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EXTENDED PROOF OF SOME RESULTS CONCERNING THE FINITE NULL DISTRIBUTION OF A RANK CORRELATION MEASURE

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Abstract

The finite null expectation and variance of a rank correlation measure proposed by Borroni and Zenga [2] are considered. A formal proof of their expressions was never provided, even if some partial results were published in a technical report (Borroni [1]), which had a limited circulation. This paper aims at filling this gap by reporting a detailed proof.

1. Introduction

A common statistical problem is to test whether two sorting criteria can be considered as independent. More specifically, take a sample of n units and denote by $R_{11}, R_{12}, \ldots, R_{1n}$ and $R_{21}, R_{22}, \ldots, R_{2n}$ the two sequences of ranks obtained after sorting the sampled units according to the two

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considered criteria. A test-statistic based on the two above sequences is often referred to as a rank correlation measure. Along with the most known Spearman's rho and Kendall's tau, other measures can be considered. This paper will focus on a measure proposed by Borroni and Zenga [2], as an application of Gini's mean difference to the total ranks $T_i = R_{1i} + R_{2i}$, $i = 1, \dots, n$. The final expression of such a measure is

$$D = \frac{3}{2(n^3 - n)} \sum_{i=1}^{n} \sum_{j=1}^{n} | R_{1i} + R_{2i} - R_{1j} - R_{2j} |. \tag{1}$$

To develop a test of independence, one obviously needs to determine the finite, or at least the asymptotical, null distribution of the chosen test-statistic. An important issue is often to know the expressions of the null expectation and variance of the statistic as a function of *n*. Borroni and Zenga [2] provided such expressions for *D*. The proof of the related theorem, which will be restated in the next section, was never reported. Some partial results appeared in a technical report (Borroni [1]), which had, however, a limited circulation. A detailed proof is hence provided in the following.

2. Expectation and Variance of *D* under Independence

Note that the value taken by *D* does not change if the terms in its summation are rearranged according to the increasing values of one of the two sequences of ranks. This means that *D* can be rewritten as

$$D = \frac{3}{2(n^3 - n)} \sum_{i=1}^{n} \sum_{j=1}^{n} | i + R_{2i}^* - j - R_{2j}^* |, \tag{2}$$

where $R_{21}^*, \dots, R_{2n}^*$ denotes the corresponding rearrangement of $R_{21}, \dots, R_{2n}$. By noticing that, under independence, the sequence $R_{21}^*, \dots, R_{2n}^*$ takes every permutation of the set $\{1, \dots, n\}$ with the same probability $1/n!$, the null distribution of *D* can be determined. Such a logic bases the proof of the following theorem:

Theorem. *Under independence,*

$$E(D) = \frac{7n^2 - 10n + 2}{10n(n-1)} \tag{3}$$

and

$$Var(D)$$

$$= \frac{1751n^6 - 3090n^5 + 233n^4 + 1428n^3 - 2173n^2 - 9822n + 4725}{12600n^3(n-2)(n-1)^2(n+1)}$$

when n is odd

$$= \frac{1751n^6 - 3090n^5 - 5020n^4 + 192n^3 - 5344n^2 - 20448n - 10008}{12600n^2(n-3)(n-1)^2(n+1)^2}$$

when n is even. (4)

Proof. For the sake of simplicity, write R_i instead of R^*_{2i}, $(i = 1, ..., n)$ and consider

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n} | \, i + R_i - j - R_j \, |$$

$$= 2\sum_{i=2}^{n}\sum_{l=1}^{i-1} | \, i + R_{2i} - l - R_{2l} \, |$$

$$= 2\sum_{i=2}^{n}\sum_{l=1}^{i} | \, R_i - R_l + (i - l) \, |. \tag{5}$$

Notice that $i - l \geq 0$ in the double summation of (5). Consider now the variables $A_{i,l}(u) = | \, R_i - R_l + u \, |$ $(i = 2, ..., n; \, l = 1, ..., i)$; for each fixed $u = 1, ..., n-1,$ these variables are, under independence, identically distributed as $A_{2,1}(u) = | \, R_2 - R_1 + u \, |$. The distribution of $A_{2,1}(u)$ depends on u. When n is odd,

- when $u = 1, \ldots, \frac{n-1}{2}$

$$
\begin{aligned}
\Pr\{A_{2,1}(u) = x\} &= \frac{n-u}{n(n-1)} && \text{if } x = 0 \\
&= \frac{2(n-u)}{n(n-1)} && \text{if } x = 1, \ldots, u-1 \\
& && (\text{except when } u = 1) \\
&= \frac{n-2u}{n(n-1)} && \text{if } x = u \\
&= \frac{2(n-x)}{n(n-1)} && \text{if } x = u+1, \ldots, n-u-1 \\
& && \left(\text{except when } u = \frac{n-1}{2}\right) \\
&= \frac{n+u-x}{n(n-1)} && \text{if } x = n-u, \ldots, n+u-1
\end{aligned}
$$

- when $u = \frac{n+1}{2}, \ldots, n-1$

$$
\begin{aligned}
\Pr\{A_{2,1}(u) = x\} &= \frac{n-u}{n(n-1)} && \text{if } x = 0 \\
&= \frac{2(n-u)}{n(n-1)} && \text{if } x = 1, \ldots, n-u-1 \\
&= \frac{n-u+x}{n(n-1)} && \text{if } x = n-u, \ldots, u-1 \\
&= \frac{n+u-x}{n(n-1)} && \text{if } x = u+1, \ldots, n+u-1.
\end{aligned}
$$

