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ESTIMATION OF $P(X < Y)$ IN BIVARIATE WEIBULL MODEL WITH BIVARIATE TYPE I CENSORED DATA

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Abstract

In this paper, we consider two components system in which the lifetimes have a bivariate weibull distribution with bivariate type I censored data. We obtain estimators for reliability of stress-strength system based on the likelihood and the relative frequency, respectively. Also we present a numerical study.

1. Introduction

In many studies for the reliability of two components system, it is more realistic to assume some forms of dependence among the components of the system. As the forms of dependence among the components in two components system, bivariate weibull (BVW) distribution is a versatile family of life distributions in view of its physical interpretation.

Lu [3] and Lu and Bhattacharyya [4, 5] introduced some new constructions of BVW distributions. Cho et al. [1] constructed large

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sample tests for independence and symmetry. Also Cho et al. [2] obtained estimator for the reliability under univariate random censored data.

In this paper, we assume that the lifetimes of components in two components system follow a BVW distribution with bivariate type I censored data. And we obtain estimators of reliability of stress-strength system based on the likelihood and the relative frequency, respectively. Also, we present a numerical example by giving a data set which is generated by computer.

2. Preliminaries

Let the random variable (X, Y) be lifetimes of two components that follow a BVW distribution with parameter $(\zeta_1, \zeta_2, \zeta_3, \psi)$. Then the joint probability density function of (X, Y) is given as

$$f(x, y : \zeta_1, \zeta_2, \zeta_3, \psi)$$

$$= \begin{cases} \zeta_1(\zeta_2 + \zeta_3)\psi^2 x^{\psi-1} y^{\psi-1} \exp[-\zeta_1 x^{\psi} - (\zeta_2 + \zeta_3) y^{\psi}], & 0 < x < y < \infty, \\ \zeta_2(\zeta_1 + \zeta_3)\psi^2 x^{\psi-1} y^{\psi-1} \exp[-(\zeta_1 + \zeta_3) x^{\psi} - \zeta_2 y^{\psi}], & 0 < y < x < \infty, \\ \zeta_3 \psi x^{\psi-1} \exp[-\zeta x^{\psi}], & 0 < x = y < \infty, \end{cases} \quad (1)$$

where $\zeta_1, \zeta_2, \zeta_3, \psi > 0$ and $\zeta = \zeta_1 + \zeta_2 + \zeta_3$.

We note that random variables X and Y are independent if and only if $\zeta_3 = 0$ and that X and Y are symmetrically distributed if and only if $\zeta_1 = \zeta_2$.

On the other hand, the probability that Y is greater than X is given by

$$R = P[X < Y] = \zeta_1/\zeta. \quad (2)$$

For $j = 1, 2;$ $k = 1, 2, 3;$ $i = 1, 2, \ldots, n,$ we use following notations in this paper.

(I) (c_x, c_y): bivariate type I censoring times for ith observation in random variable (X, Y).

(II) $G_{1i} = I(X_i > c_x)$, $G_{2i} = I(Y_i > c_y)$, $G_{ji}^* = 1 - G_{ji}$.

(III) $R_{1i} = I(X_i < Y_i), R_{2i} = I(X_i > Y_i), R_{3i} = I(X_i = Y_i), R_{ki}^* = 1 - R_{ki}$.

Suppose that there are n two components systems under study and ith pair of the components have lifetimes (x_i, y_i) with bivariate type I censored data. Then ith observed lifetime (x_i, y_i) is given by

$$(x_i, y_i) = \begin{cases} (x_i, y_i), & x_i < c_x, \quad y_i < c_y \\ (c_x, y_i), & x_i > c_x, \quad y_i < c_x \\ (x_i, c_y), & x_i < c_x, \quad y_i > c_y \\ (c_x, c_y), & x_i > c_x, \quad y_i > c_y, \end{cases} \tag{3}$$

where the bivariate type I censoring times c_x and c_y are fixed values. We note that if $c_x = c_y$, then it is type I censoring model.

