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ON IRRATIONAL NUMBERS WHOSE SUM, DIFFERENCE, QUOTIENT AND PRODUCT ARE IRRATIONAL NUMBERS

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Abstract

Suppose that A and B are positive irrational numbers. In this paper, we find the criterion that $A \pm B$, $\frac{A}{B}$ and AB are all irrational numbers.

1. Introduction

Let $a_1, a_2, a_3, \ldots;\ b_1, b_2, b_3, \ldots$ be integers with $a_1 \geq 0, a_2 > 0, a_3 > 0, \cdots; b_1 \geq 0, b_2 > 0, b_3 > 0, \cdots$ all along this note. Nettler [1] proved the following theorem:

For

$$A = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 + \cdots} \text{ and } B = b_1 + \frac{1}{b_2 +}\,\frac{1}{b_3 + \cdots},$$

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if $a_n > b_n > a_{n-1}^{(n-1)^2}$ for all n sufficiently large, then A, B, $A \pm B$, $\frac{A}{B}$ and AB are all trancendental numbers.

The aim of this note is to prove the following theorem that is relevant to the above theorem.

Theorem. *Let* $a_1, a_2, a_3, \ldots;$ $b_1, b_2, b_3, \ldots$ *be integers with* $a_1 \geq 0$, $a_2 \geq 2$, $a_3 > 0$, $a_4 > 0, \cdots;$ $b_1 \geq 0$, $b_2 > 0$, $b_3 > 0, \cdots$. *For*

$$A = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 + \cdots} \text{ and } B = b_1 + \frac{1}{b_2 +}\,\frac{1}{b_3 + \cdots},$$

if $a_n > b_n > a_{n-1}^{\gamma(n-1)}$ *for all positive integer* $n \geq 3$, *then* $A \pm B$, $\frac{A}{B}$ *and* AB *are all irrational numbers, where* γ *is any constant such that* $\gamma > 8$.

We give an elementary proof of this theorem using the method of Nettler.

2. Lemmas

Lemma 1. *If*

$$A(n) = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 + \cdots +}\,\frac{1}{a_n} = \frac{{}^aP_n}{{}^aQ_n} \tag{1}$$

$$B(n) = b_1 + \frac{1}{b_2 +}\,\frac{1}{b_3 + \cdots +}\,\frac{1}{b_n} = \frac{{}^bP_n}{{}^bQ_n} \tag{2}$$

then, for all $n \geq 1$, *we have*

$$A(n) + B(n) = a_1 + b_1 + \frac{a_2 + b_2}{a_2 b_2 +}\,\frac{a_2 b_2 F_3}{E_3 - F_3 +}\,\frac{E_3 F_4}{E_4 - F_4 +}$$

$$\frac{E_4 F_5}{E_5 - F_5 + \cdots +}\,\frac{E_{n-1} F_n}{E_n - F_n},$$

$$E_n = {}^aQ_n{}^bQ_n({}^aQ_{n-2}{}^aQ_{n-1} + {}^bQ_{n-2}{}^bQ_{n-1}),$$

$$F_n = {}^aQ_{n-2}{}^bQ_{n-2}({}^aQ_{n-1}{}^aQ_n + {}^bQ_{n-1}{}^bQ_n), \tag{3}$$

$$B(n) - A(n) = b_1 - a_1 + \frac{a_2 - b_2}{a_2b_2} \, {}_{+} \, \frac{a_2b_2H_3}{G_3 - H_3} \, {}_{+} \, \frac{G_3H_4}{G_4 - H_4} \, {}_{+}$$

$$\frac{G_4H_5}{G_5 - H_5} \, {}_{+\cdots+} \, \frac{G_{n-1}H_n}{G_n - H_n},$$

$$G_n = {}^aQ_n{}^bQ_n({}^aQ_{n-2}{}^aQ_{n-1} - {}^bQ_{n-2}{}^bQ_{n-1}),$$

$$H_n = {}^aQ_{n-2}{}^bQ_{n-2}({}^aQ_{n-1}{}^aQ_n - {}^bQ_{n-1}{}^bQ_n), \tag{4}$$

$$\frac{A(n)}{B(n)} = \frac{a_1}{b_1} + \frac{b_1b_2 - a_1a_2}{a_2b_1(b_1b_2 + 1)} \, {}_{+} \, \frac{a_2b_1(b_1b_2 + 1)J_3}{I_3 - J_3} \, {}_{+}$$

$$\frac{I_3J_4}{I_4 - J_4} \, {}_{+} \, \frac{I_4J_5}{I_5 - J_5} \, {}_{+\cdots+} \, \frac{I_{n-1}J_n}{I_n - J_n},$$

