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HARMONIC CONVEX AND GENERALIZED CONVEX FUZZY MAPPINGS

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Abstract

In the present work, we introduce a new concept of harmonic convexity (concavity) under the fuzzy environment. The relations between the harmonic convex fuzzy mappings are explicitly discussed and many important results are obtained that relate the concept to fuzzy convexity and logarithmic convexity. Some properties are also obtained that relate the concept to fuzzy quasiconvexity.

1. Introduction

The concept of convex fuzzy sets was introduced by Zadeh [14], in which a fuzzy set with membership function $\mu : R^n \to [0, 1]$ is called *convex*

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if for $0 \leq \lambda \leq 1$,

$$\mu(\lambda x + (1 - \lambda) y) \geq \min\{\mu(x), \mu(y)\} \tag{1}$$

for all $x, y \in \operatorname{supp}(\mu)$, where $\operatorname{supp}(\mu) = \{t \in R^n \mid \mu(t) > 0\}$.

A fuzzy set $\mu : R^n \to [0, 1]$ is said to be *normal*, if there exists a point $x \in R^n$ such that $\mu(x) = 1$. A fuzzy number, we study in this paper is a fuzzy set $\mu : R \to [0, 1]$, which is normal, fuzzy convex, upper semi continuous with bounded support.

We denote F, as the set of all fuzzy numbers. A mapping from any nonempty set into F will be called a *fuzzy mapping*. It is obvious that $r \in R$ can be considered as a fuzzy number. So each real-valued function can be considered as a fuzzy mapping.

In [3] Goetschel and Voxman proposed a linear ordering '$\preccurlyeq$' on F. For each fuzzy mapping $f : R \to F$, based on the linear ordering '$\preccurlyeq$', they introduced a real-valued function T_f on the domain of the fuzzy mapping f. In [13], two concepts of convexity and quasiconvexity for a fuzzy mapping f are defined through the real-valued function introduced in [3].

In this paper, we have introduced a new kind of convex fuzzy mapping called *harmonic convex fuzzy mapping* directly through the linear ordering proposed in [3]. Following [3], a ranking value function τ on F and the concept of monotonicity for the fuzzy mapping $g : F \to F$ is defined. Based on the ranking value function τ on F, the concept of harmonic convex and quasiconvex mapping is introduced. The continuity of fuzzy mapping through a metric on F is studied and Weierstrass theorem is extended from real-valued functions to fuzzy mappings. The local-global minimum property of real-valued convex functions is extended to convex fuzzy mappings. The characterization for convex fuzzy mappings, harmonic convex fuzzy mappings and quasiconvex fuzzy mappings is also given. It is proved that every strict local minimizer of a harmonic convex fuzzy mapping is a global minimizer.

2. Preliminaries

We present several definitions and results without proof.

The α-level set of a fuzzy number $\mu \in F$ is a closed and bounded interval as follows:

$$[\mu]_\alpha = [a(\alpha), b(\alpha)] = \begin{cases} \{x \in R \mid \mu(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ cl(\text{supp}(\mu)), & \text{if } \alpha = 0, \end{cases}$$

where $cl(\text{supp}(\mu))$ denotes the closure of $\text{supp}(\mu)$.

The fuzzy set $\mu : R \to [0, 1]$ is a fuzzy number if and only if

(i) $[\mu]_\alpha$ is a closed and bounded interval for each $\alpha \in [0, 1]$ and

(ii) $[\mu]_1 \neq \Phi$.

Thus, we can identify a fuzzy number μ with the parameterized triples, $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$, where $a(\alpha)$ and $b(\alpha)$ denote the left and right endpoints of $[\mu]_\alpha$, respectively.