From the distribution above, the null expected value of $A_{2,1}(u)$, which turns out to have the same expression for every $u = 1, \ldots, n-1$, can be computed as:

$$E(A_{2,1}(u)) = \frac{n^3 + n(3u^2 - 3u - 1) - u^3 + u}{3n(n-1)}.$$

We can then compute

$$E(S) = 2\sum_{i=2}^{n}\sum_{l=1}^{i-1} E(A_{i,l}(i-l))$$

$$= 2\sum_{i=2}^{n}\sum_{u=1}^{i-1} E(A_{2,1}(u))$$

$$= \frac{(n+1)(7n^2 - 10n + 2)}{15}.$$

This expression does not change when n is even. The results in (3) are then easily obtained by the definition of D.

The null variance of S is to be computed as

$$Var(S) = 4\sum_{i=2}^{n}\sum_{l=1}^{i-1} Var(A_{i,l}(i-l))$$

$$+ 4\sum_{i=2}^{n}\sum_{l=1}^{i-1}\sum_{h=2}^{n}\sum_{k=1}^{h-1} Cov(A_{i,l}(i-l),\ A_{h,k}(h-k)). \tag{6}$$

The first term in (6) can be easily obtained from the above reported distribution:

$$Var(A_{2,1}(u)) = \frac{1}{18(n-1)^2 n^2}[n^6 - 3n^5 + (6u^2 + 12u + 1)n^4$$

$$+ (4u^3 - 36u^2 - 4u + 3)n^3 - 2(3u^2 - 1)(3u^2 - 6u - 1)n^2$$

$$+ 4u(u-1)(u+1)(3u^2 - 3u - 1)n - 2u^2(u-1)^2(u+1)^2]$$

for $u = 1, \ldots, n-1$. Concerning the second term in (6), we have to compute three kinds of covariance:

$$C_0(u, v) = Cov(A_{2,1}(u), A_{4,3}(v)), \tag{7}$$

$$C_1(u, v) = Cov(A_{2,1}(u), A_{3,1}(v)), \tag{8}$$

$$C_2(u, v) = Cov(A_{2,1}(u), A_{4,2}(v)), \tag{9}$$

$C_0(u, v)$ is used when all the indexes of the summands take different values $(i \neq l \neq h \neq k)$; $C_1(u, v)$ is used when the index i equals h or the index l equals k; finally, $C_2(u, v)$ is used when $i = k$ or $l = h$.

To determine (7), (8) and (9), suitable joint distributions are to be derived. For $C_0(u, v)$, we can first determine the expressions of $\Pr\{A_{2,1}(u) = x, A_{4,3}(v) = y\}$ when n is odd, $u, v = 1, ..., (n-1)/2$ and $u < v$:

