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TOPOLINE SET GRACEFUL GRAPHS

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Abstract

In this paper, we introduce the concept of a topoline set indexer which induces a topology on the edge set of a graph. Unlike topological set indexers, not all graphs have topoline set indexers and this caused the origin of topoline graphs. Every topoline graph admits a topoline number which in turn produces topoline set graceful graphs. It is derived that all set-graceful graphs are topoline and topoline set graceful stars are topogenic. We have also identified many families of topoline as well as topoline set graceful graphs.

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1. Introduction

Labeling of graphs is an active area of research that gained a wider interest because of its practical applications. After the awakening paper of Rosa [18] in 1967, many have come up with different types of labeling for discrete structures. Following this, in 1986, Acharya [1] introduced set-valuation of graphs in which the vertices and edges of a graph are labeled using the subsets of a given set. In connection with this, he introduced the notions of set-indexing number and topological number of a graph. Further, a graph for which these two numbers are equal is termed topologically set graceful or t-set graceful.

This attracted many and a lot of significant results on set-indexers and topological numbers of various graphs were obtained through different papers [3, 4, 12, 13, 15, 16, 21, 22, 23] and [24]. In [26], we have identified certain families of graphs for which every spanning subgraph is topologically set graceful and this led to the notion of strongly t-set graceful graphs.

In this paper, we introduce the concept of a topoline set indexer analogous to that of a topological set indexer. While the latter defines a topology on the vertex set of the graph, the former induces a topology on the edge set. Unlike topological set indexers, not all graphs have topoline set indexers and this caused the origin of topoline graphs. It is noted that set-graceful graphs form a proper subfamily of topoline graphs. However, it is conjectured that a complete graph is topoline if and only if it is set-graceful. We have also identified many families of topoline graphs including trees, fans, etc.

Further, the graphs with same set-indexing number and topoline number are defined to be topoline set graceful. Even though the notions of topologically set graceful and topoline set graceful are independent in general, they coincide in the case of a star graph. Also, it is derived that all topoline set graceful stars are topogenic. Finally, several families of topoline set graceful graphs have been obtained.

2. Preliminaries

In this section, we include certain definitions and known results needed for the subsequent development of the study. Throughout this paper, m and n stand for natural numbers without restrictions unless otherwise mentioned. For a nonempty set X, the set of all subsets of X is denoted by 2^X. We always denote a graph under consideration by G and its vertex and edge sets by V and E, respectively, and G' being a subgraph of a graph G is denoted by $G' \subseteq G$. When G' is a proper subgraph of G we denote it by $G' \subset G$. Graphs considered in this paper are assumed to be nonempty unless otherwise mentioned. By the term graph we mean a simple graph and the basic notations and definitions of graph theory are assumed to be familiar to the readers.

Definition 2.1 [1]**.** Let $G = (V, E)$ be a given graph and X be a nonempty set. Then a mapping $f : V \to 2^X$ or $f : E \to 2^X$ or $f : V \cup E \to 2^X$ is, respectively, called a *set-assignment* or *set-valuation* of the vertices or *edges* or both.

Definition 2.2 [1]**.** Let G be a given graph and X be a nonempty set. Then a set-valuation $f : V \cup E \to 2^X$ is said to be a *set-indexer* of G if

(1) $f(u, v) = f(u) \oplus f(v)$; $\oplus$ is the symmetric difference and

(2) the restrictive maps $f|_V$ and $f|_E$ are both injective.

In this case, X is called an indexing set of G. Clearly, a graph can have many indexing sets and the minimum of the cardinalities of the indexing sets is said to be the *set-indexing number* of G, denoted by $\gamma(G)$. The set-indexing number of the trivial graph K_1 is defined to be zero.

Theorem 2.3 [1]**.** *Every graph has a set-indexer*.

Theorem 2.4 [1]**.** *If X is an indexing set of* $G = (V, E)$. *Then*

(i) $|E| \leq 2^{|X|} - 1$ *and*

(ii) $\lceil \log_2(|E| + 1) \rceil \leq \gamma(G) \leq |V| - 1$, *where* $\lceil \ \rceil$ *is the ceiling function*.

Theorem 2.5 [1]**.** *If* G' *is subgraph of* G, *then* $\gamma(G') \leq \gamma(G)$.

Theorem 2.6 [26]**.** *For any graph* G, $\lceil \log_2(|V|) \rceil \leq \gamma(G)$.

Theorem 2.7 [24]**.** $\gamma(P_m) = n + 1;\ 2^n \leq m \leq 2^{n+1} - 1$.

Theorem 2.8 [1]**.** *Let G be any graph and* $\Theta_X(G)$ *denote the set of all optimal set-indexers f of G with respect to a set X such that* $f(u) = \varnothing$ *for some* $u \in V(G)$. *Then* $\Theta_X(G)$ *is nonempty*.

Definition 2.9 [1]**.** A graph G is said to be *set-graceful* if $\gamma(G) = \log_2(|E| + 1)$ and the corresponding optimal set-indexer is called a *set-graceful labeling* of G.

Theorem 2.10 [15]**.** *Every cycle* $C_{2^n - 1};\ n \geq 2$ *is set-graceful*.

Theorem 2.11 [1]**.** *For any integer* $n \geq 2$, *the path* P_{2^n} *with* 2^n *vertices is not set-graceful*.