For the fuzzy numbers $\mu, \nu \in F$ represented parametrically by $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ and $\{(c(\alpha), d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ respectively, and each non-negative real number r, the addition and non-negative scalar multiplication can be defined as follows:

$$\mu + \nu = \{(a(\alpha) + c(\alpha), b(\alpha) + d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}, \tag{2.1}$$

$$r\mu = \{(ra(\alpha), rb(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}. \tag{2.2}$$

Addition and scalar multiplication on F defined by (2.1) and (2.2) are equivalent to those derived from the usual extension principle. It is evident that F is closed under addition and non-negative scalar multiplication.

According to Goetschel and Voxman [3], F can be metricized by the metric,

$$D(\{(a(\alpha), b(\alpha), \alpha) \mid 0 \le \alpha \le 1\}, \{(c(\alpha), d(\alpha), \alpha) \mid 0 \le \alpha \le 1\})$$

$$= \sup\{\max\{| a(\alpha) - c(\alpha)|, | b(\alpha) - d(\alpha)|\} \mid 0 \le \alpha \le 1\}$$

and the following ordering '$\preccurlyeq$' on F is defined.

Definition 2.1 [15]. Let $\mu, \nu \in F$ in the parameterized form. Then $\mu \preccurlyeq \nu$ if

$$\int_0^1 \alpha[a(\alpha) + b(\alpha)]d\alpha \le \int_0^1 \alpha[c(\alpha) + d(\alpha)]d\alpha.$$

The ordering '$\preccurlyeq$' is a partial ordering on F. Moreover, any two elements on F are comparable under the ordering '$\preccurlyeq$', i.e., '$\preccurlyeq$' is a linear ordering for F.

3. Definitions and Basic Results

Based on the linear ordering, we define a ranking value function and a strict ordering '$\prec$' of '$\preccurlyeq$' on F. The concept of monotonicity for a fuzzy mapping $g : F \to F$ is proposed. We also propose convexity and continuity for fuzzy mappings based on the positive ordering '$\preccurlyeq$' and the metric D, respectively.

From the notion of linear ordering '$\preccurlyeq$' on F, we define ranking value function $\tau : F \to R$ as follows:

Definition 3.1. Let $\tau : F \to R$ be defined by

$$\tau(\mu) = \int_0^1 \alpha[a(\alpha) + b(\alpha)]d\alpha \tag{3.1}$$

for each $\mu \in F$ in parametric form.

Lemma 3.1 [9]. *For* $\mu, \nu \in F$ *and* $k > 0$,

(1) $\tau(\mu + \nu) = \tau(\mu) + \tau(\nu)$, *additivity*,

(2) $\tau(k\mu) = k\tau(\mu)$, *homogeneity*.

Corollary 3.1 [9]**.** *For* $\mu, \nu \in F$ *and* $k_1, k_2 > 0$,

$$\tau(k_1\mu + k_2\nu) = k_1\tau(\mu) + k_2\tau(\nu) \ (\textit{Linearity property}). \tag{3.2}$$

Definition 3.2 [9]**.** For $\mu, \nu \in F$, we say $\mu \prec \nu$ if $\mu \preccurlyeq \nu$ and $\tau(\mu) \neq \tau(\nu)$.

From Definition 3.2, the following results can easily be established.

Lemma 3.2 [9]**.** *For* $\mu, \nu \in F$, *if* $\mu \prec \nu$, *then*

$$\mu \prec \lambda\mu + (1 - \lambda)\nu, \ \textit{for} \ \lambda \in (0, 1). \tag{3.3}$$

Lemma 3.3 [9]**.** *For* $\mu, \nu \in F$,

$$\mu \preccurlyeq \nu \Leftrightarrow \tau(\mu) \leq \tau(\nu), \tag{3.4}$$

$$\mu \prec \nu \Leftrightarrow \tau(\mu) < \tau(\nu). \tag{3.5}$$

Definition 3.3 [9]**.** A fuzzy mapping $g : F \to F$ is said to be

1. *nondecreasing* if for $\mu, \nu \in F$,

(i) $\mu \prec \nu \Rightarrow g(\mu) \preccurlyeq g(\nu)$ and

(ii) $\tau(\mu) = \tau(\nu) \Rightarrow g(\mu) = g(\nu)$.

