JP Journal of Geometry and Topology
Volume 12, Number 3, 2012, Pages 235-242
Published Online: December 2012
Available online at http://pphmj.com/journals/jpgt.htm
Published by Pushpa Publishing House, Allahabad, INDIA

REAL HYPERSURFACES IN A COMPLEX SPACE FORM WITH RESPECT TO THE STRUCTURE LIE OPERATOR

Dong Ho Lim and Woon Ha Sohn

Department of Mathematics
Hankuk University of Foreign Studies
Seoul 130-791, Republic of Korea
e-mail: dhlnys@hufs.ac.kr

Department of Mathematics
Catholic University of Daegu
Daegu, 712-702, Republic of Korea
e-mail: kumogawa@cu.ac.kr

Abstract

We characterize some real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$ in terms of the structure Lie operator L_ξ.

1. Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n(\mathbb{C})$,

2010 Mathematics Subject Classification: Primary 53C40; Secondary 53C15.

Keywords and phrases: real hypersurface, totally η-umbilical, Hopf hypersurface.

Received May 24, 2012

a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper, we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant ([7]) and that M is called a *Hopf hypersurface*.

Takagi [13] completely classified such hypersurfaces as six model spaces which are said to be A_1, A_2, B, C, D and Berndt [1] classified all homogeneous Hopf hypersurfaces in $H_n(\mathbb{C})$ as four model spaces which are said to be A_0, A_1, A_2 and B. A real hypersurface of type A_1 or A_2 in $P_n(\mathbb{C})$ or type A_0, A_1 or A_2 in $H_n(\mathbb{C})$, then M is said to be of *type A* for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [12] for $c > 0$, and Montiel and Romero [10] for $c < 0$.

Theorem A ([10, 11])**.** *Let M be a real hypersurface of* $M_n(c)$, $c \neq 0$, $n \geq 2$. *It satisfies* $A\phi - \phi A = 0$ *on M if and only if M is locally congruent to one of the model spaces of type A*.

The *totally umbilical* on M is defined by $A = aI$, where I is an identity. From a result of Tashiro and Tachibana [14], we see that there are no totally umbilical hypersurfaces in $M_n(c)$. So we consider the notion of totally η-umbilical hypersurfaces. A real hypersurface M is said to be *totally* η*-umbilical* if shape operator A is of the form $A = a + b\eta \otimes \xi$, where a and b are scalar functions on M. The classification of totally η-umbilical real hypersurfaces in a complex projective space and complex hyperbolic space is determined by Kon [7] and Montiel [9].

The induced operator L_ξ on a real hypersurface M from the 2-form $\mathcal{L}_\xi g$ is defined by $(\mathcal{L}_\xi g)(X, Y) = g\langle L_\xi X, Y\rangle$ for any vector field X and Y on M, where $\mathcal{L}_\xi$ denotes the operator of the Lie derivative with respect to the structure vector field ξ. This operator L_ξ is given by

$$L_\xi = \phi A - A\phi$$

on M, and the structure vector field ξ is Killing if $L_\xi = 0$. Then we call L_ξ *structure Lie operator* of M.

In this paper, we shall study a real hypersurface in a nonflat complex space form $M_n(c)$ whose structure Lie operator L_ξ has totally η-umbilical and in terms of the shape operator A, the structure tensor ϕ and the structure Lie operator L_ξ with respect to ξ. More specifically, we prove the following:

Theorem 1. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. If M has $L_\xi X = \rho X + \sigma\eta(X)\xi$, where ρ and σ are scalar functions, then M is a locally congruent to a real hypersurface of type A.*

Theorem 2. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. If M has $L_\xi\phi A = A\phi L_\xi$, then M is a locally congruent to a real hypersurface of type A.*

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$, we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M.

Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas the following:

$$\nabla_X \xi = \phi AX, \tag{2.1}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.2}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have Codazzi equations:

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.3}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

Let W be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$, and let μ be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish, and zero (constant function) if it vanishes. Then it is easily seen from (2.1) that

$$A\xi = \alpha\xi + \mu W, \tag{2.4}$$

where $\alpha = \eta(A\xi)$. We notice here that W is orthogonal to ξ. We put

$$\Omega = \{p \in M \mid \mu(p) \neq 0\}.$$

Then Ω is an open subset of M.

