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NUMERICAL STUDY OF MATRIX DIFFERENTIAL MODELS USING MATRIX SPLINES

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Abstract

This paper considers the solution of the matrix differential models using quadratic, cubic, quartic, and quintic splines. Also, using the Taylor's and Picard's matrix methods, three illustrative examples are included.

1. Introduction

The evaluation of matrix functions is frequent in the solution of

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differential systems. So the system

$$\dot{Y}(t) = A(t)Y(t), \quad Y(0) = Y_0, \quad \Delta = [0, 1], \tag{1}$$

where $A(t)$ is matrix and Y_0 is a vector arising of the parabolic equation. The matrix differential equation

$$\ddot{Y}(t) = A(t)Y(t), \quad Y(0) = Y_0, \quad \dot{Y}(0) = Y_1, \quad \Delta = [0, 1], \tag{2}$$

where $A(t)$ is matrix, Y_0 and Y_1 are vectors arising of the hyperbolic equation. The matrix differential equation

$$\dot{Y}(t) = A(t)Y(t) + Y(t)B(t), \quad Y(0) = Y_0, \quad \Delta = [0, 1], \tag{3}$$

where $A(t)$ and $B(t)$ are matrices appearing in systems stability and control. Consider the matrix differential equation in the form

$$\dot{Y}(t) = A(t)Y(t) + B(t), \quad Y(0) = D, \quad \Delta = [0, 1], \tag{4}$$

where $Y(t) \in C^{r\times q}$, $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are matrices. Let Δ be partition as $\Delta = \{0 = t_0 < \cdots < t_n = 1\}$. The set of matrix splines of order m defined as [1]:

$$M\_C^{r\times r}(\Delta)_{m-1}^{m} = \left\{ Q : \Delta \to C^{r\times q}; \quad \begin{Bmatrix} Q|_{[t_{i-1}, t_i]}(t) \in P_m[t], \, i \in \{1, \dots, n\} \\ Q \in C^{m-1}(\Delta) \end{Bmatrix} \right\}, \tag{5}$$

if $m = 2$, the matrix splines are called *matrix quadratic splines*, $m = 3$ called *matrix cubic splines*, $m = 4$ called *matrix quartic splines* and $m = 5$ called *matrix quintic splines*. A recent paper [2] deals with the construction of an approximate solution of the first order matrix linear differential equations using matrix cubic splines. The present paper extends the first order linear differential equations using different matrix splines and also approximates the solution by using Picard’s method and Taylor’s method which are best than all matrix splines.

2. The Matrix Spline Methods

This section gives the theoretical studies for the matrix differential equation in the form (4) using the matrix quadratic splines, matrix cubic splines, matrix quartic splines and matrix quintic splines.

2.1. The matrix quadratic splines

Consider the interval $\Delta_0 = [0, k]$, $k = \Delta t$; suppose the solution in the form

$$S_0(t) = Y(0) + \dot{Y}(0)t + \frac{1}{2}\alpha_0 t^2, \tag{6}$$

where $Y(0) = D$, $\dot{Y}(0) = A(0)Y(0) + Y(0)B(0) + C(0)$, but to find α_0, we suppose that $S_0(t)$ satisfies the matrix differential equation (4) at $t = k$, so

$$\dot{S}_0(k) = A(k)S_0(k) + B(k). \tag{7}$$

From equations (6) and (7), we get

$$k\left(I - \frac{k}{2}A(k)\right)\alpha_0 = A(k)(Y(0) + \dot{Y}(0)k) + B(k) - \dot{Y}(0), \tag{8}$$

where I is the identity matrix, from equation (8), we get α_0 and so $S_0(t)$ as in (6). Consider $\Delta_i = [ik, (i+1)k]$, $1 \le i \le n-1$; suppose the matrix quadratic solution in the form

$$S_i(t) = S_{i-1}(ik) + \dot{S}_{i-1}(ik)(t - ik) + \frac{1}{2}\alpha_i(t - ik)^2, \tag{9}$$

as above we determine α_i from the equation

$$k\left(I - \frac{k}{2}A((i+1)k)\right)\alpha_i$$

$$= A((i+1)k)(S_{i-1}(ik) + \dot{S}_{i-1}(ik)k) + B((i+1)k) - \dot{S}_{i-1}(ik), \tag{10}$$

and then $S_i(t)$ are determined for all $i = 1, ..., n.$ Note that solubility of the suggested scheme (10) is guaranteed showing that the matrix coefficient of α_i is invertible. If $M = \max\limits_{0 \le t \le 1} \| A(t) \|$, then $\left\| I - \left(I - \frac{k}{2} A((i+1)k) \right) \right\| \le 1$, so we get $k \le \frac{2}{M}$ and then equation (10) has a unique solution α_i.