2. *nonincreasing* if for $\mu, \nu \in F$,

(i) $\mu \prec \nu \Rightarrow g(\mu) \succcurlyeq g(\nu)$ and

(ii) $\tau(\mu) = \tau(\nu) \Rightarrow g(\mu) = g(\nu)$.

Let *S* be a nonempty subset of R^n. For any $x \in R^n$ and $\delta > 0$, let

$$B_\delta(x) = \{y \in R^n \,|\, \| y - x \| < \delta\},$$

where $\| \cdot \|$ is the Euclidian norm of R^n.

Definition 3.4 [9]**.** For a fuzzy mapping $f : S \to F$,

(1) an element $\widetilde{x} \in S$ is called a *local minimizer* of $f : S \to F$, if there exists a $\delta > 0$, such that $f(\widetilde{x}) \preccurlyeq f(x)$, for all $x \in S \cap B_\delta(\widetilde{x})$;

(2) an element $\hat{x} \in S$ is called a *strict local minimizer* of $f : S \to F$, if there exists a $\delta > 0$ such that $f(\hat{x}) \prec f(x)\ \forall x \neq \hat{x}$ and $x \in S \cap B_\delta(\hat{x})$;

(3) an element $x^* \in S$ is called a *global minimizer* of $f : S \to F$, if $f(x^*) \preccurlyeq f(x)\ \forall x \in S$.

Definition 3.5 [9]**.** Let $x_0 \in S$. A fuzzy mapping $f : S \to F$ is said to be *continuous* at x_0 if for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$, whenever $x \in S \cap B_\delta(x_0)$.

$f : S \to F$ is said to be *continuous* if it is continuous at each $x \in S$.

Definition 3.6. For each fuzzy mapping $f : R^n \to F$, define $T_f : R^n \to R$ by

$$T_f(x) = \int_0^1 \alpha[a(\alpha, x) + b(\alpha, x)]d\alpha, \tag{3.6}$$

where for each $x \in R^n$, $f(x)$ is parametrically represented by

$$\{(a(\alpha, x), b(\alpha, x), \alpha) \mid 0 \leq \alpha \leq 1\}.$$

From (2.3) and (3.6), for $x, y \in R^n$, we have

$$f(x) \preccurlyeq f(y) \Leftrightarrow T_f(x) \leq T_f(y). \tag{3.7}$$

Motivated by Goetschel and Voxman [3, Lemma 2.9], we have established the following.

Lemma 3.4 [9]**.** *If* $f : R^n \to F$ *is continuous, then* $T_f : R^n \to R$ *is also continuous.*

Finally, we introduce the concept of convexity, concavity for fuzzy mappings. Let C be a nonempty convex subset of F and let K be a nonempty convex subset of R^n.

Definition 3.7. Let $C \subseteq F$ be a nonempty convex subset of F.

A fuzzy mapping $g : F \to F$ is said to be

1. *convex* if for $\mu, \nu \in C$ and $\lambda \in (0, 1)$,

$$g(\lambda\mu + (1-\lambda)\nu) \preccurlyeq \lambda g(\mu) + (1-\lambda) g(\nu);$$

2. *concave* if for $\mu, \nu \in C$ and $\lambda \in (0, 1)$,

$$g(\lambda\mu + (1-\lambda)\nu) \succcurlyeq \lambda g(\mu) + (1-\lambda) g(\nu);$$

3. *logarithmic convex* if for $\mu, \nu \in C$ and $\lambda \in (0, 1)$,

$$g(\lambda\mu + (1-\lambda)\nu) \preccurlyeq (g(\mu))^{\lambda} (g(\nu))^{1-\lambda};$$

