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BLOW-UP OF SOLUTIONS FOR SOME NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract

In this paper, we give some new results of blow-up of solutions for some higher-order nonlinear Schrödinger equations. We give some new conclusion to extend blow-up of solutions for nonlinear Schrödinger equations in [12] by eigenfunction method.

1. Introduction

The Schrödinger equation is the important equation in nonlinear science and widely applied in natural sciences such as the nonlinear optics, plasma physics, biology laser-wave, communication and almost all branches of physics such as the fluid mechanics, as well as the condensed matter physics. Now, we may extend some known results as special in [12] under established past works [3, 4, 7, 8, 11], in this paper.

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As we all know, the solution of initial problem for Schrödinger equation below:

$$\begin{cases} u_t - ia^2\Delta u = f(x, t), & x \in R^n, \quad t > 0, \\ u(x, 0) = \varphi(x), & x \in R^n. \end{cases}$$

Assume the real part and imaginary part of $\varphi(x)$, $f(x, t)$ are real analytical functions for $x \in R^n$. Then this solution of the problem may be expressed in the form:

$$u(x, t) = \sum_{k=0}^{\infty} \frac{(ia^2)^k}{k!} \left(t^k \Delta^k \phi(x) + \int_0^t (t-\tau)^k \Delta_x^k f(x, \tau) d\tau \right).$$

We consider the nonlinear equation:

$$u_t - i\Delta^2 u = f(u, D_x u, D_x^2 u) + \Delta^2 g(u), \quad x \in \Omega, \ t > 0, \tag{1}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{2}$$

$$u|_{\partial\Omega} = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{3}$$

where Ω is a bounded domain in R^n with suite smooth boundary $\partial\Omega$, f and g are complex-valued functions, $u_0(x)$ is also enough smooth complex function.

By using eigenfunction method, we can get result below. That is, Theorem 1.

Lemma 1. *This eigenvalue problem* (*see* [12])

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, & x \in \Omega, \\ \partial\varphi|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

It is well known that the first eigenvalue $\lambda_1 \geq 0$ *of* (4), *the corresponding eigenfunction* $\phi_1(x) > 0$, *assume it with* $\int_\Omega \phi_1(x) dx = 1$.

2. Main Theorem

Theorem 1. *Assume that nonlinear equation of problem* (1)-(3) *satisfies*:

(i) $\frac{\partial G}{\partial n}|_{\partial\Omega} = G(0) = 0,\ G = \operatorname{Re}\Delta g(u) - \Delta \operatorname{Im} u;\ g(0) = 0;$

(ii) $A = \operatorname{Re}(f(u,\ D_x u,\ D_x^2 u) - \lambda g(u)) + \operatorname{Im} u - CF(\operatorname{Re} u),$

$$\alpha = \int_\Omega \phi \operatorname{Re} u_0 dx, \quad A \cdot \alpha \geq 0;$$

$F(u)$ *is continuous convex and even function*, $C = \begin{cases} 1, & \alpha > 0, \\ -1, & \alpha < 0, \end{cases}$

(iii) $F(s) > 0\ (s > \alpha)$ *and* $\int_\alpha^\infty \frac{ds}{F(s)} < \infty.$

Then the classical solution of problem (1)-(3) *is blow-up in finite time*.

Proof. I Step. Let $A \geq 0,\ \alpha > 0$ and $A\alpha \geq 0.$ Taking the real part of both the sides for (1), we get

$$\operatorname{Re} u_t - \operatorname{Re} i\Delta^2 u = \operatorname{Re} f(u,\ D_x u,\ D_x^2 u) + \operatorname{Re}\Delta^2 g(u),$$

$$\operatorname{Re} u_t - \Delta^2 \operatorname{Im} u = \operatorname{Re} f(u,\ D_x u,\ D_x^2 u) + \operatorname{Re}\Delta^2 g(u). \tag{5}$$

Multiplying by $\varphi(x)$ both the sides of (5) and integral on Ω for x, we have

$$\int_\Omega \phi \operatorname{Re} u_t dx = \int_\Omega \phi(-\operatorname{Im} u + \operatorname{Re} f(u,\ D_x u,\ D_x^2 u) + \Delta^2 \operatorname{Re} g(u)) dx.$$