Definition 2.12 [1]**.** A set-indexer *f* of a graph *G* with indexing set *X* is said to be a *topological set indexer* (*t-set indexer*) if $f(V)$ is a topology on *X* and *X* is called the *topological indexing set* (*t-indexing set*) of *G*. The minimum number among the cardinalities of such topological indexing sets is said to be the *topological number* (*t-number*) of *G* and is denoted by $\tau(G)$ and the corresponding t-set indexer is optimal.

Theorem 2.13 [1]**.** *Every graph with at least two vertices has a t-set indexer*.

Theorem 2.14 [1]**.** *Let G be any graph with at least two vertices. Then* $\gamma(G) \leq \tau(G)$.

Theorem 2.15 [23]**.** *If* G' *is a spanning subgraph of* G, *then* $\tau(G') \leq \tau(G)$.

Theorem 2.16 [23]**.** *If* G *be a graph of order* m; $3.2^{n-2} < m < 2^n (n \geq 3)$, *then* $\tau(G) \geq n+1$.

Definition 2.17 [1]**.** A graph G is said to be *topologically set graceful* or *t-set graceful* if $\gamma(G) = \tau(G)$, where $\gamma(G)$ and $\tau(G)$ are the set-indexing number and t-number of G.

Theorem 2.18 [26]**.** K_n is t-set graceful if and only if $2 \leq n \leq 5$.

Definition 2.19 [19]**.** The join $K_1 \vee P_{n-1}$ of K_1 and P_{n-1} is called a *fan graph* and is denoted by F_n.

Definition 2.20 [12]**.** A set-indexer f of G with indexing set X is said to be topogenic if the family $f(V) \cup f(E)$ is a topology on X, where $f(V) = \{f(v); v \in V\}$ and $f(E) = \{f(u) \oplus f(v); (u, v) \in E\}$.

Theorem 2.21 [25]**.** $T(n+1, 2^n + 2^m) \geq 1$, *where* $T(n, k)$ *is the number of topologies on n points having k open sets and* $0 \leq m \leq n-1$.

Theorem 2.22 [7]**.** *For* $n \geq 3$, *there is no topology on n points having k open sets*; $3.2^{n-2} < k < 2^n$.

Theorem 2.23 [17]**.** $T(4, 11) = 0$ *and* $T(5, 11) \geq 1$.

Definition 2.24 [16]**.** Embedding is a mapping ζ of the vertices of G into the set of vertices of a graph H such that the subgraph induced by the set $\{\zeta(u) : u \in V(G)\}$ is isomorphic to G; for all practical purposes, we shall assume then that G is indeed a *subgraph* of H.

Theorem 2.25 [16]**.** *If a* (p, q)*-graph* G *is set-graceful, then* $q = 2^m - 1$ *for some positive integer m*.

Definition 2.26 [27]**.** The double star graph $ST(m, n)$ is the graph formed by two stars $K_{1,m}$ and $K_{1,n}$ by joining their centers by an edge.

Theorem 2.27 [3]**.** *Any graph G can be embedded as an induced subgraph of a connected set-graceful graph*.

Theorem 2.28 [22]**.** $\gamma(K_{1,m}) = n + 1$ *if and only if* $2^n \leq m < 2^{n+1}$.

3. Topoline Graphs

In this section, we introduce the concept of a topoline set indexer by requiring that the edge labels together with the empty set form a topology on the indexing set. The minimum of the cardinalities of such topoline indexing sets is called the *topoline number* of the graph. In contrast with topological set indexers, not all graphs have topoline set indexers and this caused the origin of topoline graphs. It is noted that all set-graceful graphs are topoline, but the converse is not true. However, it is conjectured that, converse holds in the case of complete graphs. Further, we identify many families of topoline graphs.

Definition 3.1. A set-indexer $f : V \cup E \to 2^X$ of a nonempty graph G is said to be a *topoline set indexer* if $f(E) \cup \varnothing$ is a topology on X and X is called the *topoline indexing set* of G. The minimum among the cardinalities of such topoline indexing sets is called the *topoline number* of G and is denoted by $\tau_e(G)$. A nonempty graph G is said to be *topoline* if it has a topoline set indexer.

Theorem 3.2. $\gamma(G) \leq \tau_e(G)$.

Proof. Since every topoline set indexer is also a set-indexer, the result follows. □

The following are two easy consequences from Theorems 2.4 and 2.6.

Corollary 3.3. $\lceil \log_2 |V| \rceil \leq \tau_e(G)$.

Corollary 3.4. $\lceil \log_2(|E|+1) \rceil \leq \tau_e(G)$.

Remark 3.5. By Theorem 2.10, C_7 is set-graceful and let f be a set-graceful labeling of C_7. Then $f(E) \cup \varnothing$ is the discrete topology on a set X with $|X| = 3$. Thus, the set-graceful labeling on C_7 is a topoline set indexer and hence $\lceil \log_2 |V| \rceil = \lceil \log_2 7 \rceil = 3 = \tau_e(C_7)$ so that equality holds in Corollary 3.3. But by assigning $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$ and $X = \{a, b, c, d\}$ to the vertices of the complete graph K_6 we get a topoline set indexer of K_6 and by Theorems 2.4 and 3.2 it follows that $\tau_e(K_6) = 4$. Clearly, $\lceil \log_2 |V| \rceil = \lceil \log_2 6 \rceil = 3 < \tau_e(K_6)$ so that strict inequality holds in Corollary 3.3.