4. *logarithmic concave* if for $\mu, \nu \in C$ and $\lambda \in (0, 1)$,

$$g(\lambda\mu + (1-\lambda)\nu) \succcurlyeq (g(\mu))^{\lambda} (g(\nu))^{1-\lambda};$$

5. *harmonic convex* if for $\mu, \nu \in C$ and $\lambda \in (0, 1)$,

$$g(\lambda\mu + (1-\lambda)\nu) \preccurlyeq \left(\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right)^{-1};$$

6. *harmonic concave* if for $\mu, \nu \in C$ and $\lambda \in (0, 1)$,

$$g(\lambda\mu + (1-\lambda)\nu) \succcurlyeq \left(\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right)^{-1}.$$

Definition 3.8. A fuzzy mapping $f : K \subset R^n \to F$ is said to be

1. *convex* if for $x, y \in K$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda) y) \preccurlyeq \lambda f(x) + (1-\lambda) f(y);$$

2. *concave* if for $x, y \in K$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \succcurlyeq \lambda f(x) + (1-\lambda) f(y);$$

3. *logarithmic convex* if for $x, y \in K$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \preccurlyeq (f(x))^{\lambda}(f(y))^{1-\lambda};$$

4. *logarithmic concave* if for $x, y \in K$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \succcurlyeq (f(x))^{\lambda}(f(y))^{1-\lambda};$$

5. *harmonic convex* if for $x, y \in K$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \preccurlyeq \left(\frac{\lambda}{f(x)} + \frac{1-\lambda}{f(y)}\right)^{-1};$$

6. *harmonic concave* if for $x, y \in K$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \succcurlyeq \left(\frac{\lambda}{f(x)} + \frac{1-\lambda}{f(y)}\right)^{-1}.$$

Definition 3.9. Let $g : F \to F$ be a convex fuzzy mapping and $\mu, \nu \in F,\ \lambda \in (0, 1)$. Then

$$\lambda g(\mu) + (1-\lambda) g(\nu) \succcurlyeq (g(\mu))^{\lambda}(g(\nu))^{1-\lambda} \succcurlyeq \left(\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right)^{-1}.$$

Theorem 3.1 [9]**.** *Let* $g : F \to F$ *and* $f : K \to F$ *be convex fuzzy mappings. If g is nondecreasing, then the fuzzy mapping* $gof : K \to F$ *defined by* $(gof)(x) = g(f(x))$ *for each* $x \in K \subseteq R^n$ *is convex on K.*

4. Main Results

In this section, we extend Weierstrass theorem from real-valued functions to convex fuzzy mappings and give characterizations for convex fuzzy mappings. From the characterization of convex fuzzy mappings in terms of ranking value function, we propose the concept of quasiconvexity for fuzzy mappings by using the ranking value function $\tau : F \to R$.

Theorem 4.1. *Let* $g : K \to F$ *be a fuzzy mapping, where* $K \subset R^n$. *Then*

(i) *If g is H-convex on F, then it is L-convex and convex on F.*

(ii) *If g is concave on F, then it is L-concave and H-concave on F.*

Proof.

(i) Let $g : K \to F$ be H-convex on F. Then for $\mu, \nu \in K$, $\lambda \in (0, 1)$,

$$\begin{aligned} g(\lambda\mu + (1-\lambda)\nu) &\preccurlyeq \left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]^{-1} \\ &\preccurlyeq [g(\mu)]^{\lambda}[g(\nu)]^{1-\lambda} \\ &\preccurlyeq \lambda g(\mu) + (1-\lambda)g(\nu). \end{aligned}$$

(ii) Let $g : K \to F$ be concave on F. Then for $\mu, \nu \in K$, $\lambda \in (0, 1)$,

$$\begin{aligned} g(\lambda\mu + (1-\lambda)\nu) &\succcurlyeq \lambda g(\mu) + (1-\lambda)g(\nu) \\ &\succcurlyeq [g(\mu)]^{\lambda}[g(\nu)]^{1-\lambda} \\ &\succcurlyeq \left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]^{-1}. \end{aligned}$$

Theorem 4.2. *Let* $g : F \to F$ *be a fuzzy mapping. Then*

(i) *If g is H-convex on F, then it is L-convex and convex on F.*

(ii) *If g is concave on F, then it is L-concave and H-concave on F.*

Proof. Similar to the proof of Theorem 4.1.