Now the likelihood function of the sample of size n is given by

$$L(\underline{\zeta}) = \prod_{i=1}^{n} \{[f(x_i, y_i)]^{G_{1i}^* G_{2i}^*} \cdot [\overline{F}(x_i, y_i)]^{G_{1i} G_{2i}}$$

$$\cdot [\overline{F}_{X \mid Y=y}(x_i) f_Y(y_i)]^{G_{1i} G_{2i}^*} \cdot [\overline{F}_{Y \mid X=x}(y_i) f_X(x_i)]^{G_{1i}^* G_{2i}}\}^{(R_{1i} + R_{2i} + R_{3i})}$$

$$= \zeta_1^{D_1} \zeta_2^{D_2} \zeta_3^{D_3} (\zeta_1 + \zeta_3)^{D_4} (\zeta_2 + \zeta_3)^{D_5} \rho^{D_6}$$

$$\cdot x_i^{(\rho-1) G_{1i}^*} y_i^{(\rho-1) G_{2i}^* (1 - R_{3i} G_{1i}^*)} \exp[-\zeta_1 x_s - \zeta_2 y_s - \zeta_3 (x_s + y_s - t_s)], \tag{4}$$

where $f_X(x)$ and $f_Y(y)$ are marginal distributions of X and Y, respectively.

$$D_1 = \sum_{i=1}^{n} (R_{1i} G_{1i}^* G_{2i}^* + R_{2i}^* G_{1i}^* G_{2i}), \quad D_2 = \sum_{i=1}^{n} (R_{2i} G_{1i}^* G_{2i}^* + R_{1i}^* G_{1i} G_{2i}^*),$$

$$D_3 = \sum_{i=1}^{n} R_{3i} G_{1i}^* G_{2i}^*, \quad D_4 = \sum_{i=1}^{n} R_{2i} G_{1i}^*, \quad D_5 = \sum_{i=1}^{n} R_{1i} G_{2i}^*,$$

$$D_6 = \sum_{i=1}^{n} \{R_{3i}^* G_{1i}^* G_{2i}^* + (1 - G_{1i} G_{2i})\}, \quad x_s = \sum_{i=1}^{n} x_i^{\rho},$$

$$y_s = \sum_{i=1}^{n} y_i^{\rho}, \quad t_s = \sum_{i=1}^{n} \min(x_i, y_i)^{\rho}.$$

Notice that D_i, $i = 1, \ldots, 6$ are random variables. After some calculations, the expected value of each D_i can be obtained as follows:

$$E(D_1) = \sum_{i=1}^{n} \{\zeta_1/\zeta - \zeta_1 \exp(-\zeta c_x^\rho)/\zeta + \exp(-\zeta c_y^\rho) - \exp(-(\zeta_2 + \zeta_3)c_y^\rho)$$

$$+ (1 - \exp(-\zeta_1 c_x^\rho)) \cdot \exp(-(\zeta_2 + \zeta_3)c_y^\rho) \cdot I(c_x < c_y)$$

$$+ \zeta_3(\exp(-\zeta c_y^\rho) - \exp(-\zeta c_x^\rho))/\zeta \cdot I(c_y < c_x)\},$$

$$E(D_2) = \sum_{i=1}^{n} \{\zeta_2/\zeta - \zeta_2 \exp(-\zeta c_y^\rho)/\zeta + \exp(-(\zeta_1 + \zeta_3)c_x^\rho - \zeta_2 c_y^\rho)$$

$$- \exp(-(\zeta_1 + \zeta_3)c_x^\rho)$$

$$+ (1 - \exp(-\zeta_2 c_y^\rho)) \cdot \exp(-(\zeta_1 + \zeta_3)c_x^\rho) \cdot I(t_{y_i} < t_{x_i})$$

$$+ \zeta_3(\exp(-\zeta c_x^\rho) - \exp(-\zeta c_y^\rho))/\zeta \cdot I(c_x < c_y)\},$$

$$E(D_3) = \sum_{i=1}^{n} \{(\zeta_3 - \zeta_3 \exp(-\zeta \min(c_x^\rho, c_y^\rho)))/\zeta\},$$