- when $x = 0, ..., u-1$

$$= \frac{(n-u+x)(n-v) - 2(2n-u-2v+x)}{n(n-1)(n-2)(n-3)}$$

if $y = 0$

$$= \frac{2(n-u+x)(n-v) - 2(4n-2u-4v+2x)}{n(n-1)(n-2)(n-3)}$$

if $y = 1, ..., v-u+x-1$

$$= \frac{2(n-u+x)(n-v) - (7n-3u-8v+3x)}{n(n-1)(n-2)(n-3)}$$

if $y = v-u+x$

$$= \frac{2(n-u+x)(n-v) - 2(4n-3u-3v+3x-y)}{n(n-1)(n-2)(n-3)}$$

if $y = v-u+x+1, ..., v-1$; $x \neq u-1$

$$= \frac{(n-u+x)(n-2v) - 2(2n-u-4v+x)}{n(n-1)(n-2)(n-3)}$$

if $y = v$

$$= \frac{2(n-u+x)(n-y)-2(4n-3u-v+3x-3y)}{n(n-1)(n-2)(n-3)}$$

if $y = v+1, \ldots, u+v-x-1;\ x \neq u-1$

$$= \frac{2(n-u+x)(n-u-v+x)-(7n-11u-8v+11x)}{n(n-1)(n-2)(n-3)}$$

if $y = u+v-x$

$$= \frac{2(n-u+x)(n-y)-4(2n-u+x-2y)}{n(n-1)(n-2)(n-3)}$$

if $y = u+v-x+1, \ldots, n-1-u-v+x$

$$= \frac{2(n-u+x)(n-y)-2(3n-u+v+x-3y)}{n(n-1)(n-2)(n-3)}$$

if $y = n-u-v+x, \ldots, n-v-1$

$$= \frac{(n-u+x)(n+v-y)-2(2n-u+2v+x-2y)}{n(n-1)(n-2)(n-3)}$$

if $y = n-v, \ldots, n-1-u+v+x$

$$= \frac{(n-u+x)(n+v-y)-2(n+v-y)}{n(n-1)(n-2)(n-3)}$$

if $y = n+u-v+x, \ldots, n+v-1$

- when $x = 1, \ldots, v-u-1$

$$= \frac{(n-u-x)(n-v)-2(2n-u-2v-x)}{n(n-1)(n-2)(n-3)}$$

if $y = 0$

$$= \frac{2(n-u-x)(n-v)-4(2n-u-2v-x)}{n(n-1)(n-2)(n-3)}$$

if $y = 1, \ldots, v-u-x-1;\ x \neq v-u-1$

$$= \frac{2(n-u-x)(n-v)-(7n-3u-8v-3x)}{n(n-1)(n-2)(n-3)}$$

if $y = v - u - x$

$$= \frac{2(n-u-x)(n-v)-2(4n-3u-3v-3x-y)}{n(n-1)(n-2)(n-3)}$$

if $y = v - u - x + 1, \ldots, v - 1$

$$= \frac{(n-u-x)(n-2v)-2(2n-u-4v-x)}{n(n-1)(n-2)(n-3)}$$

if $y = v$

$$= \frac{2(n-u-x)(n-y)-2(4n-3u-v-3x-3y)}{n(n-1)(n-2)(n-3)}$$

if $y = v + 1, \ldots, u + v + x - 1$

$$= \frac{2(n-u-x)(n-u-v-x)-(7n-11-8v-11x)}{n(n-1)(n-2)(n-3)}$$

if $y = v + u + x$

$$= \frac{2(n-u-x)(n-y)-4(2n-u-x-2y)}{n(n-1)(n-2)(n-3)}$$

if $y = v + u + x + 1, \ldots, n - 1 - u - v - x$

$$= \frac{2(n-u-x)(n-y)-2(3n-u+v-x-3y)}{n(n-1)(n-2)(n-3)}$$

if $y = n - u - v - x, \ldots, n - v - 1$

$$= \frac{(n-u-x)(n+v-y)-2(2n-u+2v-x-2y)}{n(n-1)(n-2)(n-3)}$$

if $y = n - v, \ldots, n - 1 - u + v - x$

$$= \frac{(n-u-x)(n+v-y)-2(n+v-y)}{n(n-1)(n-2)(n-3)}$$

if $y = n - u + v - x, \ldots, n + v - 1$

• when $x = v - u$

$$= \frac{(n-v)^2 - (3n-5v)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = 0$$

$$= \frac{2(n-v)^2 - 2(4n-6v-y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = 1, \ldots, v-1$$

$$= \frac{(n-v)(n-2v) - 2(2n-5v)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = v$$

$$= \frac{2(n-v)(n-y) - 2(4n-4v-3y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = v+1, \ldots, 2v-1$$

$$= \frac{2(n-v)(n-2v) - (7n-13v)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = 2v$$

$$= \frac{2(n-v)(n-y) - 4(2n-v-2y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = 2v+1, \ldots, n-1-2v$$

$$= \frac{2(n-v)(n-y) - 6(n-y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = n-2v, \ldots, n-v-1$$

$$= \frac{(n-v)(n+v-y) - 2(2n+v-2y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = n-v, \ldots, n-1$$

$$= \frac{(n-v)(n+v-y) - 2(n+v-y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = n, \ldots, n+v-1$$

• when $x = v-u+1, \ldots, 2v-u-1$

$$= \frac{(n-u-x)(n-v) - 2(2n-2u-v-2x)}{n(n-1)(n-2)(n-3)}$$

if $y = 0$

$$= \frac{2(n-u-x)(n-v) - 4(2n-2u-v-2x)}{n(n-1)(n-2)(n-3)}$$

if $y = 1, \ldots, u-v+x-1;\ x \neq u-v+1$

$$= \frac{2(n-u-x)(n-v)-(7n-7u-4v-7x)}{n(n-1)(n-2)(n-3)}$$

if $y = u - v + x$

$$= \frac{2(n-u-x)(n-v)-2(4n-3u-3v-3x-y)}{n(n-1)(n-2)(n-3)}$$

if $y = u - v + x + 1, \ldots, v - 1;\ x \neq 2v - u - 1$

$$= \frac{(n-u-x)(n-2v)-2(2n-u-4v-x)}{n(n-1)(n-2)(n-3)}$$

if $y = v$

$$= \frac{2(n-u-x)(n-y)-2(4n-3u-v-3x-3y)}{n(n-1)(n-2)(n-3)}$$

if $y = v + 1, \ldots, n - 1 - u - v - x$

$$= \frac{2(n-u-x)(n-y)-2(3n-2u-2x-2y)}{n(n-1)(n-2)(n-3)}$$

if $y = n - u - v - x, \ldots, u + v + x - 1$

$$= \frac{2(n-u-x)(n-u-v-x)-(5n-7u-4v-7x)}{n(n-1)(n-2)(n-3)}$$

if $y = u + v + x$

$$= \frac{2(n-u-x)(n-y)-2(3n-u+v-x-3y)}{n(n-1)(n-2)(n-3)}$$

if $y = u + v + x + 1, \ldots, n - v - 1$

$$= \frac{(n-u-x)(n+v-y)-2(2n-u+2v-x-2y)}{n(n-1)(n-2)(n-3)}$$

if $y = n - v, \ldots, n - 1 - u + v - x$

$$= \frac{(n-u-x)(n+v-y)-2(n+v-y)}{n(n-1)(n-2)(n-3)}$$

if $y = n - u + v - x, \ldots, n + v - 1$

- when $x = 2v - u$

$$= \frac{(n-2v)(n-v) - 2(2n-5v)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = 0$$

$$= \frac{2(n-2v)(n-v) - 2(4n-10v)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = 1, \ldots, v-1$$

$$= \frac{(n-2v)^2 - (3n-10v)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = v$$

$$= \frac{2(n-2v)(n-y) - 2(4n-7v-3y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = v+1, \ldots, 3v-1$$

$$= \frac{2(n-2v)(n-3v) - (7n-30v)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = 3v$$

$$= \frac{2(n-2v)(n-y) - 8(n-v-y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = 3v+1, \ldots, n-1-3v$$

$$= \frac{2(n-2v)(n-y) - 2(3n-v-3y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = n-3v, \ldots, n-1-v$$

$$= \frac{(n-2v)(n+v-y) - 2(n+v-y)}{n(n-1)(n-2)(n-3)} \qquad \text{if } y = n-v, \ldots, n+v-1$$