Remark 3.6. By Theorem 2.11 we have $P_4 = (u_1, u_2, u_3, u_4)$ is not set-graceful and hence $\tau_e(P_4) \geq \gamma(P_4) \geq 3$. Now, by assigning the subsets $\varnothing$, $\{a\}$, $\{b\}$ and $\{a, c\}$ of the set $X = \{a, b, c\}$ to the vertices u_1, u_2, u_3, u_4 in that order, we get a topoline set indexer of P_4. Clearly, $\lceil \log_2(|E|+1) \rceil = \lceil \log_2 4 \rceil = 2 < \tau_e(P_4) = 3$ so that strict inequality holds in Corollary 3.4. Now by assigning the subsets $\varnothing$, $\{a\}$, $\{a, b\}$ and $\{c\}$ of $X = \{a, b, c\}$ to the vertices v_1, v_2, v_3, v_4 of $K_4 \backslash \{(v_1, v_2)\}$ we get a topoline set indexer. Clearly, $\lceil \log_2(|E|+1) \rceil = \lceil \log_2 6 \rceil = 3 = \tau_e(K_4 \backslash (v_1, v_2))$ so that equality holds in Corollary 3.4.

Remark 3.7. By Theorem 2.15 we know that, if G' is a spanning subgraph of G, then $\tau(G') \leq \tau(G)$. But this is not true in the case of a topoline set indexer. For example, it can be easily verified that $\tau_e(K_{1,7}) = 3$ while $\tau_e(K_{1,7} \backslash e) = 4$ by Theorem 2.22.

Theorem 3.8. *Any set-graceful labeling is a topoline set indexer. Moreover,* $\gamma(G) = \tau_e(G)$ *for a set-graceful graph G.*

Proof. Let f be a set-graceful labeling of a graph G with indexing set X. Then $\gamma(G) = \log_2(|E|+1) = |X|$ so that $|E| = 2^{|X|} - 1$. Therefore,

$f(E) \cup \varnothing = 2^X$ so that f is a topoline set indexer of G. This implies that $\tau_e(G) \leq \gamma(G)$. Now by Theorem 3.2, we have $\gamma(G) = \tau_e(G)$. □

Remark 3.9. The converse of Theorem 3.8 is not true. Consider the path $P_5 = (v_1, ..., v_5)$. By Corollary 3.3, we have $\tau_e(P_5) \geq 3$. Now by assigning the subsets $\{a\}, \varnothing, \{a, b\}, \{c\}$ and $\{b, c\}$ of $X = \{a, b, c\}$ to the vertices $v_1, ..., v_5$ in that order, we get a topoline set indexer of P_5. Then by Theorems 2.6 and 3.2, we have $\gamma(P_5) = \tau_e(P_5)$, but P_5 is not set-graceful. However, if $\tau_e(G) = \log_2(|E| + 1)$, then G is set-graceful.

Theorem 3.10. *Let f be a set-indexer of a graph G with indexing set X. Then the function g defined by* $g(v) = X \setminus f(v)$ *for all* $v \in V$ *is also a set-indexer of G with indexing set X. Moreover* $f(E) = g(E)$.

Proof. Since f is a set-indexer, $g(v) = X \setminus f(v)$ for all $v \in V$ are distinct. Clearly,

$$\begin{aligned} g(u, v) &= \{X \setminus f(u)\} \oplus \{X \setminus f(v)\} \\ &= \{(X \setminus f(u)) \cup (X \setminus f(v))\} \setminus \{(X \setminus f(u)) \cap (X \setminus f(v))\} \\ &= \{X \setminus (f(u) \cap f(v))\} \setminus \{X \setminus (f(u) \cup f(v))\} \\ &= f(u) \oplus f(v) = f(u, v). \end{aligned}$$ □

Definition 3.11. Let f be a set-indexer of a graph G with indexing set X. Then the set-indexer g defined by $g(v) = X \setminus f(v)$ for all $v \in V$ is called the *dual set-indexer* of f and is denoted by f^d.

In light of Theorem 3.10, it follows that f^d is a t-set indexer if and only if f is a t-set indexer and f^d is a topoline set indexer if and only if f is so.

Corollary 3.12. *Let G be any graph and* $\Theta'_X(G)$ *denote the set of all optimal set-indexers f of G with respect to a set X such that* $f(u) = X$ *for some* $u \in V$. *Then* $\Theta'_X(G)$ *is nonempty.*

Proof. Let $\Theta_X(G)$ denote the set of all optimal set-indexers *f* of *G* with respect to a set *X* such that $f(u) = \varnothing$ for some $u \in V$. Then by Theorem 2.8, we have $\Theta_X(G) \neq \varnothing$ and let $g \in \Theta_X(G)$, then the dual set-indexer g^d of *g* belongs to $\Theta'_X(G)$ so that $\Theta'_X(G) \neq \varnothing$. □

In contrast with t-set indexers, not all graphs have topoline set indexers.