$$E(D_4) = \sum_{i=1}^{n} \{\zeta_2/\zeta - \zeta_2 \exp(-\zeta c_y^\rho)/\zeta + \exp(-(\zeta_1 + \zeta_3)c_x^\rho - \zeta_2 c_y^\rho)$$

$$- \exp(-(\zeta_1 + \zeta_3)c_x^\rho) + [\exp(-\zeta_2 c_y^\rho) \cdot [1 - \exp(-(\zeta_1 + \zeta_3)c_x^\rho)]$$

$$+ (\zeta_1 + \zeta_3) \cdot (\exp(-\zeta c_x^\rho) - 1)/\zeta] \cdot I(c_x > c_y)\},$$

$$E(D_5) = \sum_{i=1}^{n} \{\zeta_1/\zeta - \zeta_1 \exp(-\zeta c_x^\rho)/\zeta + \exp(-\zeta c_y^\rho) - \exp(-(\zeta_2 + \zeta_3)c_y^\rho)$$

$$+ \zeta_1 \exp(-\zeta c_x^\rho)/\zeta - \exp(-(\zeta_2 + \zeta_3)c_y^\rho - \zeta_1 c_x^\rho)\}.$$

In this paper, we focus only on fixed ρ. Then the likelihood equations are given by

$$\frac{\partial}{\partial \zeta_1} \log L(\underline{\zeta}) = \frac{D_1}{\zeta_1} + \frac{D_4}{\zeta_1 + \zeta_3} - x_s = 0, \tag{5}$$

$$\frac{\partial}{\partial \zeta_2} \log L(\underline{\zeta}) = \frac{D_2}{\zeta_2} + \frac{D_5}{\zeta_2 + \zeta_3} - y_s = 0, \tag{6}$$

$$\frac{\partial}{\partial \zeta_3} \log L(\underline{\zeta}) = \frac{D_3}{\zeta_3} + \frac{D_4}{\zeta_1 + \zeta_3} + \frac{D_5}{\zeta_2 + \zeta_3} - (x_s + y_s - t_s) = 0. \tag{7}$$

The likelihood equations (5)-(7) are not easy to solve. But we can obtain MLE's $(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3)$ by either Newton-Raphson procedure or Fisher's method of scoring.

Let $\underline{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$. Then Fisher information matrix is given by $I(\underline{\zeta}) = ((I_{ij}))$. Here

$$I_{ij} = E\left[-\frac{\partial^2}{\partial \zeta_i \partial \zeta_j} \log L(\underline{\zeta})\right]; \quad i, j = 1, 2, 3,$$

where

$$I_{11} = E(D_1)/\zeta_1^2 + E(D_4)/(\zeta_1 + \zeta_3)^2, \; I_{12} = 0, \; I_{13} = E(D_4)/(\zeta_1 + \zeta_3)^2,$$

$$I_{22} = E(D_2)/\zeta_2^2 + E(D_5)/(\zeta_2 + \zeta_3)^2, \; I_{23} = E(D_5)/(\zeta_2 + \zeta_3)^2,$$

$$I_{33} = E(D_3)/\zeta_3^2 + E(D_4)/(\zeta_1 + \zeta_3)^2 + E(D_5)/(\zeta_2 + \zeta_3)^2.$$

Thus $\sqrt{n}(\hat{\underline{\zeta}}^T - \underline{\zeta}^T)$ has asymptotic trivariate normal distribution with mean vector zero and covariance matrix $I^{-1}(\underline{\zeta})$; $i, j = 1, 2, 3$, where $\hat{\underline{\zeta}} = (\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3)$.

3. Estimations of $P(X < Y)$

In this section, we obtain MLE and the relative frequency estimator for R. Also we obtain approximate confidence intervals for R based on MLE and the relative frequency estimator, respectively.