$$I_n = {}^aQ_n{}^bP_n({}^aQ_{n-2}{}^aP_{n-1} + {}^bQ_{n-2}{}^bP_{n-1}),$$

$$J_n = {}^aQ_{n-2}{}^bP_{n-2}({}^aQ_n{}^aP_{n-1} + {}^bQ_n{}^bP_{n-1}), \tag{5}$$

$$A(n)B(n) = a_1b_1 + \frac{a_1a_2 + b_1b_2 + 1}{a_2b_2} \, {}_{+} \, \frac{a_2b_2L_3}{K_3 - L_3} \, {}_{+}$$

$$\frac{K_3L_4}{K_4 - L_4} \, {}_{+} \, \frac{K_4L_5}{K_5 - L_5} \, {}_{+\cdots+} \, \frac{K_{n-1}L_n}{K_n - L_n},$$

$$K_n = {}^aQ_n{}^bQ_n({}^aQ_{n-2}{}^aP_{n-1} + {}^bQ_{n-1}{}^bP_{n-2}),$$

$$L_n = {}^aQ_{n-2}{}^bQ_{n-2}({}^aQ_n{}^aP_{n-1} + {}^bQ_{n-1}{}^bP_n). \tag{6}$$

Proof. See Theorem 2.1 in [2].

Lemma 2. *Let*

$$C = e_1 + \frac{d_2}{e_2} \, {}_{+} \, \frac{d_3}{e_3} \, {}_{+\cdots}$$

be the continued fraction expantion, given in Lemma 1, *for either* $A \pm B$, $\dfrac{A}{B}$ *or AB. And let*

$$\frac{{}^{c}P_n}{{}^{c}Q_n} = e_1 + \frac{d_2}{e_2 +}\,\frac{d_3}{e_3 +}\cdots+\,\frac{d_n}{e_n}.$$

If $a_n > b_n$ *for sufficient large n, then* ${}^{c}Q_n < {}^{a}Q_n^{\alpha n}$, *where* α *is any constant such that* $\alpha > 8$.

Proof. From (3), (4), (5) and (6), we have the following inequalities

$$\begin{aligned} {}^{c}Q_n &= e_n{}^{c}Q_{n-1} + d_n{}^{c}Q_{n-2} < {}^{c}Q_{n-1}(d_n + e_n) \\ &< \cdots < \prod_{i=2}^{n}(d_i + e_i) < \prod_{i=2}^{n}{}^{a}Q_i^{\alpha} < {}^{\alpha}Q_n^{\alpha n} \end{aligned}$$

for all n sufficient large.

2. Proof of the Theorem

Now let

$$A + B = C = e_1 + \frac{d_2}{e_2 +}\,\frac{d_3}{e_3 +}\cdots$$

and

$$\frac{{}^{a}P_n}{{}^{a}Q_n} + \frac{{}^{b}P_n}{{}^{b}Q_n} = \frac{{}^{c}P_n}{{}^{c}Q_n} \text{ for } n \geq 1.$$

Let n be a sufficiently large integer to ensure the validity of the later argument. We have

$$\begin{aligned} \left|C - \frac{{}^{c}P_n}{{}^{c}Q_n}\right| &\leq \left|A - \frac{{}^{a}P_n}{{}^{a}Q_n}\right| + \left|B - \frac{{}^{b}P_n}{{}^{b}Q_n}\right| < \frac{1}{{}^{a}Q_n{}^{a}Q_{n+1}} + \frac{1}{{}^{b}Q_n{}^{b}Q_{n+1}} \\ &< \frac{2}{{}^{b}Q_n{}^{b}Q_{n+1}} < \frac{2}{b_{n+1}\cdot{}^{b}Q_n^2} < \frac{2}{a_n^{\gamma n}\cdot{}^{b}Q_n^2}. \end{aligned}$$

And we have

$$ {}^{a}Q_n < (a_n + 1) \cdot {}^{a}Q_{n-1} \le (2a_n) \cdot {}^{a}Q_{n-1} < \prod_{i=2}^{n} (2a_i) = 2^{n-1} a_2 a_3 \cdots a_n $$

$$ < 2^{n-1} \cdot a_n^{1+\frac{1}{\gamma(n-1)}+\frac{1}{\gamma^2(n-1)(n-2)}+\cdots+\frac{1}{\gamma^{n-2}(n-1)(n-2)\cdots 3\cdot 2}} $$

$$ < 2^{n-1} \cdot a_n^{1+\frac{1}{n-1}\cdot\frac{1}{\gamma-1}} < 2^{n-1} \cdot a_n^{\frac{\gamma}{\delta}}, $$