2.2. The matrix cubic splines [2]

Consider the interval $\Delta_0 = [0, k]$; suppose the solution in the form

$$S_0(t) = Y(0) + \dot{Y}(0)t + \frac{1}{2}\ddot{Y}(0)t^2 + \frac{1}{6}\alpha_0 t^3, \tag{11}$$

where $Y(0) = D$, $\dot{Y}(0) = A(0)Y(0) + B(0)$, $\ddot{Y}(0) = A(0)\dot{Y}(0) + \dot{A}(0)Y(0) + \dot{B}(0)$ and to determine α_0, we suppose that $S_0(t)$ satisfies the matrix differential equation (4) at $t = k$, so

$$\frac{k^2}{2}\left(I - \frac{k}{3} A(k) \right)\alpha_0 = A(k)\left(Y(0) + \dot{Y}(0)k + \frac{1}{2}\ddot{Y}(0)k^2 \right)$$

$$+ B(k) - \dot{Y}(0) - \ddot{Y}(0)k, \tag{12}$$

and $S_0(t)$ as in (11). Consider $\Delta_i = [ik, (i+1)k]$, $1 \le i \le n-1$; suppose the matrix cubic solution in the form

$$S_i(t) = S_{i-1}(ik) + \dot{S}_{i-1}(ik)(t - ik) + \frac{1}{2}\dot{S}_{i-1}(ik)(t - ik)^2 + \frac{1}{6}\alpha_i (t - ik)^3, \tag{13}$$

as above we determine α_i from the equation

$$\frac{k^2}{2}\left(I - \frac{k}{3} A((i+1)k) \right)\alpha_i = A((i+1)k)\left(S_{i-1}(ik) + \dot{S}_{i-1}(ik)k + \frac{1}{2}\ddot{S}_{i-1}(ik)k^2 \right)$$

$$+ B((i+1)k) - \dot{S}_{i-1}(ik) - \dot{S}_{i-1}(ik)k, \tag{14}$$

and then $S_i(t)$ are determined for all $i = 1, \ldots, n$. Note that solubility of the suggested scheme (14) is guaranteed showing that the matrix coefficient of α_i is invertible. If $M = \max\limits_{0 \le t \le 1} \| A(t) \|$, then $\left\| I - \left(I - \frac{k}{3} A((i+1)k) \right) \right\| \le 1,$ so we get $k \le \frac{3}{M}$ and then equation (14) has a unique solution α_i.

2.3. The matrix quartic splines

Consider the interval $\Delta_0 = [0, k]$; suppose the solution in the form

$$S_0(t) = Y(0) + \dot{Y}(0)t + \frac{1}{2} \ddot{Y}(t)t^2 + \frac{1}{6} \dddot{Y}(0)t^3 + \frac{1}{24} \alpha_0 t^4, \qquad (15)$$

for this case, α_0 can be determined from the equation

$$\frac{k^3}{6}\left(I - \frac{k}{4} A(k) \right)\alpha_0 = A(k)\left(Y(0) + \dot{Y}(0)k + \frac{1}{2}\ddot{Y}(0)k^2 + \frac{1}{6}\dddot{Y}(0)k^3 \right)$$

$$+ B(k) - \dot{Y}(0) - \ddot{Y}(0)k - \dddot{Y}(0)k^2, \qquad (16)$$

and $S_0(t)$ as in (15). Consider $\Delta_i = [ik, (k+1)k]$, $1 \le i \le n-1$; suppose the matrix quartic solution in the form

$$S_i(t) = S_{i-1}(ik) + \dot{S}_{i-i}(ik)(t - ik) + \frac{1}{2}\ddot{S}_{i-1}(ik)(t - ik)^2$$

$$+ \frac{1}{6}\dddot{S}_{i-1}(ik)(t - ik)^3 + \frac{1}{24}\alpha_i (t - ik)^4, \qquad (17)$$

as above we determine α_i from the equation

$$\frac{k^3}{6}\left(I - \frac{k}{4} A((i+1)k) \right)\alpha_i$$

$$= A((i+1)k)\left(S_{i-1}(ik) + \dot{S}_{i-1}(ik)k + \frac{1}{2}\ddot{S}_{i-1}(ik)k^2 + \frac{1}{6}\dddot{S}_{i-1}(ik)k^3 \right)$$

$$+ B((i+1)k) - \dot{S}_{i-1}(ik) - \ddot{S}_{i-1}(ik)k - \frac{1}{2}\dddot{S}_{i-1}(ik)k^2, \qquad (18)$$

and then $S_i(t)$ are determined for all $i = 1, \ldots, n.$ Note that solubility of the suggested scheme (18) is guaranteed showing that the matrix coefficient of α_i is invertible. If $M = \max\limits_{0 \le t \le 1} \| A(t) \|,$ then $\left\| I - \left(I - \frac{k}{4} A((i+1)k) \right) \right\| \le 1,$ so we get $k \le \frac{4}{M}$ and then equation (18) has a unique solution $\alpha_i.$