Theorem 3.13. *K_4 is not topoline.*

Proof. If possible, let *f* be a topoline set indexer of K_4 with topoline indexing set *X*. Let A_1, A_2, A_3 and A_4 be the vertex labels of K_4 under *f*. Since $f(E) \cup \varnothing$ is a topology say τ on *X*, one of the edge labels is *X*. Without loss of generality we may assume that $A_1 \oplus A_2 = X$ so that $A_1 \cap A_2 = \varnothing$. Now the remaining edge labels are $A_1 \oplus A_3$, $A_1 \oplus A_4$, $A_2 \oplus A_3$, $A_2 \oplus A_4$ and $A_3 \oplus A_4$ and by the definition of topology, we have

$$(A_1 \oplus A_3) \cap (A_2 \oplus A_3) = (A_1 \cap A_2) \oplus A_3 = A_3 \in \tau,$$

$$(A_1 \oplus A_4) \cap (A_2 \oplus A_4) = (A_1 \cap A_2) \oplus A_4 = A_4 \in \tau,$$

$$(A_1 \oplus A_3) \cup (A_2 \oplus A_3) = (A_1 \cup A_2) \oplus A_3 = X \setminus A_3 \in \tau,$$

$$(A_1 \oplus A_4) \cup (A_2 \oplus A_4) = (A_1 \cup A_2) \oplus A_4 = X \setminus A_4 \in \tau.$$

Now we claim that $A_3 \neq \varnothing$.

For if $A_3 = \varnothing$, then the edge labels of K_4 are $X, A_1, A_2, A_4, A_1 \oplus A_4$ and $A_2 \oplus A_4$. Since *f* is an injection on both *V* and *E*, we have $A_4 \neq \varnothing$ and $A_4 \neq X$. Consequently, one of the edge labels $A_1, A_2, A_1 \oplus A_4$ and $A_2 \oplus A_4$ is $X \setminus A_4$. Now $X \setminus A_4 = A_1$ or $A_2 \Rightarrow A_4 = A_2$ or A_1, a contradiction.

$X \setminus A_4 = A_1 \oplus A_4$ or $A_2 \oplus A_4 \Rightarrow X = A_1$ or A_2, a contradiction.

So we must have $A_3 \neq \varnothing$ and hence the claim.

Further, by Theorem 3.10, we must have $A_3 \neq X$ also.

By similar arguments, we get $A_4 \neq \varnothing, X$.

Thus, we have $X, A_3, A_4, X \setminus A_3, X \setminus A_4$ and $\varnothing$ are 6 distinct elements of τ. Now the following cases arise.

Case 1. $A_3 \cap A_4 = \varnothing$. Then $A_3 \cup A_4$ and $X \setminus (A_3 \cup A_4)$ are two distinct elements of τ other than the above six elements, which is a contradiction.

Case 2. $A_3 \cap A_4 \neq \varnothing$.

Subcase 2.1. If $A_3 \subset A_4$, then we have $A_3 \cup (X \setminus A_4)$ and $A_4 \cap (X \setminus A_3)$ are two distinct elements of τ other than the above six elements, which is a contradiction. Similarly, the case $A_4 \subset A_3$ is also not possible.

Subcase 2.2. Let $A_3 \cup A_4 \neq A_3, A_4$. Then $A_3 \cap A_4$ and $X \setminus (A_3 \cap A_4)$ are two distinct elements of τ other than the above six elements, which is a contradiction.

Thus, we arrive at contradictions in all the possibilities. Consequently, K_4 has no topoline set indexer. □

Theorem 3.14 [15]**.** *The complete graph* K_n *is set-graceful if and only if* $n \in \{2, 3, 6\}$.

Remark 3.15. Since K_2, K_3 and K_6 are set-graceful, by Theorem 3.8 they are topoline. We strongly feel that no other complete graphs are topoline and put forward the following:

Conjecture 3.16. *The complete graph* K_n; $n > 1$ *is topoline only if it is set-graceful*.

Theorem 3.17. $\tau_e(K_2 \cup (n-2)K_1) = \lceil \log_2 n \rceil$.

Proof. Consider the graph $G = K_2 \cup (n-2)K_1$; $V = \{v_1, ..., v_n\}$ and $K_2 = (v_1, v_2)$. Now we can find a topoline set indexer *f* of *G* with topoline indexing set $X = \{x_1, ..., x_{\lceil \log_2 n \rceil}\}$ as follows: Assign $\varnothing$ and X to v_1 and v_2. Now assign the $n-2$ distinct unassigned subsets of X to the remaining vertices of G. Clearly, f is a set-indexer and $f(E) \cup \varnothing$ is the indiscrete topology on X. □

Corollary 3.18. *Let G be a topoline graph of order n. Then* $\tau_e(G) \geq \tau_e(K_2 \cup (n-2)K_1)$.

Proof. Follows from Theorem 3.17 and Corollary 3.3.

Theorem 3.19. *Every nonempty graph has a topoline embedding.*

Proof. Follows from Theorem 3.17. □

Remark 3.20. A graph even with a connected topoline embedding need not be topoline. For example, consider the graph $K_4 \backslash \{e\}$; $V = \{v_1, ..., v_4\}$ and $e = (v_1, v_2)$. Assigning $\varnothing, \{c\}, \{a\}$ and $\{a, b\}$ to the vertices $v_1, ..., v_4$ in that order, we get a topoline set indexer of $K_4 \backslash \{e\}$. But by Theorem 3.13, we have K_4 is not topoline.

Theorem 3.21. *Every star is topoline. Moreover* $\tau_e(K_{1,n}) = \tau(K_{1,n})$.

Proof. Let $V(K_{1,n}) = \{v_0, v_1, ..., v_n\}$; v_0 be the central vertex. Let f be a t-set indexer of $K_{1,n}$ with t-indexing set say Y. Let $A_0, A_1, ..., A_n$ be the vertex labels of $K_{1,n}$ under f. Without loss of generality we may assume that $A_0 = \varnothing$. Consider a set-indexer $g : V \to 2^Y$ by $g(v_i) = A_i$; $0 \leq i \leq n$. Clearly then $\varnothing \cup g(E) = g(V) = f(V)$ so that g is a topoline set indexer. Also, this implies that $\tau_e(K_{1,n}) \leq \tau(K_{1,n})$.