Taking $a(t) = \int_\Omega \varphi \operatorname{Re} u dx,$ then $a'(t) = \int_\Omega \varphi \operatorname{Re} u_t dx,$ and that

$$a'(t) = \int_\Omega \phi(-\Delta^2 \operatorname{Im} u + \operatorname{Re} f(u,\ D_x u,\ D_x^2 u) + \Delta^2 \operatorname{Re} g(u)) dx$$

$$= \int_\Omega (\phi\Delta(\Delta \operatorname{Re} g(u) - \Delta \operatorname{Im} u) - \Delta\phi(\operatorname{Re}\Delta g(u) - \Delta(\operatorname{Im} u))) dx$$

$$+ \int_\Omega (\phi \operatorname{Re} f(u,\ D_x u,\ D_x^2 u) - \Delta\phi(\Delta \operatorname{Im} u - \Delta \operatorname{Re} g(u))) dx. \tag{6}$$

By $g(u)|_{\partial\Omega} = g(0) = 0$ and Green's second formula, we have

$$\int_\Omega \phi\Delta(\Delta \operatorname{Re} g(u) - \Delta \operatorname{Im} u)dx = \int_\Omega \Delta\phi(\Delta \operatorname{Re} g(u) - \Delta(\operatorname{Im} u))dx. \tag{7}$$

Substituting (7) into (6), we get

$$\begin{aligned} a'(t) &= \int_\Omega (\phi \operatorname{Re} f(u, D_x u, D_x^2 u) - \Delta\phi(\Delta \operatorname{Im} u - \Delta \operatorname{Re} g(u)))dx \\ &= \int_\Omega \phi(\operatorname{Re} f(u, D_x u, D_x^2 u) + \lambda(\Delta \operatorname{Im} u - \Delta \operatorname{Re} g(u)))dx \\ &= \int_\Omega (\lambda\phi(\operatorname{Re} g(u) - \operatorname{Im} u) - \lambda\Delta\phi(\operatorname{Im} u - \operatorname{Re} g(u)))dx \\ &\quad + \int_\Omega (\phi(\operatorname{Re} f(u, D_x u, D_x^2 u)) - \lambda\Delta\phi(\operatorname{Re} g(u) - \operatorname{Im} u))dx \\ &= \int_\Omega \phi(\operatorname{Re} f(u, D_x u, D_x^2 u) + \lambda^2 g(u) - \lambda^2 \operatorname{Im} u)dx. \end{aligned} \tag{8}$$

From $A \geq 0$, $A = \operatorname{Re}(f(u, D_x u, D_x^2 u) + \lambda^2 g(u)) - \lambda^2 \operatorname{Im} u - CF(\operatorname{Re} u) \geq 0$, then

$$\operatorname{Re}(f(u, D_x u, D_x^2 u) + \lambda^2 g(u)) \geq \lambda^2 \operatorname{Im} u + CF(\operatorname{Re} u). \tag{9}$$

Combining (8) and (9) and Jensen inequality, we obtain

$$a'(t) = \int_\Omega \phi(\operatorname{Re} u)dx \geq F\left(\int_\Omega \phi \operatorname{Re} u dx\right) = F(a(t)). \tag{10}$$

Here

$$F(a(t)) \leq \frac{da}{dt}.$$

So, $t \leq \int_\alpha^{a(t)} \frac{da}{F(a)}$, there exists $T \leq \int_\alpha^\infty \frac{da}{F(a)} < +\infty$ such that

$$\lim_{t \to T} a(t) = +\infty. \tag{11}$$

From $a(t) = \int_\Omega \varphi \operatorname{Re} u dx$ and Holder inequality, we get $(1/p + 1/q = 1)$

$$a(t) = \int_\Omega \varphi \operatorname{Re} u dx \le \| \varphi \|_{L^q(\Omega)} \cdot \| \operatorname{Re} u \|_{L^p(\Omega)},$$

that is, $\dfrac{a(t)}{\| \varphi \|_{L^q(\Omega)}} \le \| \operatorname{Re}(u) \|_{L^p(\Omega)}.$

Therefore,

$$\lim_{t \to T} \frac{a(t)}{\| \phi \|_{L^q(\Omega)}} \le \lim_{t \to T} \| \operatorname{Re} u \|_{L^p(\Omega)}.$$

Hence,

$$\lim_{t \to T} \| \operatorname{Re} u \|_{L^p(\Omega)} = +\infty, \quad \forall 1 \le p \le +\infty.$$

