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SOME SIMPLE CRITERIA FOR EXISTENCE OF COMMON LINEAR COPOSITIVE LYAPUNOV FUNCTIONS OF 2-DIMENSIONAL DUAL POSITIVE SWITCHED SYSTEMS

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Abstract

Positive switched systems have attracted increasing attention from researchers. This paper addresses the existence problem of common linear copositive Lyapunov function for positive linear switched systems, both discrete- and continuous-time cases are considered. If the system matrices of two linear systems are mutually transposed, then these two systems are said to be mutually dual, or one is the duality of the other one. This paper proposes some simple necessary and sufficient conditions which can determine whether or not two-dimensional dual systems (continuous- or discrete-time) simultaneously possess corresponding common linear copositive Lyapunov function. A numerical example is provided to demonstrate the theoretical results.

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1. Introduction

There are many real variables in various systems which can only take nonnegative values, otherwise, they are meaningless, for example, population levels, absolute temperature, concentration and density of substances, and data packets flowing in a network. A dynamic system only involving such kind of states can be modeled as dynamic positive system. A system is said to be *positive* if its state and output are nonnegative whenever the initial condition and input are nonnegative [1, 2]. Because positive systems have broad applications in many areas [3-6] and important properties [7, 8], in recent years, researchers from different fields have paid much attention to analyze and synthesize positive systems [9-11].

Positive systems have many important and interesting properties, among which the most fundamental one is their stability. For example, for a linear time-invariant positive system, its asymptotic stability and diagonal quadratic stability are equivalent [5]; for positive systems with delays, the asymptotic stability is completely determined by system matrices, and has nothing to do with delays [12-16]. When dealing with the stability issues of positive systems with delays, especially in the case where the delays are time-varying, we frequently use the “comparison approach” [14] rather than the popular Lyapunov-Krasovskii or Razumikhin method [17]. During the past few years, many researchers have also paid their attention to positive switched systems - a combination of positive systems and switched systems [18-20]. Recently, Liu and Dang [14] have extended some interesting properties of positive systems to positive switched systems with delays.

Generally speaking, when we consider the stability property of a dynamic system, it is sometimes more convenient to establish the results by studying another system related to the original system. One of such approaches is using the equivalence of stability between two systems under duality. Two systems are said to be *mutually dual* if their system matrices are mutually transposed [21, 22]. It is well known that a linear time-invariant system and its dual system have the same stability properties, that is, the system $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$ is (asymptotically) stable if and only if the system $\dot{\boldsymbol{x}}(t) = A^T \boldsymbol{x}(t)$ is. Based on this simple property, Ait Rami and Tadeo [21] proposed a new stability condition for positive systems ([21, Theorem 2.4]). It is this simple result that motivates some new ideas, with which [12, 23] established some necessary and sufficient stability conditions for positive systems with time-varying delays. As a matter of fact, duality is widely used in a variety of fields such as circuits, systems, and control theory [24].

When stability of positive systems is considered, it is natural to adopt linear copositive Lyapunov function. The approach of linear copositive Lyapunov function relies on the following fact: a positive system is stable, if there is a function $V(\boldsymbol{x})$ which is positive definite and its derivative (for continuous-time systems) or difference (for discrete-time systems) taken along the system's trajectories is negative definite, when $\boldsymbol{x}$ is in the positive orthant. This approach captures the nature of positivity, and has been shown a powerful tool to treat the positive systems. For example, using common linear copositive Lyapunov functional (a generalization of linear copositive Lyapunov function), [22, 25] have successfully dealt with the analysis and synthesis problems of positive systems with delays. In the context of switched linear positive systems, the common linear copositive Lyapunov function has attracted more and more attention and a massive literature on it has appeared; see [18, 26-29] and the references therein for details.

On the ground above-mentioned, this paper considers the following problems: for switched linear positive systems, either continuous- or discrete-time, under what conditions a system and its dual system admit corresponding common linear copositive Lyapunov functions? Some necessary and sufficient criteria are provided to answer the question.

The rest of this paper is organized as follows: Section 2 gives two necessary and sufficient conditions to determine when a system and its duality admit corresponding common linear copositive Lyapunov functions, Section 3 verifies the main results of this paper by a numerical example, and Section 4 concludes this paper.

