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A CERTAIN QUADRUPLE FAMILY OF ELLIPTIC CURVES ASSOCIATED WITH $(2, 2)$-EXTENSION

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Abstract

We construct a certain family of products of twists of an elliptic curve such that each rank of twisted elliptic curves is positive.

1. Introduction

In the previous paper [4], we investigate a generalization of the following problem:

Problem 1. For a field K finitely generated over a prime field, construct polynomials $f(t), g(t) \in K[t]$ of degree three such that each of the three twists of $E : y^2 = f(x)$;

$$f(t)y^2 = f(x), \quad g(t)y^2 = f(x), \quad f(t)g(t)y^2 = f(x),$$

has positive Mordell-Weil rank over $K(t)$.

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This and related problems are investigated in [3] by focusing on the geometry of elliptic surfaces, and in [2] by relating them to a certain threefold. In [4] however, we consider a generalized problem where y^2 is replaced by y^p with p a prime. There we recognize that the theory of twists, developed in [1] plays a central role, since it provides us with a unified view point to study the original problem as well as the generalized one.

Based on the theory of twists, as another way of generalization of Problem 1, we construct in this paper polynomials $f(t), g(t), h(t) \in K[t]$ of degree three and $c \in K$ such that each of the four twists of $E : y^2 = f(x)$;

$$f(t)y^2 = f(x),\ g(t)y^2 = f(x),\ h(t)y^2 = f(x),\ cg(t)h(t)y^2 = f(x), \quad (1)$$

has positive Mordell-Weil rank over $K(t)$. Moreover we show that one can eliminate c from the last equation. Actually our field K has four independent parameters. Hence by specialization we can show that for any elliptic curve $E : y^2 = f(x)$ over $\mathbf{Q}$ with $E_2(\mathbf{Q}) \cong (\mathbf{Z}/2\mathbf{Z})^2$, there exists a pair $g(t), h(t) \in \mathbf{Q}[t]$ such that all of twists in (1) have positive Mordell-Weil rank over $\mathbf{Q}(t)$.

2. Main Theorem and its Proof

Let k be a field of characteristic $\neq 2, 3$ and let α, β, m_1, m_2 be independent variables. Let $K = k(\alpha, \beta, m_1, m_2)$ and $f(t) = t(t-\alpha)(t-\beta)$. Let E be an elliptic curve over K defined by the equation $y^2 = f(x)$. Put $m_3 = (\beta m_1 - \beta m_2 + \alpha m_2)/\alpha \in K$, and let

$$g(t) = \frac{1}{\alpha}(m_1 - m_2)(t - m_1)(t - m_2)(t - m_3),$$

$$h(t) = -\frac{\alpha(2\alpha - \beta)}{(\alpha - \beta)^4 (m_1 - m_2)^7} \times (\alpha(2\alpha - \beta)t + (-\alpha^2 - \alpha\beta + \beta^2)m_1 - (\alpha - \beta)^2 m_2)$$

$$\times (\alpha(2\alpha - \beta)t + (\alpha - \beta)^2 m_1 + (-3\alpha^2 + 3\alpha\beta - \beta^2) m_2)$$
$$\times ((2\alpha - \beta)t - (\alpha - \beta) m_1 - \alpha m_2).$$

Then we have the following theorem:

Theorem 1. *Let* C_f, C_g, C_h, C_{gh} *be elliptic* (*or hyperelliptic*) *curves defined by the following equations*:

$$C_f : u^2 = f(t), \tag{2}$$

$$C_g : u^2 = g(t), \tag{3}$$

$$C_h : u^2 = h(t), \tag{4}$$

$$C_{gh} : u^2 = -\frac{\alpha(\alpha - \beta)}{2\alpha - \beta} g(t) h(t). \tag{5}$$

Let E_f (*resp.* E_g, E_h, E_{gh}) *be the twist of E by the quadratic extension* $K(C_f)/K(\mathbf{P}^1)$ (*resp.* $K(C_g)/K(\mathbf{P}^1), K(C_h)/K(\mathbf{P}^1), K(C_{gh})/K(\mathbf{P}^1)$). *Then each rank of* $E_f(K(t)), E_g(K(t)), E_h(K(t)), E_{gh}(K(t))$ *is* ≥ 1.

For the proof, we recall the following result in [1].