2.4. The matrix quintic splines

Consider the interval $\Delta_0 = [0, k];$ suppose the solution in the form

$$S_0(t) = Y(0) + \dot{Y}(0)t + \frac{1}{2}\ddot{Y}(0)t^2 + \frac{1}{6}\dddot{Y}(0)t^3 + \frac{1}{24}\ddddot{Y}(0)t^4 + \frac{1}{120}\alpha_0 t^5, \quad (19)$$

for this case, α_0 can be determined from the equation

$$\frac{k^4}{24}\left(I - \frac{k}{5} A(k) \right)\alpha_0$$

$$= A(k)\left(Y(0) + \dot{Y}(0)k + \frac{1}{2}\ddot{Y}(0)k^2 + \frac{1}{6}\dddot{Y}(0)k^3 + \frac{1}{24}\ddddot{Y}(0)k^4 \right)$$

$$+ B(k) - \dot{Y}(0) - \ddot{Y}(0)k - \dddot{Y}(0)k^2 - \ddddot{Y}(0)k^3, \quad (20)$$

and $S_0(t)$ as in (19). Consider $\Delta_i = [ik, (i+1)k], \ 1 \le i \le n-1;$ suppose the matrix quintic solution in the form

$$S_i(t) = S_{i-1}(tk) + \dot{S}_{i-1}(ik)(t - ik) + \frac{1}{2}\ddot{S}_{i-1}(ik)(t - ik)^2 + \frac{1}{6}\dddot{S}_{i-1}(ik)(i - ik)^3$$

$$+ \frac{1}{24}\ddddot{S}_{i-1}(ik)(t - ik)^4 + \frac{1}{120}\alpha_i (t - ik)^5, \quad (21)$$

as above we determine α_i from the equation

$$\frac{k^4}{24}\left(I - \frac{k}{5}A((i+1)k)\right)\alpha_i$$

$$= A((i+1)k)\Bigg(S_{i-1}(ik) + \dot{S}_{i-1}(ik)k + \frac{1}{2}\ddot{S}_{i-1}(ik)k^2$$

$$+ \frac{1}{6}\dddot{S}_{i-1}(ik)k^3 + \frac{1}{24}\ddddot{S}_{i-1}(ik)k^4\Bigg) + B((i+1)k) - \dot{S}_{i-1}(ik)$$

$$- \ddot{S}_{i-1}(ik)k - \frac{1}{2}\dddot{S}_{i-1}(ik)k^2 - \frac{1}{6}\ddddot{S}_{i-1}(ik)k^3, \tag{22}$$

and then $S_i(t)$ are determined for all $i = 1, \dots, n.$ Note that solubility of the suggested scheme (22) is guaranteed showing that the matrix coefficient of α_i is invertible. If $M = \max_{0 \le t \le 1} \| A(t) \|,$ then $\left\| I - \left(I - \frac{k}{5}A((i+1)k)\right)\right\| \le 1,$ so we get $k \le \frac{5}{M}$ and then equation (22) has a unique solution α_i.

3. The Matrix Picard's Method

In this section, we see the Picard's method for the matrix differential equation in the form (4), then the first approximation is

$$Y_{i+1}(t) = Y_0(t) + \int_0^t (A(t)Y_i(t) + B(t))dt, \tag{23}$$

where $Y_0(t) = D,\ i = 0, 1, 2, \dots.$ As in ordinary differential equation, we get a sequence $\{Y_i(t)\}|_0^\infty$ which is convergent to the exact solution.

4. The Matrix Taylor's Method

Suppose the approximate solution for the matrix differential equation (4) takes the form

$$Y_n(t) = Y(0) + \dot{Y}(0)t + \frac{1}{2}\ddot{Y}(0)t^2 + \cdots + \frac{1}{n!}Y^{(n)}(0)t^n, \tag{24}$$

where $Y(0), \dot{Y}(0), \ldots, Y^{(n)}(0)$ all can be determined from the matrix differential equation (4).

5. Illustration of the Analysis

In this section, distinct matrix differential equations will be tested by using the proposed methods.

Example 1. We first consider the matrix differential equation in the form [6]:

$$\begin{pmatrix} \dot{y}_1(t) \\ y_2(t) \end{pmatrix} = \frac{1}{t^3 - t - 1}\begin{pmatrix} 2t^2 - 1 & t^2 - 2t - 1 \\ -t - 1 & t^3 + t^2 - t - 1 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad 0 \le t \le 1,$$

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in C^2, \tag{25}$$

this matrix differential equation has the exact solution $\begin{pmatrix} e^t \\ te^t \end{pmatrix}$, in the following table, we see the matrix splines methods.