2. Main Results

Nomenclature

$A \succeq 0 \ (\succ 0)$	Matrix A with nonnegative (positive) elements
$A \preceq 0 \ (\prec 0)$	Matrix A with nonpositive (negative) elements
A^T	Transpose of matrix A
$\mathbb{R} \ (\mathbb{R}_{0,+})$	The set of all real (nonnegative) numbers
$\mathbb{R}^n$	The set of n-dimensional real vectors
$\mathbb{R}^{n \times m}$	The set of all real matrices of $n \times m$ -dimension
$\mathbb{N}$	$\{1, 2, 3, \dots\}$
$\mathbb{N}_0$	$\{0\} \cup \mathbb{N}$
$\underline{p}$	$\{1, 2, \dots, p\}$, where $p \in \mathbb{N}$
$\mathbb{M}$	The set of Metzler matrices of appropriate dimension, i.e., the set of $n \times n$ real matrices with nonnegative off diagonal entries

Throughout this paper, the dimensions of matrices and vectors will not be explicitly mentioned if clear from context. In addition, we use the convention that $\frac{a}{0} = +\infty$ if $a > 0$ and that $\frac{a}{0} = -\infty$ if $a < 0$.

First, let us consider the following system:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t), \quad t \geq 0. \tag{2.1}$$

Definition 1 [5]**.** System (2.1) is said to be *positive* if $\boldsymbol{x}(0) \succeq 0$ always implies that $\boldsymbol{x}(t) \succeq 0$ holds for any $t > 0$.

Lemma 1 [5]**.** *System* (2.1) *is positive if and only if the system matrix A is a Metzler matrix.*

In this paper, we are concerned only with the 2-dimensional systems, that is, A is of the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

The following lemma is crucial to establish our main results.

Lemma 2. *Suppose that system* (2.1) *is positive. Then there exists a positive vector* $\boldsymbol{\lambda} \succ 0$ *such that* $A\boldsymbol{\lambda} \prec 0$ *holds if and only if the following condition holds*:

$$\frac{a_{12}}{a_{11}} > \frac{a_{22}}{a_{21}}. \tag{2.2}$$

Proof. Clearly, there exists a vector $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]^T \succ 0$ satisfying $A\boldsymbol{\lambda} \prec 0$ is equivalent to the fact that

$$\begin{aligned} & A\boldsymbol{\lambda} \\ & = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \\ & = \begin{bmatrix} a_{11}\lambda_1 + a_{12}\lambda_2 \\ a_{21}\lambda_1 + a_{22}\lambda_2 \end{bmatrix} \\ & \prec 0 \end{aligned}$$

in other words,

$$a_{11}\lambda_1 + a_{12}\lambda_2 < 0, \quad a_{21}\lambda_1 + a_{22}\lambda_2 < 0. \tag{2.3}$$

Since system (2.1) is positive, by Lemma 1, A is a Metzler matrix, (2.3) clearly implies that $a_{11} < 0$ and $a_{22} < 0$ necessarily hold by the fact that $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]^T \succ 0.$ As a result, (2.3) amounts to saying that

$$-\frac{a_{12}}{a_{11}}\lambda_2 < \lambda_1 < -\frac{a_{22}}{a_{21}}\lambda_2$$

which is just the same as the following inequality:

$$-\frac{a_{12}}{a_{11}} < \frac{\lambda_1}{\lambda_2} < -\frac{a_{22}}{a_{21}}. \tag{2.4}$$

Hence, existing a positive vector $\boldsymbol{\lambda} \succ 0$ satisfying $A\boldsymbol{\lambda} \prec 0$ means that $-\frac{a_{12}}{a_{11}} < -\frac{a_{22}}{a_{21}}$, or equivalently, (2.2) holds.

Conversely, if (2.2) holds, then we can always find out, not unique, a vector $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]^T \succ 0$ satisfying (2.4), and (2.4) indicates that $A\boldsymbol{\lambda} \prec 0$.