Further, let $A_1, A_2, ..., A_n$ be the edge labels of $K_{1,n}$ under any topoline set indexer. Then by assigning $\varnothing$, $A_1, ..., A_n$ to the vertices $v_0, v_1, ..., v_n$ in that order, we get a topoline set indexer of $K_{1,n}$ so that $\tau(K_{1,n}) \leq \tau_e(K_{1,n})$. Thus for a star graph, $\tau_e = \tau$. □

Theorem 3.22. *The complete bipartite graph* $K_{m,n}$ *is topoline.*

Proof. Let $V(K_{m,n}) = \{u_1, ..., u_m, v_1, ..., v_n\}$; $d(u_i) = n$ and $d(v_j) = m$, $1 \leq i \leq m$, $1 \leq j \leq n$. Let X be a set in which there is a topology with m open sets say $A_1, ..., A_m$. Consider the set $Y = \{y_1, ..., y_n\}$ which is disjoint from X. Now we can define a topoline set indexer f of $K_{m,n}$ with topoline indexing set $X \cup Y$ as follows: Assign $A_1, ..., A_m$ to the vertices $u_1, ..., u_m$, respectively, and assign $\{y_1, ..., y_i\}$ to the vertex v_i; $1 \leq i \leq n$. Clearly, the edge labels of $K_{m,n}$ together with ϕ will form a topology on $X \cup Y$. □

Theorem 3.23. $\tau_e(K_{2^m+2^n,\, p}) \leq m + p + 1$; $0 \leq n \leq m$.

Proof. Let $V(K_{2^m+2^n,\, p}) = \{u_1, ..., u_{2^m+2^n}, v_1, ..., v_p\}$; $d(u_i) = p$ and $d(v_j) = 2^m + 2^n$, $1 \leq i \leq 2^m + 2^n$, $1 \leq j \leq p$. Now we can define a topoline set indexer f of $K_{2^m+2^n,\, p}$ with topoline indexing set $X \cup Y$; $X = \{x_1, ..., x_{m+1}\}$ and $Y = \{y_1, ..., y_p\}$ as follows: Assign the $2^m + 2^n$ distinct open sets of X to the vertices $u_1, ..., u_{2^m+2^n}$ and assign $\{y_1\}, \{y_1, y_2\}, ..., \{y_1, ..., y_p\}$ to the vertices $v_1, ..., v_p$ in that order. Clearly, the edge labels will form a topology on $X \cup Y$ and the result follows. □

Theorem 3.24. *All paths are topoline.*

Proof. Assigning the subsets $\varnothing, \{x_1\}, \{x_2\}, \{x_1, x_3\}, \{x_2, x_4\}, \{x_1, x_3, x_5\}, ...$ of the set $X = \{x_1, ..., x_{n-1}\}$ to the vertices of the path $P_n = (v_1, ..., v_n)$ in that order, we get a topoline set indexer of P_n. □

Theorem 3.25. *All trees are topoline.*

Proof. The proof is by induction on the order n of trees.

Clearly, by Theorem 3.24, K_2 and P_3 are topoline and hence the result is true for $n = 2$ and 3.

Suppose that any tree of order m is topoline.

Let T be a tree of order $m+1$ and u be a pendant vertex of T. By assumption $T\backslash\{u\}$ is topoline, let f be a topoline set indexer of $T\backslash\{u\}$ with $X = \{x_1, ..., x_k\}$ as the topoline indexing set. Now we can define a topoline set indexer say g of T with topoline indexing set $Y = X \cup \{x_{k+1}\}$ as follows: $g(v) = f(v)$ for all $v \in V(T\backslash u)$ and $g(u) = Y \backslash f(w)$, $w \in N(u)$. Thus T, the tree of order $m+1$ is topoline and hence the result is true for $n = m+1$ also. □

Theorem 3.26. *Nonempty forests are topoline.*

Proof. The proof is by induction on the order n of the forest.

Clearly, $K_2 \cup K_1$ and P_3 are topoline, follows from Theorems 3.17 and 3.24 and hence the result is true for $n = 3$.

Assume that all forests of order m are topoline.

Let G be a forest of order say $m+1$ and let u be a pendant vertex of G. Then $G\backslash\{u\}$ is a forest of order m and by assumption let f be a topoline set indexer of $G\backslash\{u\}$ with topoline indexing set say $X = \{x_1, ..., x_k\}$. Now assigning $g(v) = f(v)$ for all $v \in V(G)$ and $g(u) = Y \backslash f(w)$; $w \in N(u)$, we get a topoline set indexer say g of G with topoline indexing set $Y = X \cup \{x_{k+1}\}$. Hence the result is true for $n = m+1$ also.