- when $x = 2v - u + 1, \ldots, n - u - 2v - 1$

$$= \frac{(n-u-x)(n-v) - 2(2n-2u-v-2x)}{n(n-1)(n-2)(n-3)}$$

if $y = 0$

$$= \frac{2(n-u-x)(n-v) - 2(4n-4u-2v-4x)}{n(n-1)(n-2)(n-3)}$$

if $y = 1, \ldots, v-1$

$$= \frac{(n-u-x)(n-2v) - 2(2n-2u-2v-2x)}{n(n-1)(n-2)(n-3)}$$

if $y = v$

$$= \frac{2(n-u-x)(n-y)-2(4n-4u-4x-2y)}{n(n-1)(n-2)(n-3)}$$

if $y = v+1, \ldots, n-1-u-v-x;\ x \neq 2v-u+1;\ x \neq n-u-2v-1$

$$= \frac{2(n-u-x)(n-y)-2(3n-3u+v-3x-y)}{n(n-1)(n-2)(n-3)}$$

if $y = n-u-v-x, \ldots, u-v+x-1;\ x \neq 2v-u+1$

$$= \frac{2(n-u-x)(n-u+v-x)-(5n-7u+4v-7x)}{n(n-1)(n-2)(n-3)}$$

if $y = u-v+x$

$$= \frac{2(n-u-x)(n-y)-2(3n-2u-2x-2y)}{n(n-1)(n-2)(n-3)}$$

if $y = u-v+x+1, \ldots, n-1-u+v-x$

$$= \frac{2(n-u-x)(n-y)-2(2n-u-v-x-y)}{n(n-1)(n-2)(n-3)}$$

if $y = n-u+v-x, \ldots, u+v+x-1$

$$= \frac{2(n-u-x)(n-u-v-x)-(3n-3u-4v-3x)}{n(n-1)(n-2)(n-3)}$$

if $y = u+v+x$

$$= \frac{2(n-u-x)(n-y)-4(n-y)}{n(n-1)(n-2)(n-3)}$$

if $y = u+v+x+1, \ldots, n-v-1;\ x \neq n-u-2v-1$

$$= \frac{(n-u-x)(n+v-y)-2(n+v-y)}{n(n-1)(n-2)(n-3)}$$

if $y = n-v, \ldots, n+v-1$

- when $x = n - u - 2v, \ldots, n - 1 - u - v$

$$= \frac{(n-u-x)(n-v) - 2(2n-2u-v-2x)}{n(n-1)(n-2)(n-3)}$$

if $y = 0$

$$= \frac{2(n-u-x)(n-v) - 4(2n-2u-v-2x)}{n(n-1)(n-2)(n-3)}$$

if $y = 1, \ldots, n-1-u-v-x;\ x =\neq n-1-u-v$

$$= \frac{2(n-u-x)(n-v) - 2(3n-3u-v-3x+y)}{n(n-1)(n-2)(n-3)}$$

if $y = n-u-v-x, \ldots, v-1;\ x \neq n-u-2v$

$$= \frac{(n-u-x)(n-2v) - 2(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = v$

$$= \frac{2(n-u-x)(n-y) - 2(3n-3u+v-3x-y)}{n(n-1)(n-2)(n-3)}$$

if $y = v+1, \ldots, n-1-u+v-x$

$$= \frac{2(n-u-x)(n-y) - 4(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = n-u+v-x, \ldots, u-v+x-1$

$$= \frac{2(n-u-x)(n-u+v-x) - 3(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = u-v+x$

$$= \frac{2(n-u-x)(n-y) - 2(2n-u-v-x-y)}{n(n-1)(n-2)(n-3)}$$

if $y = u-v+x+1, \ldots, n-v-1$

$$= \frac{(n-u-x)(n+v-y)-2(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = n-v, \ldots, u+v+x-1$

$$= \frac{(n-u-x)^2-(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = u+v+x$

$$= \frac{(n-u-x)(n+v-y)-2(n+v-y)}{n(n-1)(n-2)(n-3)}$$

if $y = u+v+x+1, \ldots, n+v-1$

- when $x = n-u-v, \ldots, n-1-u$

$$= \frac{(n-u-x)(n-v)-2(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = 0$

$$= \frac{2(n-u-x)(n-v)-4(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = 1, \ldots, u+v+x-n;\ x \neq n-u-v$

$$= \frac{2(n-u-x)(n-v)-2(3n-3u-v-3x+y)}{n(n-1)(n-2)(n-3)}$$

if $y = u+v+x-n+1, \ldots, v-1;\ x \neq n-1-u$

$$= \frac{(n-u-x)(n-2v)-2(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = v$