Table 1

$[t_i, t_{i-1}]$	Quadratic	Cubic	Quartic	Quintic
$[0, 0.1]$	3.06573E-4	6.33769E-6	1.14628E-7	1.7956E-9
$[0.1, 0.2]$	7.11688E-4	6.33769E-6	8.81776E-7	5.7101E-8
$[0.2, 0.3]$	12.397E-4	8.32925E-6	2.2721E-6	5.46782E-7

In the following table, we see the approximation solution using quadratic matrix method in some intervals.

Table 2

$[t_i, t_{i-1}]$	Quadratic
$[0, 0.1]$	$\begin{pmatrix} 1 + t + 0.527889t^2 \\ t + 1.0804t^2 \end{pmatrix}$
$[0.1, 0.2]$	$\begin{pmatrix} 1.00056 + 0.988792t + 0.583928t^2 \\ 0.00172163 + 0.965567t + 1.25257t^2 \end{pmatrix}$
$[0.2, 0.3]$	$\begin{pmatrix} 1.00304 + 0.964035t + 0.645822t^2 \\ 0.009578 + 0.887004t + 1.44897t^2 \end{pmatrix}$

Example 2. We next consider the matrix differential equation in the form [6]:

$$\dot{\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \\ y_5 & y_6 \end{pmatrix}} = \begin{pmatrix} -1-t & 0 & -1+e^t+t \\ e^t & -t & 1 \\ 0 & -1 & e^t \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \\ y_5 & y_6 \end{pmatrix}$$

$$+ \begin{pmatrix} -2+(-3+e^t)t & -1-t-t^2-e^t(2+t) \\ t+e^t(1+t) & e^{2t}+te^t-(5+t)(1+t^2) \\ -1+te^t & 1-t(5+t) \end{pmatrix}, \qquad (26)$$

this matrix differential equation has the exact solution

$$\begin{pmatrix} 1+t & e^t+t \\ 0 & -1+5t+t^2 \\ t & 0 \end{pmatrix},$$

in the following table, we see the matrix splines methods.

Table 3

$[t_i, t_{i-1}]$	Quadratic	Cubic	Quartic	Quintic	Picard	Taylor
[0, 0.1]	8.31824E-5	1.39565E-6	2.09742E-8	2.79779E-10	2.77528E-8	8.1057E-17
[0.1, 0.2]	1.66464E-4	1.43614E-6	1.71722E-7	9.09382E-9	1.63446E-8	5.4655E-18
[0.2, 0.3]	2.50312E-4	1.43614E-6	5.158566E-7	9.35249E-9	2.20588E-8	4.74378E-17

In the following table, we see the approximation solution using quadratic matrix method in some intervals.

Table 4

$[t_i, t_{i-1}]$	Quadratic
[0, 0.1]	$\begin{pmatrix} 1+t & 1+2t-0.525398t^2 \\ 0 & -1+5t+1.00046t^2 \\ t & -0.000024106t^2 \end{pmatrix}$
[0.1, 0.2]	$\begin{pmatrix} 1+t-2.095E-15t^2 & 1.005+1.988t+0.580t^2 \\ 9.69E-18-1.939E-16t+6.699E-18t^2 & -0.999+4.999t+1.000t^2 \\ t-5.165E-17t^2 & -5.580E-7+0.00001t-7E-5t^2 \end{pmatrix}$
[0.2, 0.3]	$\begin{pmatrix} 1+t+2.08E-15t^2 & 1.002+1.964t+0.641t^2 \\ -2.356E-17+1.386E-16t+1.382E-16t^2 & -0.999+4.99t+1.0005t^2 \\ t-7.413E-18t^2 & -3.48E-6+4E-5t-15E-5t^2 \end{pmatrix}$

Example 3. Consider the matrix differential equation in the Sylvester problem form

$$\dot{\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}} = \begin{pmatrix} 0 & te^t \\ t & 0 \end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} -e^{-t}(1+t^2)t & -t(e^t + e^{-t}) \\ 1-te^{-t} & -t^2 \end{pmatrix}, \tag{27}$$

this matrix differential equation has the exact solution $\begin{pmatrix} e^{-t} & 0 \\ t & 1 \end{pmatrix}$, in the following table, we see Picard's method and Taylor's method, we note that for this example, we cannot use the matrix spline method since the multiplication of matrices is not commutative and this is an open problem which is how the application of the matrix spline method for the nonlinear matrix differential models of the first order and there is other open problem is using the suggested methods for higher order matrix differential models.

Table 5

$[t_i, t_{i-1}]$	Picard	Taylor
$[0, 0.1]$	6.63265E-13	5.09563E-17
$[0.1, 0.2]$	7.23197E-13	3.02364E-17
$[0.2, 0.3]$	2.16118E-12	1.83892E-17

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