where δ is any constant such that $8 < \delta < \gamma$. Hence

$$ a_n^{\gamma n} > \frac{{}^{a}Q_n^{\delta n}}{2^{\delta n(n-1)}}. $$

From Lemma 2, we obtain

$$ \left| C - \frac{{}^{c}P_n}{{}^{c}Q_n} \right| < \frac{2}{{}^{a}Q_n^{\delta n} \dfrac{{}^{b}Q_n^2}{2^{\delta n(n-1)}}} < \frac{2}{{}^{c}Q_n \dfrac{{}^{b}Q_n^2}{2^{\delta n(n-1)}}} = \frac{1}{{}^{c}Q_n M_n}, $$

where $M_n = \frac{1}{2} \cdot \frac{{}^{b}Q_n^2}{2^{\delta n(n-1)}}$. And we obtain

$$ {}^{b}Q_n > b_n \cdot {}^{b}Q_{n-1} > a_{n-1}^{\gamma(n-1)} \cdot {}^{b}Q_{n-1} > \cdots > (a_{n-1}^{n-1} a_{n-2}^{n-2} \cdots a_2^2)^{\gamma} $$

$$ > (2^{2!+\gamma\cdot 3!+\gamma^2\cdot 4!+\cdots+\gamma^{n-3}\cdot(n-1)!})^{\gamma} > 2^{\gamma^{n-2}\cdot(n-1)!}. $$

Then,

$$ \lim_{n\to\infty} \frac{1}{M_n} = 0. $$

Therefore $A + B$ is an irrational number. Similarly, it can be proven easily that $A - B$ is also an irrational number.

To prove that $\frac{A}{B}$ is an irrational number, let

$$\frac{A}{B} = C = e_1 + \frac{d_2}{e_2 +}\,\frac{d_3}{e_3 +\cdots}$$

and

$$\frac{{}^aP_n}{{}^aQ_n}\bigg/\frac{{}^bP_n}{{}^bQ_n} = \frac{{}^cP_n}{{}^cQ_n} \text{ for } n \ge 1.$$

As

$$\frac{{}^aP_n}{{}^aQ_n} \le 2a_1 \text{ and } \frac{1}{2b_2} \le \frac{{}^bP_n}{{}^bQ_n}$$

for $n \ge 3.$

Let n be an integer with $n \ge 3.$ We obtain

$$\left| C - \frac{{}^cP_n}{{}^cQ_n} \right| = \left| \frac{A}{B} - \frac{{}^aP_n/{}^aQ_n}{{}^bP_n/{}^bQ_n} \right|$$

$$= \frac{\left| \frac{{}^bP_n}{{}^bQ_n}\left(A - \frac{{}^aP_n}{{}^aQ_n}\right) + \frac{{}^aP_n}{{}^aQ_n}\left(\frac{{}^bP_n}{{}^bQ_n} - B\right) \right|}{B({}^bP_n/{}^bQ_n)}$$

$$\le \frac{\left| A - \frac{{}^aP_n}{{}^aQ_n} \right|}{B} + \frac{2a_1\left| B - \frac{{}^bP_n}{{}^bQ_n} \right|}{B/(2b_2)}$$

$$< \frac{4a_1b_2 + 1}{B} \cdot \frac{1}{{}^bQ_n{}^bQ_{n+1}} = o\left(\frac{1}{{}^cQ_n}\right) \text{ as } n \to \infty.$$

Therefore $\frac{A}{B}$ is an irrational number. Similarly, it can be proven easily that AB is also an irrational number.

3. Example

Let

$$A = 2^{2!} + \frac{1}{2^{4!}} + \frac{1}{2^{6!}} + \cdots + \frac{1}{2^{(2n)!}} + \cdots$$

and

$$B = 2^{5\cdot 1!} + \frac{1}{2^{5\cdot 3!}} + \frac{1}{2^{5\cdot 5!}} + \cdots + \frac{1}{2^{5\cdot(2n-1)!}} + \cdots.$$

Then $A \pm B$, $\frac{A}{B}$ and AB are all irrational numbers.

Proof. Now we put $a_n = 2^{(2n)!}$, $b_n = 2^{5\cdot(2n-1)!}$. First we can see easily that $a_n > b_n$ for $n \geq 3$. And we have

$$\frac{\log b_n}{(n-1)\log a_{n-1}} = \frac{5\cdot(2n-1)!}{(n-1)\cdot(2n-2)!} = \frac{5(2n-1)}{n-1} > 10$$

for $n \geq 2$. Therefore $b_n > a_{n-1}^{18(n-1)}$ for $n \geq 3$. From the Theorem, $A \pm B$, $\frac{A}{B}$ and AB are all irrational numbers.

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