Corollary 3.27. *Every spanning subgraph of a tree is topoline.*

Proof. Follows from Theorem 3.26. □

Theorem 3.28. *All fans are topoline.*

Proof. Let $F_n = P_{n-1} \vee \{u\}$; $P_{n-1} = (v_1, ..., v_{n-1})$. Now assign the subsets $\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_3\}, \{x_2, x_4\}, \{x_1, x_3, x_5\}, ...$ of the set $X = \{x_1, ..., x_{n-1}\}$ to the vertices $u, v_1, ..., v_{n-1}$ in that order. Clearly,

$$f(v_{n-1}) = \begin{cases} \{x_1, x_3, ..., x_{n-3}, x_{n-1}\} & \text{if } n-1 \text{ is odd,} \\ \{x_2, x_4, ..., x_{n-3}, x_{n-1}\} & \text{if } n-1 \text{ is even} \end{cases}; \quad n \geq 2.$$

Thus, we get a topoline set indexer of F_n with X as the topoline indexing set. □

Theorem 3.29. *The complete tripartite graph $K_{1,m,n}$ is topoline.*

Proof. Let $V(K_{1,m,n}) = \{u, v_1, ..., v_m, w_1, ..., w_n\}$; $d(u) = m + n$, $d(v_i) = n + 1$, $d(w_j) = m + 1$, $1 \leq i \leq m$ and $1 \leq j \leq n$. Now assign $\varnothing$ to u. Let X be a nonempty set in which there is a topology say τ_1 on X with $m + 1$ open sets say $A_1, ..., A_{m+1}$. Similarly, let Y be a nonempty set disjoint from X such that there is a topology say τ_2 on Y with $n + 1$ open sets say $B_1, ..., B_{n+1}$. Without loss of generality we may assume that $A_{m+1} = B_{n+1} = \varnothing$. Now define a topoline set indexer f of $K_{1,m.n}$ with topoline indexing set $X \bigcup Y$ as follows: $f(u) = \varnothing$, $f(v_i) = A_i$; $1 \leq i \leq m$ and $f(w_j) = B_j$; $1 \leq j \leq n$. □

Theorem 3.30. *Every graph can be embedded into a connected topoline graph as an induced subgraph.*

Proof. Let $\{v_1, ..., v_n\}$ be the vertex set of the given graph G. Now take a new vertex say u and join it with all the vertices of G. Consider the set $X = \{x_1, ..., x_n\}$. Let $m = 2^n - (|E| + n) - 1$. Take m new vertices $u_1, ..., u_m$ and join them with u. A topoline set indexer of the resulting graph G' can be defined as follows: Assign $\varnothing$ to u, $\{x_i\}$ to v_i; $1 \leq i \leq n$. Let $S = \{f(e); e \in E\} \bigcup \{\{x_i\}/1 \leq i \leq n\}$. Note that $|S| = |E| + n$. Now by assigning the m elements of $2^X \setminus (S \bigcup \varnothing)$ to the vertices $u_1, ..., u_m$ in any order, we get a topoline set indexer of G' with X as the topoline indexing set. □

Lemma 3.31. *Let f be a topoline set indexer of a graph G with topoline indexing set X. If $f(V)$ is a topology on X with $f(E) \subset f(V)$, then $G \vee K_n^c$ is topoline for each positive integer m.*

Proof. We can find a topoline set indexer say g of $G \vee K_n^c$; $V(K_n^c) = \{v_1, ..., v_n\}$ with topoline indexing set $Y = X \cup \{x_1, ..., x_n\}$ as follows: $g(u) = f(u)$ for all $u \in V(G)$ and $g(v_i) = \{x_1, ..., x_i\}$; $1 \leq i \leq n$. □

Theorem 3.32. *Let G be a set graceful tree. Then* $G \vee K_n^c$ *is topoline.*

Proof. By Theorem 2.25, we have $|E(G)| = 2^m - 1$ for some positive integer m. Since G is a tree, $|V(G)| = 2^m$. Let f be a set-graceful labeling of G with indexing set X of cardinality m. By the definition of set-graceful labeling, $\gamma(G) = \log_2(|E(G)| + 1) = m = |X|$. Then we must have $f(V) = f(E) \cup \varnothing = 2^X$ so that by Lemma 3.31, $G \vee K_n^c$ is topoline. □

Remark 3.33. Let f be a topoline set indexer of K_3 such that $f(V)$ is also a topology on the topoline indexing set say X. Then f is given by $f(v_1) = \varnothing$, $f(v_2) = A$ and $f(v_3) = X$ for some $A \subset X$; $V(K_3) = \{v_1, v_2, v_3\}$. Then clearly $f(E) \not\subset f(V)$. Note that here $K_3 \vee K_1^c = K_4$ is not topoline by Theorem 3.13.

4. Topoline Set Graceful Graphs

In this section, we define topoline set graceful graphs - graphs with set-indexing number and topoline number are equal. Clearly, all set-graceful graphs are topoline set graceful but the converse is not true. Further we note that topologically set graceful and topoline set graceful are two independent notions. However, a star is topologically set graceful if and only if it is topoline set graceful.

Definition 4.1. A graph G is said to be *topoline set graceful* if it is topoline and $\gamma(G) = \tau_e(G)$.

By Theorem 3.8, we have all set-graceful graphs are topoline set graceful. But the converse is not true as already seen in Remark 3.9.

Theorem 4.2. $K_2 \cup (n-2)K_1$ *is topoline set graceful.*

Proof. Follows from Theorems 2.4, 3.2 and 3.17. □

Remark 4.3. The notions of t-set graceful graphs and topoline set graceful graphs are not related to each other. For the complete graph K_6, we have $\gamma(K_6) = 4 = \tau_e(K_6)$ and hence it is topoline set graceful. But K_6 is not t-set graceful by Theorem 2.18. Now the complete graph K_4 is t-set graceful, but by Theorem 3.13 K_4 is not even topoline. But K_3 is both t-set graceful and topoline set graceful.