□

Now, consider the following system:

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k), \quad \forall k \in \mathbb{N}_0. \tag{2.5}$$

Definition 2 [5]**.** System (2.5) is said to be *positive* if $\boldsymbol{x}(0) \succeq 0$ always implies that $\boldsymbol{x}(k) \succeq 0$ holds for any $k \in \mathbb{N}$.

Lemma 3 [5]**.** (2.5) *is positive if and only if* $A \succeq 0$.

We have the following lemma.

Lemma 4. *Consider positive system* (2.5). *There exists a positive vector* $\boldsymbol{\lambda} \succ 0$ *such that* $(A - I)\boldsymbol{\lambda} \prec 0$ *holds if and only if the following condition holds*:

$$\frac{a_{12}}{a_{11} - 1} > \frac{a_{22} - 1}{a_{21}}.$$

Proof. Since system (2.5) is positive, $A - I$ is necessarily a Metzler matrix. Applying Lemma 2, one knows that the lemma holds. □

Consider the following continuous-time switched linear system:

$$\dot{\boldsymbol{x}}(t) = A_{\rho(t)}\boldsymbol{x}(t), \quad t \geq 0, \tag{2.6}$$

where $\rho : \mathbb{R}_{0,+} \to \underline{m}$ is a switching signal. In what follows, it is always assumed that the switching signal $\rho(t) \in \mathcal{S}_C$, where $\mathcal{S}_C$ is the set of all switching signals with the property that there are only finitely many switches on any finite interval.

System (2.6) is positive if $\boldsymbol{x}(0) \succeq 0$ always implies that $\boldsymbol{x}(t) \succeq 0$ holds for any $t > 0$ and any switching signal $\rho(t)$. By Lemma 1, system (2.6) is positive if and only if all system matrices A_l's are Metzler matrices.

The dual system of (2.6) is given by

$$\dot{\boldsymbol{x}}(t) = A^T_{\rho(t)(t)}\boldsymbol{x}(t), \quad t \geq 0, \quad \rho(t) \in \mathcal{S}_C. \tag{2.7}$$

Denote $A_l = [a_{ij}^{(l)}]$. Now, we will consider the following question: under what conditions, there exist two positive vectors $\boldsymbol{\lambda} \succ 0$ and $\boldsymbol{\eta} \succ 0$ such that

$$A_l\boldsymbol{\lambda} \prec 0, \quad \forall l \in \underline{m} \tag{2.8}$$

and

$$A_l^T\boldsymbol{\eta} \prec 0, \quad \forall l \in \underline{m} \tag{2.9}$$

hold simultaneously?

Theorem 1. *Consider the* 2*-dimensional positive systems* (2.6) *and* (2.7). *There exist two positive vectors* $\boldsymbol{\lambda} \succ 0$ *and* $\boldsymbol{\eta} \succ 0$ *such that* (2.8) *and* (2.9) *hold if and only if*

$$\min_{l\in\underline{m}}\left\{\frac{a_{12}^{(l)}}{a_{11}^{(l)}}\right\} > \max_{l\in\underline{m}}\left\{\frac{a_{22}^{(l)}}{a_{21}^{(l)}}\right\} \tag{2.10}$$

and

$$\min_{l\in\underline{m}}\left\{\frac{a_{21}^{(l)}}{a_{11}^{(l)}}\right\} > \max_{l\in\underline{m}}\left\{\frac{a_{22}^{(l)}}{a_{12}^{(l)}}\right\} \tag{2.11}$$

hold simultaneously.

Proof. The proof is divided into two parts: Necessity and Sufficiency.

Necessity. Suppose that there exists a positive vector $\boldsymbol{\lambda} \succ 0$ such that $A_l \boldsymbol{\lambda} \prec 0$, $\forall l \in \underline{m}$ hold. By Lemma 2, for each fixed $l \in \underline{m}$, $A_l \boldsymbol{\lambda} \prec 0$ is equivalent to

$$\frac{a_{12}^{(l)}}{a_{11}^{(l)}} > \frac{a_{22}^{(l)}}{a_{21}^{(l)}}.$$

Since the vector $\boldsymbol{\lambda}$ is common to all $l \in \underline{m}$, inequality (2.10) necessarily holds.