3. Proof of Theorems

In this section, we shall prove that ξ is principal under the our assumptions.

Proof of Theorem 1. Let M be a real hypersurface in complex space form $M_n(c)$, $c \neq 0$ satisfying $L_\xi X = \rho X + \sigma\eta(X)\xi$. This condition implies that

$$(\phi A - A\phi)X = \rho X + \sigma\eta(X)\xi \tag{3.1}$$

for any vector field X on Ω.

If we put $X = \xi$ into (3.1) and make use of (2.4), then we have

$$\mu = 0 \text{ and } \rho + \sigma = 0. \tag{3.2}$$

Thus M is a Hopf hypersurface and $\rho + \sigma = 0$. Since M is a Hopf hypersurface and $\rho + \sigma = 0$, it follows from (3.2) that

$$(\phi A - A\phi)X = \rho(X - \eta(X))\xi. \tag{3.3}$$

Putting $X = W$ into (3.3), we obtain

$$(\phi A - A\phi)W = \rho W \tag{3.4}$$

and by putting $X = \phi W$ into (3.3) and make use of (2.1), we have

$$-A\phi W + \phi AW = -\rho W. \tag{3.5}$$

From (3.4), (3.5) and (3.3), we have

$$\rho = 0 \text{ and } \sigma = 0. \tag{3.6}$$

Therefore, we have $L_\xi = \phi A - A\phi = 0$ on M. The statement Theorem 1 follows immediately from Theorem A. □

Proof of Theorem 2. Let M be a real hypersurface in complex space form $M_n(c)$, $c \neq 0$ satisfying $L_\xi \phi A = A \phi L_\xi$. This condition implies that

$$(\phi A \phi A + A \phi A \phi + 2A^2)X = 2\eta(AX)A\xi. \tag{3.7}$$

If we put $X = \xi$ into (3.7) and make use of (2.4), then we have

$$\mu A \phi W + 2\mu AW = 0. \tag{3.8}$$

Taking inner product of (3.8) with ξ and using (2.4), we get $\mu = 0$ on Ω, and it is a contradiction. Thus M is a Hopf hypersurface. Thus, the assumption $L_\xi \phi A = A \phi L_\xi$ is equivalent to

$$(\phi A \phi A + A \phi A \phi + 2A^2)X = 2\alpha^2 \eta(X)\xi. \tag{3.9}$$

On the other hand, if we differentiate $A\xi = \alpha\xi$ covariantly and make use of equation (2.3) of Codazzi, then we have

$$A\phi A - \frac{\alpha}{2}(\phi A + A\phi) - \frac{c}{4}\phi = 0. \tag{3.10}$$

For any vector field X on M such that $AX = \lambda X$, it follows from (3.10) that

$$\left(\lambda - \frac{\alpha}{2}\right) A\phi X = \frac{1}{2}\left(\alpha\lambda + \frac{c}{2}\right)\phi X. \tag{3.11}$$

We can choose an orthonormal frame field

$$\{X_0 = \xi, X_1, X_2, \ldots, X_{n-1}, \phi X_1, \ldots, \phi X_{n-1}\}$$

on M such that $AX_i = \lambda_i X_i$ for $1 \leq i \leq (n-1)$. If $\lambda_i \neq \frac{\alpha}{2}$ for $1 \leq i \leq (n-1)$, then we see from (3.11) that ϕX_i is also a principal direction , say $A\phi X_i = \mu_i \phi X_i$. From (3.9) and (3.11), we have