Theorem 2 ([1])**.** *Let C be a hyperelliptic curve over k defined by the equation* $y^2 = P(x)$ *and let E be an elliptic curve over k. Let* E_P *denote the twist of E defined by the quadratic extension* $k(C)/k(\mathbf{P}^1)$ *so that* E_P *is an elliptic curve over* $k(\mathbf{P}^1) = k(t)$. *Then we have an isomorphism of abelian groups*,

$$E_P(k(t)) \cong Hom_k(J(C), E) \oplus E(k)_2, \tag{6}$$

where $E(k)_2$ *denotes the group of k-rational* 2-*division points on E*.

Proof of Theorem 1. The defining equations of E_f, E_g, E_h, E_{gh} are given by

$$E_f : f(t)y^2 = f(x), \tag{7}$$

$$E_g : g(t)y^2 = f(x), \tag{8}$$

$$E_h : h(t)y^2 = f(x), \tag{9}$$

$$E_{gh} : -\frac{\alpha(\alpha-\beta)}{2\alpha-\beta} g(t)h(t)y^2 = f(x). \tag{10}$$

We will construct good rational points on these four curves separately.

(i) $E_f(K(t))$: The isomorphism of (6) shows that the $K(t)$-rational point $(t, 1)$ on E_f is a free generator of $E_f(K(t))$.

(ii) $E_{gh}(K(t))$: Put

$$q(t) = \frac{(\alpha t - \beta m_1 - (\alpha-\beta)m_2)((2\alpha-\beta)t - (\alpha-\beta)m_1 - \alpha m_2)}{(\alpha-\beta)(m_1-m_2)^2}.$$

Then we can simplify the factors (multiplied by $(\alpha-\beta)(m_1-m_2)^2$) of $f(q(t)) = q(t)(q(t)-\alpha)(q(t)-\beta)$ as follows:

$$\begin{aligned}
&(\alpha-\beta)(m_1-m_2)^2 q(t)\\
&= (\alpha t - \beta m_1 - (\alpha-\beta)m_2)((2\alpha-\beta)t - (\alpha-\beta)m_1 - \alpha m_2)\\
&= (\alpha t - \alpha m_3)((2\alpha-\beta)t - (\alpha-\beta)m_1 - \alpha m_2)\\
&= \alpha(t - m_3)((2\alpha-\beta)t - (\alpha-\beta)m_1 - \alpha m_2),
\end{aligned}$$

$$\begin{aligned}
&(\alpha-\beta)(m_1-m_2)^2 (q(t)-\alpha)\\
&= (\alpha-\beta)(m_1-m_2)^2\\
&\quad\times\left(\frac{(\alpha t - \beta m_1 - (\alpha-\beta)m_2)((2\alpha-\beta)t - (\alpha-\beta)m_1 - \alpha m_2)}{(\alpha-\beta)(m_1-m_2)^2} - \alpha\right)
\end{aligned}$$

$$= (\alpha t - \beta m_1 - (\alpha - \beta) m_2)((2\alpha - \beta)t - (\alpha - \beta) m_1 - \alpha m_2)$$

$$- \alpha(\alpha - \beta)(m_1 - m_2)^2$$

$$= \alpha(2\alpha - \beta)t^2 + ((-\alpha^2 - \alpha\beta + \beta^2) m_1 - (3\alpha^2 - 3\alpha\beta + \beta^2) m_2)t$$

$$- (\alpha - \beta)^2 m_1^2 + (3\alpha^2 - 3\alpha\beta + \beta^2) m_1 m_2$$

$$= (t - m_1)(\alpha(2\alpha - \beta)t + (\alpha - \beta)^2 m_1 + (-3\alpha^2 + 3\alpha\beta - \beta^2) m_2),$$

$$(\alpha - \beta)(m_1 - m_2)^2 (q(t) - \beta)$$

$$= (\alpha - \beta)(m_1 - m_2)^2$$

$$\times \left(\frac{(\alpha t - \beta m_1 - (\alpha - \beta) m_2)((2\alpha - \beta)t - (\alpha - \beta) m_1 - \alpha m_2)}{(\alpha - \beta)(m_1 - m_2)^2} - \beta \right)$$

$$= (\alpha t - \beta m_1 - (\alpha - \beta) m_2)((2\alpha - \beta)t - (\alpha - \beta) m_1 - \alpha m_2)$$

$$- \beta(\alpha - \beta)(m_1 - m_2)^2)$$

$$= \alpha(2\alpha - \beta)t^2 + ((-\alpha^2 - \alpha\beta + \beta^2) m_1 + (-3\alpha^2 + 3\alpha\beta - \beta^2) m_2)t$$

$$+ (\alpha^2 + \alpha\beta - \beta^2) m_1 m_2 + (\alpha - \beta)^2 m_2^2$$

$$= (t - m_2)(\alpha(2\alpha - \beta)t + (-\alpha^2 - \alpha\beta + \beta^2) m_1 - (\alpha - \beta)^2 m_2).$$

