_{n+1}} x_n - z_t \rangle \leq\ & \frac{t}{2}\| z_t - T_n^{\lambda_{n+1}} x_n \|^2 \\ & + \frac{\| T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n \|}{2t} \\ & \cdot (2\| z_t - T_n^{\lambda_{n+1}} x_n \| + \| T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n \|). \quad (3.3)\end{aligned}$$

Since

$$\begin{aligned}& \| T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n - [T(T_n x_n) - T_n x_n] \| \\ &= \| T(T_n x_n) - T_n x_n - T(T_n x_n - \lambda_{n+1}\mu F(T_n x_n)) + (T_n x_n - \lambda_{n+1}\mu F(T_n x_n)) \| \\ &\leq \| T(T_n x_n) - T(T_n x_n - \lambda_{n+1}\mu F(T_n x_n)) \| + \lambda_{n+1}\mu \| F(T_n x_n) \| \\ &\leq 2\lambda_{n+1}\mu \| F(T_n x_n) \| \to 0 (n \to \infty),\end{aligned}$$

and

$$\begin{aligned}& \| T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n \| - \| T(T_n x_n) - T_n x_n \| \\ & \leq \| T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n - [T(T_n x_n) - T_n x_n] \|,\end{aligned}$$

we have

$$0 \leq \lim_{n\to\infty} \| T(T_n^{\lambda_{n+1}} x_n) - T_n^{\lambda_{n+1}} x_n \| \leq \lim_{n\to\infty} \| T(T_n x_n) - T_n x_n \| = 0.$$

Taking lim sup as $n \to \infty$ in (3.3) yields

$$\limsup_{n\to\infty} \langle u - z_t, T_n^{\lambda_{n+1}} x_n - z_t \rangle \leq \limsup_{n\to\infty} \frac{t}{2} \| z_t - T_n^{\lambda_{n+1}} x_n \|^2. \tag{3.4}$$

Letting $t \to 0$, noting the fact that $z_t \to z$ in norm, we have

$$\limsup_{n\to\infty} \langle u - z, T_n^{\lambda_{n+1}} x_n - z \rangle \leq 0. \tag{3.5}$$

Notice that $(d/dt)\| x + ty \|^2 = 2\langle y, x + ty \rangle$, we have

$$\| x + ty \|^2 = \| x \|^2 + 2\int_0^1 \langle y, x + ty \rangle dt, \ \forall x, y \in H.$$

It follows that

$$\begin{aligned}
\| x_{n+1} - z \|^2 &= \| (1 - \alpha_n)(T_n^{\lambda_{n+1}} x_n - z) + \alpha_n(u - z) \|^2 \\
&\leq (1 - \alpha_n)^2 (\| T_n^{\lambda_{n+1}} x_n - z \|)^2 \\
&\quad + 2\alpha_n \int_0^1 \langle u - z, (1 - \alpha_n)(T_n^{\lambda_{n+1}} x_n - z) + t\alpha_n(u - z) \rangle dt \\
&\leq (1 - \alpha_n)((1 - \lambda_{n+1}\tau)\| x_n - z \| + \lambda_{n+1}\mu \| F(z) \|)^2 \\
&\quad + 2\alpha_n \int_0^1 \langle u - z, (1 - \alpha_n)(T_n^{\lambda_{n+1}} x_n - z) + t\alpha_n(u - z) \rangle dt \\
&\leq (1 - \alpha_n)(1 - \lambda_{n+1}\tau)^2 \| x_n - z \|^2 \\
&\quad + 2\lambda_{n+1}\mu(1 - \lambda_{n+1}\tau)\| x_n - z \| \| F(z) \| + \lambda_{n+1}^2 \mu^2 \| F(z) \|^2 \\
&\quad + 2\alpha_n \int_0^1 \langle u - z, (1 - \alpha_n)(T_n^{\lambda_{n+1}} x_n - z) + t\alpha_n(u - z) \rangle dt \\
&\leq (1 - \alpha_n)\| x_n - z \|^2 + \alpha_n\beta_n + \gamma_n,
\end{aligned}$$

where

$$\beta_n := 2\int_0^1 \langle u - z, (1-\alpha_n)(T_n^{\lambda_{n+1}}x_n - z) + t\alpha_n(u - z)\rangle dt$$

and

$$\gamma_n := 2\lambda_{n+1}\mu(1-\lambda_{n+1}\tau)\| x_n - z \| \| F(z) \| + \lambda_{n+1}^2\mu^2 \| F(z) \|^2.$$

Since

$$(T_n^{\lambda_{n+1}}x_n - z) - [(1-\alpha_n)(T_n^{\lambda_{n+1}}x_n - z) + t\alpha_n(u - z)] \to 0 (n \to \infty)$$

uniformly in $t \in [0, 1]$, so we can get

$$\limsup_{n\to\infty} \beta_n = 2\limsup_{n\to\infty} \langle u - z, T_n x_n - z\rangle \le 0.$$

Then apply Lemma 2.4 to get $\lim_{n\to\infty} \| x_n - z \| = 0$, that is, $x_n \to z$. The proof is completed.

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