$$\lambda_i(\lambda_i - \mu_i) = 0. \tag{3.12}$$

If $\lambda_i = 0,$ then by the assumption, we obtain $\alpha \neq 0$ and $\mu_i = -\frac{c}{2\alpha}.$ Therefore, since M has at most three distinct principal curvatures, M is locally congruent to either one of type A_2 or type B. In $M_n(c),$ if $\lambda_i = 0,$ then $\alpha = 0$ or α is not defined. Since we have α does not vanish, it is a contradiction. From (3.12), we have $\lambda_i = \mu_i$ and hence $A\phi X_i = \phi A X_i$ for $1 \leq i \leq (n-1)$. If $\lambda_i \neq \frac{\alpha}{2}$ and $\lambda_j = \frac{\alpha}{2},$ then we can choose an orthonormal frame field $\{X_0 = \xi, X_1, X_2, ..., X_p, \phi X_1, ..., \phi X_p, X_{2p+1}, ..., X_{2(n-1)}\}$ on M such that $AX_i = \lambda_i X_i,$ $A\phi X_i = \mu_i \phi X_i$ and $AX_j = \frac{\alpha}{2} X_j$ for $1 \leq i \leq p$ and $p + 1 \leq j \leq 2(n-1).$ Then it follows from (3.9) that

$$(\phi A \phi A + A\phi A\phi + 2A^2) X_j = 0. \tag{3.13}$$

Taking inner product of (3.13) with $X_i,$ we obtain

$$\mu_i(\lambda_i - \mu_i) g(\phi X_j, X_i) = 0. \tag{3.14}$$

Also, from (3.11), we obtain $c = -\alpha^2.$ If $\mu_i = 0,$ then we have $\lambda_i = \frac{\alpha}{2}.$ But by the assumption $\lambda_i \neq \frac{\alpha}{2},$ it is a contradiction. If $\lambda_i = \mu_i$ and using (3.11), then we have $\lambda_i = \mu_i = \frac{\alpha}{2}.$ By the virtue of $\lambda_i \neq \frac{\alpha}{2},$ it is a contradiction. Thus the vector field ϕX_j is expressed by the linear combination of X_j's only, which implies $A\phi X_j = \frac{\alpha}{2} \phi X_j = \phi A X_j.$ If $\lambda_j = \frac{\alpha}{2}$ for $1 \leq j \leq 2(n-1),$ then it is easily seen that $A\phi X_j = \phi A X_j$ for all j. Therefore, we have $L_\xi = \phi A - A\phi = 0$ on M. The results of Theorem 2 follow immediately from Theorem A. □

References

[1] J. Berndt, Real hypersurfaces with constant principal curvatures in complex hyperbolic space, J. Reine Angew. Math. 395 (1989), 132-141.

[2] T. E. Cecil and P. J. Ryan, Focal sets and real hypersurfaces in a complex projective space, Trans. Amer. Math. Soc. 269 (1982), 481-499.

[3] U.-H. Ki and Y. J. Suh, On real hypersurfaces of a complex space form, J. Okayama Univ. 32 (1990), 207-221.

[4] I.-B. Kim, K. H. Kim and W. H. Sohn, Characterizations of real hypersurfaces in a complex space form, Canad. Math. Bull. 50 (2007), 97-104.

[5] M. Kimura, Real hypersurfaces and complex submanifolds in a complex projective space, Trans. Amer. Math. Soc. 296 (1986), 137-149.

[6] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. 202 (1989), 299-311.

[7] M. Kon, Pseudo-Einstein real hypersurfaces in a complex projective space, J. Differential Geom. 14 (1979), 339-354.

[8] S. H. Kon and T. H. Loo, Real hypersurfaces in a complex space form with η-parallel shape operator, Math. Z. 269 (2011), 47-58.

[9] S. Montiel, Real hypersurfaces of a complex hyperbolic space, J. Math. Soc. Japan 37 (1985), 515-535.

[10] S. Montiel and A. Romero, On some real hypersurfaces of a complex hyperbolic space, Geom. Dedicata 20 (1986), 245-261.

[11] R. Niebergall and P. J. Ryan, Real hypersurfaces in complex space forms, Tight and Taut Submanifolds, pp. 233-305, Cambridge Univ. Press, Cambridge, 1997.

[12] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. 212 (1975), 355-364.

[13] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. 10 (1973), 495-506.

[14] Y. Tashiro and S. Tachibana, On Fubinian and *C*-Fubinian manifolds, Kodai Math. Sem. Rep. 15 (1963), 176-183.