Consequently,

$$f(q(t)) = q(t)(q(t) - \alpha)(q(t) - \beta)$$

$$= \frac{\alpha}{(\alpha - \beta)^3 (m_1 - m_2)^6} (t - m_1)(t - m_2)(t - m_3)$$

$$\times ((2\alpha - \beta)t - (\alpha - \beta) m_1 - \alpha m_2)$$

$$\times (\alpha(2\alpha - \beta)t + (\alpha - \beta)^2 m_1 + (-3\alpha^2 + 3\alpha\beta - \beta^2) m_2)$$

$$\times\,(\alpha(2\alpha-\beta)t+(-\alpha^2-\alpha\beta+\beta^2)m_1-(\alpha-\beta)^2m_2)$$

$$=\frac{\alpha}{(m_1-m_2)^6(\alpha-\beta)^3}$$

$$\times\left(\frac{\alpha}{m_1-m_2}\,g(t)\right)\times\left(-\frac{(\alpha-\beta)^4(m_1-m_2)^7}{\alpha(2\alpha-\beta)}\,h(t)\right)$$

$$=-\frac{\alpha(\alpha-\beta)}{2\alpha-\beta}\,g(t)h(t).$$

Therefore $(q(t), 1)$ is a $K(t)$-rational point on E_{gh} and this point is free in $E_{gh}(K(t))$ by (6).

(iii) $E_g(K(t))$: One can check that C_g is isomorphic to E over K.

In fact, put $c_1=(m_1-m_2)/\alpha,\ t=c_1x+m_2,\ u=-c_1^2y$ in (3), then

$$(-c_1^2y)^2=\frac{1}{\alpha}(m_1-m_2)(c_1x+m_2-m_1)(c_1x)(c_1x+m_2-m_3)$$

$$=c_1^2x(c_1x-c_1\alpha)\left(c_1x+\frac{-\beta m_1+\beta m_2}{\alpha}\right)$$

$$=c_1^4x(x-\alpha)(x-\beta),$$

hence

$$y^2=x(x-\alpha)(x-\beta)=f(x).$$

Consequently, one has

$$rank(Hom_K(C_g,\ E))\geq 1,$$

and by (6), $rank(E_g(K(t)))\geq 1,$ and that the K-rational point $((t-m_2)/c_1,$ $-1/c_1^2)$ is free in $E_g(K(t))$.

(iv) $E_h(K(t))$: One can also check that C_h is isomorphic to E over K. In fact, put

$$t = -c_1 x + \frac{1}{\alpha(2\alpha - \beta)}((\alpha^2 + \alpha\beta - \beta^2)m_1 + (\alpha - \beta)^2 m_2),$$

$$u = \left(\frac{2\alpha - \beta}{(\alpha - \beta)(m_1 - m_2)}\right)^2 y$$

in (4). Then the equation becomes

$$\left(\frac{2\alpha - \beta}{(\alpha - \beta)(m_1 - m_2)}\right)^4 y^2$$

$$= -\frac{\alpha(2\alpha - \beta)}{(\alpha - \beta)^4 (m_1 - m_2)^7}$$

$$\times \Bigg(\alpha(2\alpha - \beta)\left(-c_1 x + \frac{1}{\alpha(2\alpha - \beta)}((\alpha^2 + \alpha\beta - \beta^2)m_1 + (\alpha - \beta)^2 m_2)\right)$$

$$+ (-\alpha^2 - \alpha\beta + \beta^2)m_1 - (\alpha - \beta)^2 m_2\Bigg)$$

$$\times \Bigg(\alpha(2\alpha - \beta)\left(-c_1 x + \frac{1}{\alpha(2\alpha - \beta)}((\alpha^2 + \alpha\beta - \beta^2)m_1 + (\alpha - \beta)^2 m_2)\right)$$

$$+ (\alpha - \beta)^2 m_1 + (-3\alpha^2 + 3\alpha\beta - \beta^2)m_2\Bigg)$$

$$\times \Bigg((2\alpha - \beta)\left(-c_1 x + \frac{1}{\alpha(2\alpha - \beta)}((\alpha^2 + \alpha\beta - \beta^2)m_1 + (\alpha - \beta)^2 m_2)\right)$$