Similarly, there exists a positive vector $\boldsymbol{\eta} \succ 0$ satisfying $A_l \boldsymbol{\eta} \prec 0$ $(\forall l \in \underline{m})$ implies that inequality (2.11) holds.

Sufficiency. Suppose that (2.10) holds. By Lemma 2, for any $l \in \underline{m}$, there exists a positive vector $\boldsymbol{\lambda}$ such that $A_l \boldsymbol{\lambda} \prec 0$ holds. As a result, there exists a common vector $\boldsymbol{\lambda}$ such that $A_l \boldsymbol{\lambda} \prec 0$ holds for all $l \in \underline{m}$.

Analogously, condition (2.11) implies that there exists a positive vector $\boldsymbol{\eta}$ such that $A_l^T \boldsymbol{\eta} \prec 0$ holds for all $l \in \underline{m}$.

The proof is completed. □

Now, consider the following discrete-time switched linear system:

$$\boldsymbol{x}(k+1) = A_{\sigma(k)} \boldsymbol{x}(k), \quad k \in \mathbb{N}_0, \tag{2.12}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state variable, the map $\sigma : \mathbb{N}_0 \to \underline{m}$ is an arbitrary switching signal with m being the number of subsystems, $A_l \in \mathbb{R}^{n \times n}$, $l \in \underline{m}$, are system matrices. A subsystem, say the lth one, is activated at instant k if and only if $\sigma(k) = l$. For convenience, denote the set of arbitrary switching signals for system (2.12) by $\mathcal{S}_D$.

For system (2.12), if $\boldsymbol{x}(0) \succeq 0$ always implies that $\boldsymbol{x}(k) \succeq 0$ holds for any $k \in \mathbb{N}$ and any switching signal $\sigma(k)$, then it is positive. What is more, according to Lemma 3, it is not difficult to see that (2.12) is positive if and only if all the system matrices, A_l's, are nonnegative, that is, $A_l \succeq 0$, $\forall l \in \underline{m}$.

The dual system of (2.12) is

$$\boldsymbol{x}(k+1) = A_{\sigma(k)}^T \boldsymbol{x}(k), \quad k \in \mathbb{N}_0. \tag{2.13}$$

The following theorem holds.

Theorem 2. *Consider the* 2*-dimensional positive systems* (2.12) *and* (2.13). *There exist two positive vectors* $\boldsymbol{\lambda} \succ 0$ *and* $\boldsymbol{\eta} \succ 0$ *such that*

$$A_l \boldsymbol{\lambda} \prec \boldsymbol{\lambda}, \quad A_l^T \boldsymbol{\eta} \prec \boldsymbol{\eta}, \quad \forall l \in \underline{m} \tag{2.14}$$

hold if and only if

$$\min_{l \in \underline{m}} \left\{ \frac{a_{12}^{(l)}}{a_{11}^{(l)} - 1} \right\} > \max_{l \in \underline{m}} \left\{ \frac{a_{22}^{(l)} - 1}{a_{21}^{(l)}} \right\}$$

and

$$\min_{l \in \underline{m}} \left\{ \frac{a_{21}^{(l)}}{a_{11}^{(l)} - 1} \right\} > \max_{l \in \underline{m}} \left\{ \frac{a_{22}^{(l)} - 1}{a_{12}^{(l)}} \right\}$$

hold simultaneously.

Proof. Condition (2.14) is equivalent to the following one:

$$(A_l - I)\boldsymbol{\lambda} \prec 0, \quad (A_l^T - I)\boldsymbol{\eta} \prec 0, \quad \forall l \in \underline{m}.$$

Then following a similar line in the proof of Theorem 1, we can, analogously, complete the remaining proof without any difficulty. □

3. Numerical Example

This section provides an example to test the validity of the main results of this paper.

Example 1. Consider the following system:

$$\dot{\boldsymbol{x}}(t) = A_{\rho(t)}\boldsymbol{x}(t), \quad t \geq 0, \tag{3.1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^2$ and $\rho : \mathbb{N}_0 \mapsto \{1, 2, 3\}$, and the system matrices are as follows:

$$A_1 = \begin{bmatrix} -1.5 & 0.1 \\ 0.4 & -0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.8 & a \\ 0.2 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -2 & 1.8 \\ 0.2 & -1 \end{bmatrix}$$

with a in A_2 a positive parameter.