$$= \frac{2(n-u-x)(n-y)-2(3n-3u+v-3x-y)}{n(n-1)(n-2)(n-3)}$$

if $y = v+1, \ldots, n-1-u+v-x;\ x \neq n-1-u$

$$= \frac{2(n-u-x)(n-y)-4(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = n-u+v-x, \ldots, u-v+x-1$

$$= \frac{2(n-u-x)(n-u+v-x)-3(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = u-v+x$

$$= \frac{2(n-u-x)(n-y)-2(2n-u-v-x-y)}{n(n-1)(n-2)(n-3)}$$

if $y = u-v+x+1, \ldots, n-v-1;\ x \neq n-1-u$

$$= \frac{(n-u-x)(n+v-y)-2(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = n-v, \ldots, u+v+x-1$

$$= \frac{(n-u-x)^2-(n-u-x)}{n(n-1)(n-2)(n-3)}$$

if $y = u+v+x$

$$= \frac{(n-u-x)(n+v-y)-2(n+v-y)}{n(n-1)(n-2)(n-3)}$$

if $y = u+v+x+1, \ldots, n+v-1;\ x \neq n-1-u.$

The remaining part of the distribution can be determined by the above probabilities, by noting that, for every y in the support of $A_{4,3}(v)$,

$$\Pr\{A_{2,1}(u) = x,\ A_{4,3}(v) = y\}$$

$$= \Pr\{A_{2,1}(u) = 2u-x,\ A_{4,3}(v) = y\} \text{ when } x = u+1, \ldots, 2u$$

$$= \Pr\{A_{2,1}(u) = x-2u,\ A_{4,3}(v) = y\} \text{ when } x = 2u+1, \ldots, n+u-1.$$

By the described distribution, after some calculations, we get

$$C_0(u, v) = -\frac{1}{45(n-1)^2(n-2)(n-3)n^2}$$
$$\times [n^7 - 6n^6 + 10n^5 - 10(u - 2 + v)(u^2 - u - uv - v + v^2)n^4$$
$$+ (15u^4 + 15v^4 - 90u^2v - 30v^3 - 15u^2 + 90uv - 15v^2$$
$$+ 30v - 11)n^3 - (3u^5 + 15u^4v - 30u^3v^2 + 6v^5 + 15u^4$$
$$- 30u^3v - 90u^2v^2 + 5u^3 - 15u^2v + 120uv^2 - 40v^3 + 15u^2$$
$$+ 30uv + 30v^2 - 38 + 4v - 6)n^2 - (20u^3v^3 - 3u^5 - 15u^4v$$
$$+ 60u^3v^2 + 30u^2v^3 - 6v^5 - 20u^3v - 50uv^3 + 15v^4 - 15u^3 - 15u^2v$$
$$- 60uv^2 + 50uv - 15v^2 + 18u + 6v)n + 30uv(v^2 - 1)(u^2 - 1)].$$

It can be shown that the above formula remains valid, for every n, whenever $u \le v$ and $u + v \le n + 2.$ By similar computations, when $u \le v$ and $u + v \ge n - 2,$ we get:

$$C_0(u, v) = -\frac{(v - n + 1)(v - n)(v - n - 1)}{45(n-1)^2(n-2)(n-3)n^2}[2n^4 - (15u + 9v - 3)n^3$$
$$+ (30u^2 + 15uv - 3v^2 + 15u + 24v - 23)n^2$$
$$- (20u^3 + 60u^2 + 15uv - 3v^2 - 50u + 15v - 18)n + 30u(u^2 - 1)].$$

Note that when $C_0(u, v)$ needs to be computed for $u > v,$ it is sufficient to interchange the roles of u and v in the above formulas. In addition, note that the two formulas give the same results when $u + v = n - 2, ..., n + 2.$

Turning to $C_1(u, v),$ it can be derived from the joint probability $\Pr\{A_{2,1}(u) = x, A_{3,1}(v) = y\},$ whose expressions are reported in the following when n is odd, $u, v = 1, ..., \frac{n-1}{2}$ and $u < v$:

• when $x = 0, \ldots, u-1$

$$= \frac{n-v}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n-v)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v-u+x-1$$

$$= \frac{n+u-2v-x}{n(n-1)(n-2)} \qquad \text{if } y = v-u+x$$

$$= \frac{2n-u-v+x-y}{n(n-1)(n-2)} \qquad \text{if } y = v-u+x+1, \ldots, v-1;\ x \neq u-1$$

$$= \frac{n-2v}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n-u+x-2y}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \ldots, n-v-1$$

$$= \frac{n-u+v+x-y}{n(n-1)(n-2)} \qquad \text{if } y = n-v, \ldots, n-u+v+x-1$$

• when $x = 1, \ldots, v-u-1;\ v-u \neq 1$

$$= \frac{n-v}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n-v)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v-u-x-1;\ x \neq v-u-1$$

$$= \frac{n+u-2v-x}{n(n-1)(n-2)} \qquad \text{if } y = v-u-x$$

$$= \frac{2n-u-v-x-y}{n(n-1)(n-2)} \qquad \text{if } y = v-u-x+1, \ldots, v-1;$$

$$= \frac{n-2v}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n-u-x-2y}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \ldots, n-v-1$$

$$= \frac{n-u+v-x-y}{n(n-1)(n-2)} \qquad \text{if } y = n-v, \dots, n-u+v-x-1$$

• when $x = v-u$

$$= \frac{2n-2v-y}{n(n-1)(n-2)} \qquad \text{if } y = 1, \dots, v-1$$

$$= \frac{n-2v}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n-v-2y}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \dots, n-v-1$$