$$- (\alpha - \beta)m_1 - \alpha m_2\Bigg). \tag{11}$$

Multiplying both hand sides of (11) by $(\alpha - \beta)^4 (m_1 - m_2)^4 / (2\alpha - \beta)$, we have

$$(2\alpha - \beta)^3 y^2$$

$$= -\frac{\alpha}{(m_1 - m_2)^3} \times (2\alpha - \beta)(m_2 - m_1)x$$

$$\times((2\alpha-\beta)(m_2-m_1)x+(2\alpha^2-\alpha\beta)m_1+(-2\alpha^2+\alpha\beta)m_2)$$

$$\times\frac{1}{\alpha}((2\alpha-\beta)(m_2-m_1)x+(2\alpha\beta-\beta^2)m_1+(-2\alpha\beta+\beta^2)m_2)$$

$$=\frac{2\alpha-\beta}{(m_1-m_2)^2}x$$

$$\times((2\alpha-\beta)(m_2-m_1)x+\alpha(2\alpha-\beta)m_1+\alpha(-2\alpha+\beta)m_2)$$

$$\times((2\alpha-\beta)(m_2-m_1)x+\beta(2\alpha-\beta)m_1+\beta(-2\alpha+\beta)m_2)$$

$$=\frac{(2\alpha-\beta)^3}{(m_1-m_2)^2}x((m_2-m_1)x+\alpha(m_1-m_2))$$

$$\times((m_2-m_1)x+\beta(m_1-m_2))$$

$$=(2\alpha-\beta)^3x(x-\alpha)(x-\beta),$$

hence

$$y^2=x(x-\alpha)(x-\beta)=f(x).$$

Consequently, one has

$$rank(Hom_K(C_h, E))\geq 1$$

and by (6), $rank(E_h(K(t)))\geq 1.$ Then the K-rational point

$$\left(\frac{-\alpha(2\alpha-\beta)t+(\alpha^2+\alpha\beta-\beta^2)m_1+(\alpha-\beta)^2m_2}{(2\alpha-\beta)(m_1-m_2)}, \frac{(\alpha-\beta)^2(m_1-m_2)^2}{(2\alpha-\beta)^2}\right)$$

is free in $E_h(K(t))$. □

Corollary 3. *Put* $\beta=\dfrac{\alpha(\alpha+2\gamma^2)}{\alpha+\gamma^2}$, $K=k(\alpha, \gamma, m_1, m_2)$, *and let*

$$E_f: f(t)y^2=f(x),\quad E_g: g(t)y^2=f(x),$$

$$E_h: h(t)y^2=f(x),\quad E_{gh}: g(t)h(t)y^2=f(x).$$

Then the rank of each elliptic curve over $K(t)$ *is positive.*

Proof. If $\beta = \dfrac{\alpha(\alpha + 2\gamma^2)}{\alpha + \gamma^2}$, then we can compute the coefficient $c = -\dfrac{\alpha(\alpha - \beta)}{2\alpha - \beta}$ on the left side of (10) as follows:

$$c = -\frac{\alpha(\alpha - \beta)}{2\alpha - \beta} = -\frac{\alpha\left(\alpha - \dfrac{\alpha(\alpha + 2\gamma^2)}{\alpha + \gamma^2}\right)}{2\alpha - \dfrac{\alpha(\alpha + 2\gamma^2)}{\alpha + \gamma^2}}$$

$$= -\frac{\alpha(\alpha(\alpha + \gamma^2) - \alpha(\alpha + 2\gamma^2))}{2\alpha(\alpha + \gamma^2) - \alpha(\alpha + 2\gamma^2)} = -\frac{-\alpha^2\gamma^2}{\alpha^2} = \gamma^2.$$

Hence $c \in K^{*2}$ and the curve (10) is isomorphic to $g(t)h(t)y^2 = f(x)$. □

3. Examples

By substituting suitable values into α, β, m_1, m_2, we give some examples of a family of twisted elliptic curves over $\mathbf{Q}(t)$. Moreover by specializing t, we also obtain that over $\mathbf{Q}$.