Theorem 4.4. *A star is t-set graceful if and only if it is topoline set graceful.* □

Proof. Follows from Theorem 3.21.

Remark 4.5. In 1986, Acharya [1] conjectured that the cycle C_{2^n-1}; $n \geq 2$ is set-graceful: and in 1989, Mollard and Payan [15] settled this in the affirmative. The idea of their proof is the following:

Consider the field $GF(2^n)$ constructed by a binary primitive polynomial say $p(x)$ of degree *n*. Let α be a root of $p(x)$ in $GF(2^n)$. Then $GF(2^n) = \{0, 1, \alpha, \alpha^2, ..., \alpha^{2^n-2}\}$. Now by assigning $\alpha^{i-1} \bmod p(\alpha)$, $1 \leq i \leq 2^n - 1$, to the vertices v_i of the cycle $C_{2^n-1} = (v_1, ..., v_{2^n-1}, v_1)$, we get a set-graceful labeling of C_{2^n-1} with indexing set $X = \{1, \alpha, \alpha^2, ..., \alpha^{n-1}\}$. Note that here $\alpha^j \bmod p(\alpha) = a_0\alpha^0 + a_1\alpha^1 + \cdots + a_{n-1}\alpha^{n-1}$; $a_i = 0$ or 1 for $0 \leq i \leq n-1$ with $\alpha^0 = 1$ and we identify it as $\{a_i\alpha^i / a_i = 1;\ 0 \leq i \leq n-1\}$ which is a subset of *X*.

Theorem 4.6. *Paths P_{2^n+1} and P_{2^n+2} are topoline set graceful.*

Proof. The cycle $C_{2^n-1} = (v_1, ..., v_{2^n-1}, v_1)$ has a set-graceful labeling *f* as described in Remark 4.5. Consider the path $C_{2^n-1} \backslash (v_1, v_2)$. Take two new vertices u_1 and u_2 and form the path $P_{2^n+1} = \{C_{2^n-1} \backslash (v_1, v_2)\} \cup \{(v_1, u_1), (u_1, u_2)\}$. We can define a topoline set indexer g on P_{2^n+1} with topoline indexing set $Y = X \cup \{y\}$ as follows: $g(v_i) = f(v_i)$; $1 \le i \le 2^n - 1$, $g(u_1) = Y \backslash \{1\}$ and $g(u_2) = Y \backslash \{\alpha\}$. Now by Theorems 2.4 and 3.2, we have $n + 1 = \lceil \log_2(|E(P_{2^n+1})| + 1) \rceil \le \gamma(P_{2^n+1}) \le \tau_e(P_{2^n+1}) \le n + 1$ and hence P_{2^n+1} is topoline set graceful.

Take a new vertex u_3 and form the path $P_{2^n+2} = P_{2^n+1} \cup (u_2, u_3)$. We can define a topoline set indexer h on P_{2^n+2} with topoline indexing set Y as follows: $h(u) = g(u)$ for all $u \in V(P_{2^n+1})$ and $h(u_3) = \varnothing$. Now by Theorems 2.4 and 3.2, we have $n + 1 = \lceil \log_2(|E(P_{2^n+2})| + 1) \rceil \le \gamma(P_{2^n+2}) \le \tau_e(P_{2^n+2}) \le n + 1$ and hence P_{2^n+2} is topoline set graceful. □

Lemma 4.7. $T(n + 1, 2^n + 3) = 0$; $n \ge 2$.

Theorem 4.8. *The path* P_{2^n+3}; $n \ge 2$ *is not topoline set graceful*.

Proof. By Lemma 4.7 and Theorem 3.25, we have $\tau_e(P_{2^n+3}) \ge n + 2$. Now the result follows from Theorem 2.7. □

Theorem 4.9. C_{2^n+3} *is topoline set graceful*.

Proof. The cycle $C_{2^n-1} = (v_1, ..., v_{2^n-1}, v_1)$ has a set-graceful labeling f as described in Remark 4.5. Consider the path $C_{2^n-1} \backslash (v_1, v_2)$. Take four new vertices u_1, u_2, u_3 and u_4 and form the cycle $C_{2^n+3} = \{C_{2^n-1} \backslash (v_1, v_2)\} \cup \{(v_1, u_1), (u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, v_2)\}$. We can define a topoline set indexer g on C_{2^n+3} with topoline indexing set $Y = X \cup \{y\}$ as follows: $g(v_i) = f(v_i)$; $1 \le i \le 2^n - 1$, $g(u_1) = Y \backslash \{1\}$, $g(u_2) = Y \backslash \{\alpha\}$, $g(u_3) = \varnothing$ and $g(u_4) = Y \backslash \{1, \alpha\}$. Now by Theorems 2.4 and 3.2, we have $n + 1 =$

$\lceil \log_2(|E(C_{2^n+3})|+1)\rceil \leq \gamma(C_{2^n+3}) \leq \tau_e(C_{2^n+3}) \leq n+1$ and hence C_{2^n+3} is topoline set graceful. □

Theorem 4.10. $K_{2^m, 2}$ *is topoline set graceful*.