$$= \frac{n-y}{n(n-1)(n-2)} \qquad \text{if } y = n-v, \dots, n-1$$

• when $x = v-u+1, \dots, 2v-u-1$

$$= \frac{n-u-x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n-u-x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \dots, u-v+x-1;\ x \neq v-u+1$$

$$= \frac{n-u-x}{n(n-1)(n-2)} \qquad \text{if } y = u-v+x$$

$$= \frac{2n-u-v-x-y}{n(n-1)(n-2)} \qquad \text{if } y = u-v+x+1, \dots, v-1;\ x \neq 2v-u-1$$

$$= \frac{n-2v}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n-u-x-2y}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \dots, n-v-1;$$

$$= \frac{n-u+v-x-y}{n(n-1)(n-2)} \qquad \text{if } y = n-v, \dots, n-u+v-x-1$$

• when $x = 2v-u$

$$= \frac{n-2v}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n-2v)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v-1$$

$$= \frac{2(n-v-y)}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \ldots, n-v-1$$

• when $x = 2v - u + 1, \ldots, v - u + (n-1)/2$

$$= \frac{n-u-x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n-u-x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v-1$$

$$= \frac{n-u-x}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n-2u+v-2x-y}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \ldots, u-v+x-1;\ x \neq 2v-u+1$$

$$= \frac{n-u+v-x-y}{n(n-1)(n-2)} \qquad \text{if } y = u-v+x$$

$$= \frac{2n-u-x-2y}{n(n-1)(n-2)} \qquad \text{if } y = u-v+x+1, \ldots, n-u+v-x-1;$$

$$= \frac{n-v-y}{n(n-1)(n-2)} \qquad \text{if } y = n-u+v-x, \ldots, n-v-1$$

• when $x = v - u + (n-1)/2 + 1, \ldots, n-u-1$

$$= \frac{n-u-x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n-u-x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v-1$$

$$= \frac{n-u-x}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n-2u+v-2x-y}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \ldots, n-u+v-x-1;\ x \neq n-u-1$$

$$= \frac{n-u-x}{n(n-1)(n-2)} \qquad \text{if } y = n-u+v-x, \ldots, u-v+x-1$$

$$= \frac{n-v-y}{n(n-1)(n-2)} \qquad \text{if } y = u-v+x+1, \ldots, n-v-1;\ x \neq n-u-1$$

- when $x = u+1, \ldots, (n-1)/2+u-v$

$$= \frac{n+u-v-x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n+u-v-x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v-1$$

$$= \frac{n+u-2v-x}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n+2u-v-2x-y}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \ldots, v-u+x-1;\ x \neq u+1$$

$$= \frac{n+2u-2v-2x}{n(n-1)(n-2)} \qquad \text{if } y = v-u+x$$

$$= \frac{2n+u-x-2y}{n(n-1)(n-2)} \qquad \text{if } y = v-u+x+1, \ldots, n+u-v-x-1;$$

$$x \neq (n-1)/2+u-v$$

$$= \frac{n+v-y}{n(n-1)(n-2)} \qquad \text{if } y = n+u-v-x, \ldots, n+v-1$$

- when $x = (n-1)/2+u-v+1, \ldots, n+u-2v-1$

$$= \frac{n+u-v-x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n+u-v-x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v-1$$

$$= \frac{n+u-2v-x}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n + 2u - v - 2x - y}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, n + u - v - x - 1;$$

$$x \neq n + u - 2v - 1$$

$$= \frac{n + u - x}{n(n-1)(n-2)} \qquad \text{if } y = n + u - v - x, \ldots, v - u + x - 1$$

$$= \frac{n + v - y}{n(n-1)(n-2)} \qquad \text{if } y = v - u + x + 1, \ldots, n + v - 1$$

- when $x = n + u - 2v, \ldots, n + u - v - 1$

$$= \frac{n + u - v - x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n + u - v - x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, n + u - v - x - 1;\ x \neq n + u - v - 1$$

$$= \frac{n + u - v - x + y}{n(n-1)(n-2)} \qquad \text{if } y = n + u - v - x, \ldots, v - 1;\ x \neq n + u - 2v$$

$$= \frac{n + u - x}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, v - u + x - 1$$

$$= \frac{n + v - y}{n(n-1)(n-2)} \qquad \text{if } y = v - u + x + 1, \ldots, n + v - 1$$

- when $x = n + u - v, \ldots, n + u - 1$

$$= \frac{n + u - v - x + y}{n(n-1)(n-2)} \qquad \text{if } y = v - u - n + x + 1, \ldots, v - 1;\ x \neq n + u - 1$$

$$= \frac{n + u - x}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, v - u + x - 1$$

$$= \frac{n + v - y}{n(n-1)(n-2)} \qquad \text{if } y = v - u + x + 1, \ldots, n + v - 1;\ x \neq n + u - 1.$$