We first derive the MLE of R and construct approximated confidence interval for R based on MLE. The MLE for R based on $(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3)$ is given by

$$\hat{R}_M = P[X < Y] = \hat{\zeta}_1/\hat{\zeta}, \quad \hat{\zeta} = \hat{\zeta}_1 + \hat{\zeta}_2 + \hat{\zeta}_3. \tag{8}$$

By consistency of MLE and delta method, we can see that the asymptotic distribution of $\hat{R}_M$ is a normal distribution with mean R and variance $\Delta \cdot [I^{-1}(\zeta_1, \zeta_2, \zeta_3)/n] \cdot \Delta'$, where $\Delta = \left(\frac{\zeta_2 + \zeta_3}{\zeta^2}, -\frac{\zeta_1}{\zeta^2}, -\frac{\zeta_1}{\zeta^2}\right)$.

Therefore, $100(1-\alpha)\%$ approximated confidence interval for R based on MLE is as follows:

$$\left(\hat{R}_M - z_{\alpha/2} \cdot \sqrt{\hat{\Delta} \cdot I(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3) \cdot \hat{\Delta}'/n},\ \hat{R}_M + z_{\alpha/2} \cdot \sqrt{\hat{\Delta} \cdot I(\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3) \cdot \hat{\Delta}'/n}\right). \tag{9}$$

We next obtain the relative frequency estimate and construct approximate confidence interval based on the relative frequency estimate. Let $L = \sum_{i=1}^{n} R_{1i}$. Then L is the number of observations with $X_i < Y_i$ in the sample. We can see that the distribution of L is binomial (n, R). Hence, the relative frequency estimate of R is given by $\hat{R}_{RF} = L/n$, which is asymptotic normal distribution with mean R and variance $R(1-R)/n$. Therefore, $100(1-\alpha)\%$ approximated confidence interval for R based on $\hat{R}_{RF}$ is as follows:

$$\left(\hat{R}_{RF} - z_{\alpha/2} \cdot \sqrt{\hat{R}_{RF} \cdot (1-\hat{R}_{RF})/n},\ \hat{R}_{RF} + z_{\alpha/2} \cdot \sqrt{\hat{R}_{RF} \cdot (1-\hat{R}_{RF})/n}\right). \tag{10}$$

4. Numerical Example

In this section, we present a numerical example by giving a data set which is generated by a computer. We generate a random sample of size 30 from BVW distribution with parameter $(\zeta_1 = 1.5, \zeta_2 = 1.4, \zeta_3 = 0.2)$. Also, we set bivariate type I censoring times $(c_x, c_y) = (1.2247, 1.1832)$. Then the true value of R is 0.4838. The data is given as follows, where $*$ indicates censored data.

(0.4083, 0.3199), (0.7354, 0.4891), (1.0722, 0.8106), (0.6453, 0.5875), (1.2166, 0.7953), (1.2247*, 0.9185), (0.6128, 0.4346), (0.0985, 0.0985), (0.5352, 0.3154), (0.7827, 1.1832*), (0.3241, 0.5148), (0.7827, 0.4182), (0.5487, 0.1339), (0.4381, 1.1832*), (0.6212, 0.6649), (0.1601, 0.7212), (0.5514, 0.2459), (0.9058, 0.3971), (0.7476, 1.1832*), (0.6608, 0.7463), (0.6822, 0.7936), (0.0300, 1.1832*), (0.9692, 1.1832*), (1.0366, 0.2821), (0.5303, 0.6050), (0.7089, 0.4062), (0.5517, 0.5517), (0.4734, 0.8971), (0.5134, 0.5944), (1.2247*, 0.6148).

MLE's of the parameters in BVW distribution are obtained by $\hat{\zeta}_1 = 1.5472$, $\hat{\zeta}_2 = 1.4298$, $\hat{\zeta}_3 = 0.2011$. And we obtain $L = 13$. Hence, the MLE and relative frequency estimator of R are $\hat{R}_M = 0.4868$ and $\hat{R}_{RF} = 0.4333$, respectively. Also 95% approximated confidence intervals for R based on $\hat{R}_M$ and $\hat{R}_{RF}$ are (0.3401, 0.6334) and (0.2560, 0.6106), respectively.

Hence, we note that estimates based on MLE's perform better than those based on natural estimates, more or less.

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