Proof. Let $V = \{u, v, w_1, ..., w_{2^m}\}$; $d(u) = d(v) = 2^m$ and $d(w_i) = 2$, $1 \leq i \leq 2^m$. Define a topoline set indexer *f* of $K_{2^m, 2}$ with topoline indexing set $X = \{x_1, ..., x_{m+2}\}$ as follows: Assign the distinct 2^m subsets of $X \setminus \{x_{m+1}, x_{m+2}\}$ to the vertices $w_1, ..., w_{2^m}$ in any order and finally assign $\{x_{m+1}\}$ and $\{x_{m+1}, x_{m+2}\}$ to the vertices *u* and *v*, respectively. By Theorems 3.2 and 2.4, we have $\lceil \log_2(|E|+1)\rceil = \lceil \log_2(2^{m+1}+1)\rceil = m+2 \leq \gamma(K_{2^m, 2})$ $\leq \tau_e(K_{2^m, 2}) \leq m+2$. □

Theorem 4.11. *A graph G of size* $m-1$; $2^{n-1}+2^{n-2} < m < 2^n$ *and* $\gamma(G) = n\ (\geq 3)$ *is not topoline set graceful*.

Proof. Follows from Theorem 2.22. □

Remark 4.12. From the above theorem it follows easily that the class of graphs given by P_{2^n-1}, $K_{1, 2^n-2}$, $ST(m_1, m_1)$; $m_1 + m_2 = 2^n - 3$ are not topoline set graceful.

Theorem 4.13. $K_{1, 2^n-1, 2^m-1}$ *is topoline set graceful*.

Proof. Let $V(K_{1, 2^n-1, 2^m-1}) = \{u, v_1, ..., v_{2^n-1}, w_1, ..., w_{2^m-1}\}$; $d(u) = 2^n + 2^m - 2$, $d(v_i) = 2^m$; $1 \leq i \leq 2^n - 1$ and $d(w_j) = 2^n$; $1 \leq j \leq 2^m - 1$. Consider the sets $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$. Now we can define a topoline set indexer *f* of $K_{1, 2^n-1, 2^m-1}$ with topoline indexing set $X \cup Y$ as follows: Assign $\varnothing$ to *u*, the distinct nonempty sets of *X* to v_i; $1 \leq i \leq 2^n - 1$

and the distinct nonempty subsets of Y to the vertices $w_j;\ 1 \le j \le 2^m - 1$. Clearly, by Theorems 3.2 and 2.4, we have

$$\lceil \log_2(|E| + 1) \rceil = \lceil \log_2(2^n - 1 + 2^m - 1 + 2^{n+m} - 2^n - 2^m + 1 + 1) \rceil$$

$$= \lceil \log_2(2^{m+n}) \rceil = m + n \le \gamma(K_{1,\, 2^n - 1,\, 2^m - 1})$$

$$\le \tau_e(K_{1,\, 2^n - 1,\, 2^m - 1}) \le n + m. \qquad \square$$

Theorem 4.14. $K_{1,\, 2^n,\, 2^m - 1}$ *is topoline set graceful*.

Proof. Let $V(K_{1,\, 2^n,\, 2^m - 1}) = \{u, v_1, ..., v_{2^n}, w_1, ..., w_{2^m - 1}\};\ d(u) = 2^n + 2^m - 1,\quad d(v_i) = 2^m;\quad 1 \le i \le 2^n$ and $d(w_j) = 2^n + 1;\quad 1 \le j \le 2^m - 1$. Consider the sets $X = \{x_1, ..., x_{n+1}\}$ and $Y = \{y_1, ..., y_m\}$. Now we can define a topoline set indexer f of $K_{1,\, 2^n,\, 2^m - 1}$ with topoline indexing set $X \cup Y$ as follows: Assign $\varnothing$ to u, the distinct nonempty sets of $X \setminus \{x_{n+1}\}$ to $v_i;\ 1 \le 1 \le 2^n - 1,$ the distinct nonempty subsets of Y to the vertices $w_j;$ $1 \le j \le 2^m - 1$ and finally assign X to v_{2^n}. Clearly, by Theorems 3.2 and 2.4, we have

$$\lceil \log_2(|E| + 1) \rceil = \lceil \log_2(2^n + 2^m - 1 + 2^{n+m} - 2^n + 1) \rceil$$

$$= m + n + 1 \le \gamma(K_{1,\, 2^n,\, 2^m - 1})$$

$$\le \tau_e(K_{1,\, 2^n,\, 2^m - 1}) \le n + m + 1. \qquad \square$$

Since all set-graceful graphs are topoline set graceful, by Theorem 2.27, we get the following.

Theorem 4.15. *Any graph G can be embedded as an induced subgraph of a connected topoline set graceful graph.*

Theorem 4.16. *All t-set graceful stars are topogenic*.

Proof. Let $K_{1,n}$ be a t-set graceful star with t-indexing set X. Then by Theorem 2.28, $\gamma(K_{1,n}) = \tau(K_{1,n}) = \lceil \log_2(n+1) \rceil = |X|$. Let f be the corresponding t-set indexer and let $f(v_i) = A_i$, $0 \le i \le n$; $V(K_{1,n}) = \{v_0, v_1, ..., v_n\}$ with $d(v_0) = n$. Assigning $\varnothing$ to v_0 and the other distinct nonempty elements of $\{A_i\}$ to the other vertices of $K_{1,n}$ in any order, we get another t-set indexer g on $K_{1,n}$ such that $g(V) = f(V) = g(E) \cup \varnothing$. Consequently $K_{1,n}$ is topogenic. □

Corollary 4.17. *All topoline set graceful stars are topogenic*.

Proof. Follows from Theorems 4.16 and 3.21. □

Remark 4.18. Not all topogenic stars are t-set graceful. For example, consider the star $K_{1,6}$, it is topogenic but it is not t-set graceful.

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