First, take $a = 0.7$. Simple computation shows that

$$\min_{l \in \underline{3}} \left\{ \frac{a_{12}^{(l)}}{a_{11}^{(l)}} \right\} = -0.9, \quad \max_{l \in \underline{m}} \left\{ \frac{a_{22}^{(l)}}{a_{21}^{(l)}} \right\} = -1$$

and hence (2.10) holds. We can, similarly, verify that (2.11) also holds. By Theorem 1, there exist two positive vectors $\boldsymbol{\lambda} \succ 0$ and $\boldsymbol{\eta} \succ 0$ such that (2.8) and (2.9) hold. Indeed, using linear programming tool in MATLAB (the function linprog from the Optimization Toolbox is used here), we find out two vectors:

$$\boldsymbol{\lambda} = [113.7996,\ 114.4751]^T, \quad \boldsymbol{\eta} = [70.0732,\ 155.5659]^T$$

with the following property:

$$A_1\boldsymbol{\lambda} = [-159.2519,\ -0.2702]^T \prec 0, \quad A_2\boldsymbol{\lambda} = [-10.9071,\ -91.7152]^T \prec 0,$$

$$A_3\boldsymbol{\lambda} = [-21.5441,\ -91.7152]^T \prec 0, \quad A_1^T\boldsymbol{\eta} = [-42.8834,\ -55.2190]^T \prec 0,$$

$$A_2^T\boldsymbol{\eta} = [-24.9454,\ -106.5147]^T \prec 0, \quad A_3^T\boldsymbol{\eta} = [-109.0332,\ -29.4342]^T \prec 0.$$

Now, take $a = 0.9$. It is easy to verify that

$$\min_{l \in \underline{m}} \left\{ \frac{a_{21}^{(l)}}{a_{11}^{(l)}} \right\} > \max_{l \in \underline{m}} \left\{ \frac{a_{22}^{(l)}}{a_{12}^{(l)}} \right\}$$

does hold, but $\min_{l \in \underline{3}} \left\{ \frac{a_{12}^{(l)}}{a_{11}^{(l)}} \right\} > \max_{l \in \underline{3}} \left\{ \frac{a_{22}^{(l)}}{a_{21}^{(l)}} \right\}$ does not hold. According to Theorem 1, it is impossible to exist two positive vectors $\boldsymbol{\lambda} \succ 0$ and $\boldsymbol{\eta} \succ 0$ satisfying (2.8) and (2.9) simultaneously.

4. Conclusions

The existence problem of common linear copositive Lyapunov functions for two-dimensional dual switched linear positive systems is studied. Some simple necessary and sufficient conditions are established to determine whether or not such functions exist. A numerical example is carried out to show that the obtained results are effective.

References

[1] A. Berman, M. Neumann and R. J. Stern, Nonnegative Matrices in Dynamic Systems, Wiley, New York, 1989.

[2] T. Kaczorek, Positive 1D and 2D Systems, Springer, London, 2002.

[3] L. Benvenuti and L. Farina, A tutorial on the positive realization problem, IEEE Trans. on Automatic Control 49(5) (2004), 651-664.

[4] T. Hayakawa, Compartmental modeling and adaptive control of stochastic nonnegative systems, Positive Systems, Lecture Notes in Control and Information Sciences 341, 2006, pp. 351-358.

[5] L. Farina and S. Rinaldi, Positive Linear Systems: Theory and Applications, John Wiley & Sons, Inc., New York, 2000.

[6] L. Benvenuti, A. D. Santis and L. Farina, eds., Positive Systems, Lecture Notes in Control and Information Sciences, 294, Springer-Verlag, Berlin, 2003.

[7] M. Margaliot and M. S. Branicky, Nice reachability for planar bilinear control systems with applications to planar linear switched systems, IEEE Trans. on Automatic Control 54(6) (2009), 1430-1435.

[8] H. Gao, J. Lam, C. Wang and S. Xu, Control for stability and positivity: equivalent conditions and computation, IEEE Trans. on Circuits and Systems II 52(9) (2005), 540-544.