Example 4. If $k = \mathbf{Q}$, $\alpha = 1$, $\beta = -1$, $m_1 = 3$, $m_2 = 5$, then we have

$$m_3 = 7,$$

$$f(t) = t(t-1)(t+1),$$

$$g(t) = -2(t-3)(t-5)(t-7),$$

$$h(t) = \frac{3}{2048}(3t - 11)(3t - 17)(3t - 23).$$

By Theorem 1, $rank(E_f(\mathbf{Q}(t)))$, $rank(E_g(\mathbf{Q}(t)))$, $rank(E_h(\mathbf{Q}(t)))$, $rank(E_{gh}(\mathbf{Q}(t))) \geq 1$. The free $\mathbf{Q}(t)$-rational points of respective Mordell-Weil group found in the proof of Theorem 1 become

$$(t, 1), \left(\frac{-t+5}{2}, -\frac{1}{4}\right), \left(\frac{3t-17}{6}, \frac{16}{9}\right), \left(\frac{3t^2 - 32t + 77}{8}, 1\right).$$

Example 5. Specialize t of E_f, E_g, E_h, E_{gh} in Example 4. Let $\bar{c}$ be the coefficient of y^2 in the equation $\bar{c}y^2 = x(x-1)(x+1)$. Put $t = 2$, then $c = -\frac{\alpha(\alpha-\beta)}{2\alpha-\beta} = -2/3$. Table 1 shows the values of $\bar{c}$, minimal forms, and free points on E_f, E_g, E_h, E_{gh}.

Table 1. Minimal forms of twisted elliptic curves and free points on them

	$\bar{c}$	minimal form	free point
E_f	6	$y^2 = x^3 - 36x$	$(12, 36)$
E_g	30	$y^2 = x^3 - 900x$	$(45, -225)$
E_h	$-\frac{2805}{2048}$	$y^2 = x^3 - 31472100x$	$(10285, 874225)$
E_{gh}	$\frac{14025}{512}$	$y^2 = x^3 - 1258884x$	$\left(\frac{14025}{4}, \frac{1573605}{8}\right)$

These four elliptic curves are not isomorphic to each other over **Q**. One can check that each rational point of the rightmost column of the above table is free in respective Mordell-Weil group over **Q**.

Example 6. If $k = \mathbf{Q}$, $\alpha = 1$, $\gamma = 2$, $m_1 = 3$, $m_2 = 5$, then we have

$$\beta = \frac{9}{5}, \quad m_3 = \frac{7}{5},$$

$$f(t) = t(t-1)\left(t - \frac{9}{5}\right),$$

$$g(t) = -2(t-3)(t-5)\left(t - \frac{7}{5}\right),$$

$$h(t) = \frac{1}{32768}\left(t - \frac{47}{5}\right)\left(t - \frac{57}{5}\right)(t - 13).$$

By Corollary 3, $\operatorname{rank}(E_f(\mathbf{Q}(t)))$, $\operatorname{rank}(E_g(\mathbf{Q}(t)))$, $\operatorname{rank}(E_h(\mathbf{Q}(t)))$, $\operatorname{rank}(E_{gh}(\mathbf{Q}(t))) \geq 1$. The free $\mathbf{Q}(t)$-rational points of respective Mordell-Weil group are

$$(t, 1), \left(\frac{-t+5}{2}, -\frac{1}{4}\right), \left(\frac{5t-47}{10}, 64\right), \left(\frac{-5t^2+72t-91}{80}, 2\right).$$

Example 7. Specialize t of E_f, E_g, E_h, E_{gh} in Example 6. Let $\bar{c}$ be the coefficient of y^2 in the equation $\bar{c}y^2 = x(x-1)(x-9/5)$. Put $t = 2$. Table 2 shows the values of $\bar{c}$, minimal forms, and free points on E_f, E_g, E_h, E_{gh}.

Table 2. Minimal forms of twisted elliptic curves and free points on them

	$\bar{c}$	minimal form	free point
E_f	$\frac{2}{5}$	$y^2 = x^3 - x^2 - 81x + 81$	$(11, 20)$
E_g	$-\frac{18}{5}$	$y^2 = x^3 + x^2 - 81x - 81$	$(-6, -15)$
E_h	$-\frac{19129}{819200}$	$y^2 = x^3 - x^2 - 744034570033x - 47182629708158063$	$(4431552, 9147966025)$
E_{gh}	$\frac{172161}{2048000}$	$y^2 = x^3 + x^2 - 7440345700x + 47180397604448$	$\left(-\frac{797047}{16}, \frac{1097755923}{64}\right)$

These four elliptic curves are not isomorphic to each other over $\mathbf{Q}$. One can check that each rational point of the rightmost column of the above table is free in respective Mordell-Weil group over $\mathbf{Q}$.

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