The expression of $C_1(u, v)$ derived from the above distribution can be proved to be valid for all $u \leq v$; this unique expression is:

$$C_1(u,\ v) = \frac{1}{180(n-1)^2(n-2)n^2} \times [n^7 - 11n^6 + (60uv + 25)n^5$$

$$- (10u^3 + 10v^3 - 60u^2 + 240uv - 60v^2 + 50u - 50v - 5)n^4$$

$$+ (15u^4 - 60u^3v - 60uv^3 + 15v^4 + 50u^3 - 180u^2v + 180uv^2$$

$$- 70v^3 - 15u^2 + 420uv - 15v^2 - 50u + 70v - 26)n^3 + (60u^3v^2$$

$$+ 30uv^4 - 6v^5 - 45u^4 + 120u^3v + 180u^2v^2 + 15v^4 - 80u^3$$

$$- 450uv^2 + 70v^3 - 15u^2 - 60uv - 75v^2 + 140u - 4v + 16)n^2$$

$$- (20u^3v^3 + 120u^3v^2 + 60u^2v^3 + 30uv^4 - 6v^5 - 30u^4 - 20u^3v$$

$$- 140uv^3 + 30v^4 - 40u^3 - 60u^2v - 150uv^2 - 10v^3 + 30u^2$$

$$+ 140uv - 30v^2 + 40u + 16v)n + 40uv(u^2 - 1)(v^2 - 1)].$$

Finally, $C_2(u,\ v)$ is to be determined from the joint probability $\Pr\{A_{2,1}(u) = x,\ A_{3,2}(v) = y\}$, whose expressions when n is odd, $u, v = 1, ..., \frac{n-1}{2}$ and $u < v$, are:

- when $x = 0, ..., u - 1$

$$= \frac{n - u - v + x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n - u - v + x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, ..., v - 1$$

$$= \frac{n - u - 2v + x}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n - 2u - v + 2x - y}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, ..., u + v - x - 1$$

$$= \frac{n - 2u - 2v + 2x}{n(n-1)(n-2)} \qquad \text{if } y = u + v - x$$

$$= \frac{2n - u + x - 2y}{n(n-1)(n-2)} \qquad \text{if } y = u + v - x + 1, \ldots, n - u - v + x - 1$$

$$= \frac{n + v - y}{n(n-1)(n-2)} \qquad \text{if } y = n - u - v + x, \ldots, n + v - 1$$

- when $x = 1, \ldots, (n-1)/2 - u - v$

$$= \frac{n - u - v - x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n - u - v - x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v - 1$$

$$= \frac{n - u - 2v - x}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n - 2u - v - 2x - y}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, u + v + x - 1$$

$$= \frac{n - 2v - 2u - 2x}{n(n-1)(n-2)} \qquad \text{if } y = u + v + x$$

$$= \frac{2n - u - x - 2y}{n(n-1)(n-2)} \qquad \text{if } y = u + v + x + 1, \ldots, n - u - v - x - 1$$

$$= \frac{n + v - y}{n(n-1)(n-2)} \qquad \text{if } y = n - u - v - x, \ldots, n + v - 1$$

- when $x = (n-1)/2 - u - v + 1, \ldots, n - u - 2v - 1$

$$= \frac{n - u - v - x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n - u - v - x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v - 1$$

$$= \frac{n - u - 2v - x}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n - 2u - v - 2x - y}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, n - u - v - x - 1$$

$$= \frac{n - u - x}{n(n-1)(n-2)} \qquad \text{if } y = n - u - v - x, \ldots, u + v + x - 1$$

$$= \frac{n + v - y}{n(n-1)(n-2)} \qquad \text{if } y = u + v + x + 1, \ldots, n + v - 1$$

- when $x = n - u - 2v, \ldots, n - u - v - 1$

$$= \frac{n - u - v - x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n - u - v - x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, n - u - v - x - 1$$

$$= \frac{n - u - v - x + y}{n(n-1)(n-2)} \qquad \text{if } y = n - u - v - x, \ldots, v - 1$$

$$= \frac{n - u - x}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, u + v + x - 1$$

$$= \frac{n + v - y}{n(n-1)(n-2)} \qquad \text{if } y = u + v + x + 1, \ldots, n + v - 1$$

- when $x = n - v - u, \ldots, n - u - 1$

$$= \frac{n - u - v - x + y}{n(n-1)(n-2)} \qquad \text{if } y = u + v - n + x + 1, \ldots, v - 1$$

$$= \frac{n - u - x}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, u + v + x - 1$$

$$= \frac{n + v - y}{n(n-1)(n-2)} \qquad \text{if } y = u + v + x + 1, \ldots, n + v - 1$$

- when $x = u + 1, \ldots, u + v - 1$

$$= \frac{n - v}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n-v)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, u+v-x-1$$

$$= \frac{n-u-2v+x}{n(n-1)(n-2)} \qquad \text{if } y = u+v-x$$

$$= \frac{2n+u-v-x-y}{n(n-1)(n-2)} \qquad \text{if } y = u+v-x+1, \ldots, v-1$$

$$= \frac{n-2v}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n+u-x-2y}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \ldots, n-v-1$$

$$= \frac{n+u+v-x-y}{n(n-1)(n-2)} \qquad \text{if } y = n-v, \ldots, n+u+v-x-1$$

- when $x = u + v$

$$= \frac{2n-2v-y}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v-1$$

$$= \frac{n-2v}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n-v-2y}{n(n-1)(n-2)} \qquad \text{if } y = v+1, \ldots, n-v-1$$

$$= \frac{n-y}{n(n-1)(n-2)} \qquad \text{if } y = n-v, \ldots, n-1$$

- when $x = u + v + 1, \ldots, u + 2v - 1$

$$= \frac{n+u-x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n+u-x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, x-u-v-1$$

$$= \frac{n+u-x}{n(n-1)(n-2)} \qquad \text{if } y = x-u-v$$

$$= \frac{2n + u - v - x - y}{n(n-1)(n-2)} \qquad \text{if } y = x - u - v + 1, \ldots, v - 1$$

$$= \frac{n - 2v}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n + u - x - 2y}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, n - v - 1$$

$$= \frac{n + u + v - x - y}{n(n-1)(n-2)} \qquad \text{if } y = n - v, \ldots, n + u + v - x - 1$$