[9] D. Hinrichsen, N. K. Son and P. H. A. Ngoc, Stability radii of higher order positive difference systems, Systems and Control Letters 49(5) (2003), 377-388.

[10] T. J. Laffey and H. Šmigoc, Nonnegative realization of spectra having negative real parts, Linear Algebra and its Applications 416(1) (2006), 148-159.

[11] L. Fainshil, M. Margaliot and P. Chigansky, On the stability of positive linear switched systems under arbitrary switching laws, IEEE Trans. on Automatic Control 54(4) (2009), 897-899.

[12] X. Liu, W. Yu and L. Wang, Stability analysis of positive systems with bounded time-varying delays, IEEE Trans. on Circuits and Systems II 56(7) (2009), 600-604.

[13] X. Liu and S. Zhong, Stability analysis of positive systems with unbounded delays, 2010 Chinese Control and Decision Conference, Xuzhou, China, 2010, pp. 193-198.

[14] X. Liu and C. Dang, Stability analysis of positive switched linear systems with delays, IEEE Trans. on Automatic Control 56(7) (2011), 1684-1690.

[15] X. Liu, W. Yu and L. Wang, Necessary and sufficient asymptotic stability criterion for 2-D positive systems with time-varying state delays described by Roesser model, IET Control Theory and Applications 5(5) (2011), 663-668.

[16] L. Wu, J. Lam, Z. Shu and B. Du, On stability and stabilizability of positive delay systems, Asian J. of Control 11(2) (2009), 226-234.

[17] X. Jiao and T. Shen, Adaptive feedback control of nonlinear time-delay systems: the LaSalle-Razumikhin-based approach, IEEE Trans. on Automatic Control 50(11) (2005), 1909-1913.

[18] O. Mason and R. Shorten, On linear copositive Lyapunov functions and the stability of switched positive linear systems, IEEE Trans. on Automatic Control 52(7) (2007), 1346-1349.

[19] X. Liu, Stability analysis of switched positive systems: a switched linear copositive Lyapunov function method, IEEE Trans. on Circuits and Systems II 56(5) (2009), 414-418.

[20] A. Zappavigna, P. Colaneri, J. C. Geromel and R. Middleton, Stabilization of continuous-time switched linear positive systems, Proceedings of the 2010 American Control Conference, Baltimore, USA, 2010, pp. 3275-3280.

[21] M. Ait Rami and F. Tadeo, Controller synthesis for positive linear systems with bounded controls, IEEE Trans. on Circuits and Systems II 54(2) (2007), 151-155.

[22] M. Ait Rami, U. Helmke and F. Tadeo, Positive observation problem for time-delays linear positive systems, Proceedings of the 15th Mediterranean Conference on Control and Automation, Athens, Greece, 2007, pp. 1-6.

[23] X. Liu, W. Yu and L. Wang, Stability analysis for continuous-time positive systems with time-varying delays, IEEE Trans. on Automatic Control 55(4) (2010), 1024-1028.

[24] V. Moreno-Font, A. El Aroudi, J. Calvente, R. Giral and L. Benadero, Dynamics and stability issues of a single-inductor dual-switching DC-DC converter, IEEE Trans. on Circuits and Systems I 57(2) (2010), 415-426.

[25] W. M. Haddad and V. Chellaboina, Stability theory for nonnegative and compartmental dynamical systems with time delay, Systems and Control Letters 51(5) (2004), 355-361.

[26] L. Gurvits, R. Shorten and O. Mason, On the stability of switched positive linear systems, IEEE Trans. on Automatic Control 52(6) (2007), 1099-1103.

[27] O. Mason and R. Shorten, The geometry of convex cones associated with the Lyapunov inequality and the common Lyapunov function problem, Electronic J. of Linear Algebra 12 (2005), 42-63.

[28] O. Mason and R. Shorten, Some results on the stability of positive switched linear systems, 43rd IEEE Conference on Decision and Control, 2004, pp. 4601-4606.

[29] O. Mason and R. Shorten, Quadratic and copositive Lyapunov functions and the stability of positive switched linear systems, Proceedings of the 2007 American Control Conference, 2007, pp. 657-662.