II Step. When $A \le 0$, $\alpha < 0$;

$$a_1(t) = -a(t), \quad a_1'(t) = -a'(t), \quad \alpha_1 = -\alpha > 0. \tag{12}$$

Combining (1)-(8) and $A \le 0$, $(C = -1)$, we obtain

$$-a_1'(t) \le -\int_\Omega \phi F(-\operatorname{Re} u_1) dx, \text{ that is also } a'(t) \ge \int_\Omega \phi F(-\operatorname{Re} u_1) dx.$$

From Jensen inequality and $F(s)$ is even function that

$$F(a_1) = F(-a_1) \le \frac{da_1}{dt},$$

then

$$dt \le \frac{da_1}{F(a_1)}. \tag{13}$$

From (13) and similar I Step, we can get $\lim_{t \to T} \| \operatorname{Re} u_1 \|_{L^p(\Omega)} = +\infty$, $\forall 1 \le p \le +\infty$,

$$\lim_{t \to T} \| \operatorname{Re} u(t) \|_{L^p(\Omega)} = \lim_{t \to T} \| -\operatorname{Re} u(t) \|_{L^p(\Omega)} = +\infty, \quad \forall 1 \le p \le +\infty.$$

Combining I and II, we complete the proof of Theorem 1.

Theorem 2. *The nonlinear equation of problem* (1)-(3) *satisfies*:

(i) $\frac{\partial G}{\partial n}|_{\partial\Omega} = G(0) = 0,\ G = \operatorname{Re}\Delta g(u) - \Delta \operatorname{Im} u;$

(ii) $B = \frac{\operatorname{Im}(f(u,\ D_x u,\ D_x^2 u) + \lambda^2 g(u)) - \lambda^2(\operatorname{Re} u)}{F(\operatorname{Im} u)},\quad |B| - 1 \geq 0,$

$$\beta = \int_\Omega \phi \operatorname{Im} u_0 dx < 0,$$

where $F(s)$ *is continuous*, *convex and even function* (*see* [7]).

(iii) $F(s) > 0,\ (s > \alpha)$ *and* $\int_\beta^{+\infty} \frac{ds}{F(s)} < +\infty.$

Then the classical solution of problem (1)-(3) *is blow-up in finite time*.

Proof. From $|B| - 1 \geq 0,$ we discuss the two cases:

I. $B - 1 \geq 0,\ \beta < 0,\ u(x,\ t) = i\bar{u}_2(x,\ t),$ then $\operatorname{Im} u = \operatorname{Re}\bar{u}_2.$

Taking the imaginary part of (1), similar the method of Theorem 1, we can easily have

$$\lim_{t\to T} \| \operatorname{Re} u_2 \|_{L^p(\Omega)} = +\infty, \quad \forall 1 \leq p \leq +\infty.$$

So, we get that

$$\lim_{t\to T} \| \operatorname{Im} u(t) \|_{L^p(\Omega)} = +\infty, \quad \forall 1 \leq p \leq +\infty.$$

II. $B + 1 \leq 0,\ \beta < 0,$ we may let $u(x,\ t) = -\bar{u}_3(x,\ t).$ Then $\operatorname{Im} u = \operatorname{Im}\bar{u}_3.$

So,

$$\lim_{t\to T} \| \operatorname{Im} u_3(t) \|_{L^p(\Omega)} = +\infty, \quad \forall 1 \leq p \leq +\infty.$$

We get that

$$\lim_{t\to T} \| \operatorname{Im} u(t) \|_{L^p(\Omega)} = +\infty, \quad \forall 1 \leq p \leq +\infty.$$

Combining I and II, we can complete the proof of Theorem 2.

3. Concluding Remarks

Recently, the higher-order partial differential equations are also a very interesting topic, and we may see [1, 2, 5, 7, 8, 10-12]. By using of wave function as solution of the Schrödinger differential equation, in non-relativistic quantum mechanics, and corresponding engineering for a physical systems contain all possible information about the physical properties of a system. The corresponding system with some special case that the exact solutions of the Schrödinger equation for different potentials have been derived and widely used in adverse fields of molecular, atomic, nuclear, and subnuclear physics (see [6, 9, 13]).

In our future work, we will study some problem in this field and will obtain some better results.

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