- when $x = u + 2v$

$$= \frac{n - 2v}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n - 2v)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v - 1$$

$$= \frac{2(n - v - y)}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, n - v - 1$$

- when $x = u + 2v + 1, \ldots, n + u - 1$

$$= \frac{n + u - x}{n(n-1)(n-2)} \qquad \text{if } y = 0$$

$$= \frac{2(n + u - x)}{n(n-1)(n-2)} \qquad \text{if } y = 1, \ldots, v - 1$$

$$= \frac{n + u - x}{n(n-1)(n-2)} \qquad \text{if } y = v$$

$$= \frac{2n + 2u + v - 2x - y}{n(n-1)(n-2)} \qquad \text{if } y = v + 1, \ldots, x - u - v - 1$$

$$= \frac{n + 2u + 2v - 2x}{n(n-1)(n-2)} \qquad \text{if } y = x - u - v$$

$$= \frac{2n + u - x - 2y}{n(n-1)(n-2)} \qquad \text{if } y = x - u - v + 1, \ldots, n + u + v - x - 1$$

$$= \frac{n - v - y}{n(n-1)(n-2)} \qquad \text{if } y = n + u + v - x, \ldots, n - v - 1.$$

The following expression for $C_2(u, v)$ is derived from the above distribution. The expression is valid for $u \leq v$ and $u + v \leq n + 2$. Another expression for $u + v \geq n - 2$ can be determined but, as below explained, it is not needed to complete the proof:

$$
\begin{aligned}
C_2(u, v) = \frac{1}{180(n-1)^2(n-2)n^2}&[n^7 - 11n^6 - (60uv - 25)n^5 \\
&- (10u^3 + 10v^3 - 60u^2 - 240uv - 60v^2 + 50u + 50v + 5)n^4 \\
&+ (15u^4 + 60u^3v + 60uv^3 + 15v^4 - 70u^3 - 180u^2v - 180uv^2 \\
&- 70v^3 - 15u^2 - 60uv - 15v^2 + 70u + 70v - 26)n^3 - (6u^5 + 30u^4v \\
&+ 30uv^4 + 6v^5 - 15u^4 - 180u^2v^2 - 15v^4 - 70u^3 - 30u^2v - 30uv^2 \\
&- 70v^3 + 75u^2 + 60uv + 75v^2 + 4u + 4v - 16)n^2 - (20u^3v^3 - 6u^5 \\
&- 30u^4v + 60u^3v^2 + 60u^2v^3 - 30uv^4 - 6v^5 + 30u^4 - 20u^3v \\
&- 20uv^3 + 30v^4 - 10u^3 - 30u^2v - 30uv^2 - 10v^3 - 30u^2 + 20uv \\
&- 30v^2 + 16u + 16v)n + 40uv(v^2 - 1)(u^2 - 1)].
\end{aligned}
$$

Now note that, as covariance is a symmetric operator, (6) can be computed by reducing the number of terms in the summation.

As a consequence,

$$
\begin{aligned}
Var(S \mid H_0) = 4\sum_{i=2}^{n}\sum_{u=1}^{i-1} Var(A_{2,1}(u)) + 8\sum_{i=2}^{n}\sum_{l=1}^{i-1}\Bigg[&\sum_{m=l+1}^{i-1} C_1(i-l,\, i-m) \\
&+ \sum_{h=i+1}^{n}\sum_{m=1}^{l-1} C_0(i-l,\, h-m) + \sum_{h=i+1}^{n} C_1(i-l,\, h-l) \\
&+ \sum_{h=i+1}^{n}\sum_{\substack{m=l+1 \\ (m \neq i)}}^{h-1} C_0(i-l,\, h-m) + \sum_{h=i+1}^{n} C_2(i-l,\, h-i)\Bigg].
\end{aligned}
$$

The terms in the summations of the above formula can be rearranged so that the reported formulas of covariance can be applied; for instance, the arguments of $C_0(i - l, h - m)$ should be switched when $i - l > h - m$. In addition, note that, in the last summation, a further expression for $C_2(u, v)$ for $u + v \geq n + 2$ is not needed, as the arguments $i - l$ and $h - i$ are such that $(i - l) + (h - i) \leq n - 1$. The result of this last summation is different when *n* is even or odd; this explains why the final expression for $Var(D|H_0)$ differs when *n* is odd or even.

After some calculations, we finally get

$$Var(S)$$

$$= \frac{(n+1)(1751n^6 - 3090n^5 - 9822n - 2173n^2 + 233n^4 + 1428n^3 + 4725)}{28350n(n-2)}$$

when *n* is odd

$$= \frac{1751n^6 - 3090n^5 - 5020n^4 + 192n^3 - 5344n^2 - 20448n - 10008}{28350(n-3)}$$

when *n* is even

from which, (4) is easily obtained.

References

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[2] C. G. Borroni and M. Zenga, A test of concordance based on Gini’s mean difference, Stat. Methods Appl. 16 (2007), 289-308.