Theorem 4.3 [1]**.** *Let* $g : F \to F$ *and* $f : K \to F$ *be harmonic convex fuzzy mappings. If g is nondecreasing, then the fuzzy mapping* $(gof) : K \to F$ *defined by* $(gof)(x) = g(f(x))$ *for each* $x \in K$ *is harmonic convex.*

Proof. Let $x, y \in K$ and $\lambda \in (0, 1)$.

Since $f : K \to F$ is harmonic convex, we have

$$f(\lambda x + (1-\lambda)y) \preccurlyeq \left[\frac{\lambda}{f(x)} + \frac{1-\lambda}{f(y)}\right]^{-1},$$

$$f(\lambda x + (1-\lambda)y) \prec \lambda f(x) + (1-\lambda)f(y).$$

Since g is nondecreasing and harmonic convex, it follows that

$$g(f(\lambda x + (1-\lambda)y)) \preccurlyeq g\left(\left[\frac{\lambda}{f(x)} + \frac{1-\lambda}{f(y)}\right]^{-1}\right)$$

$$\preccurlyeq \lambda g\left(\frac{1}{f(x)}\right) + (1-\lambda)g\left(\frac{1}{f(y)}\right),$$

$$g(f(\lambda x + (1-\lambda)y)) \preccurlyeq g(\lambda f(x) + (1-\lambda)f(y))$$

$$\preccurlyeq \left[\frac{\lambda}{g(f(x))} + \frac{1-\lambda}{g(f(y))}\right]^{-1} \text{ (as } g \text{ is } H\text{-convex)}$$

$$= \left[\frac{\lambda}{gof(x)} + \frac{1-\lambda}{gof(y)}\right]^{-1}.$$

So $gof : K \to F$ is H-convex.

Theorem 4.4. *Let* $g_i : K \to F$, $i = 1, 2, ..., m$, *where* $K \subset R^n$ *be* H*-convex* (H*-concave*) *mappings and let* $k_i > 0$, $i = 1, ..., m$ *be positive constants*. *Then*

$$g = \prod_{i=1}^{m} [g_i]^{k_i}$$

is H*-convex* (H*-concave*) *on* F.

Proof. Let $\mu, \nu \in K, \lambda \in (0, 1)$ and let $g_i, i = 1, ..., m$ be H-convex on F. Then

$$g[\lambda\mu + (1-\lambda)\nu] = \prod_{i=1}^{m} g_i[\lambda\mu + (1-\lambda)\nu]^{k_i},$$

$$g_i(\lambda\mu + (1-\lambda)\nu) \preccurlyeq \left[\frac{\lambda}{g_i(\mu)} + \frac{1-\lambda}{g_i(\nu)}\right]^{-1}.$$

Now

$$\tau[g(\lambda\mu + (1-\lambda)\nu)] = \tau\left[\prod_{i=1}^{m} \{g_i[\lambda\mu + (1-\lambda)\nu]^{k_i}\}\right]$$

$$= \prod_{i=1}^{m} [\tau\{g_i[\lambda\mu + (1-\lambda)\nu]\}]^{k_i}$$

$$\leq \prod_{i=1}^{m} \tau\left\{\left[\frac{\lambda}{g_i(\mu)} + \frac{1-\lambda}{g_i(\nu)}\right]^{-k_i}\right\}$$

$$\leq \left[\prod_{i=1}^{m} \tau\left\{\frac{\lambda}{g_i(\mu)} + \frac{1-\lambda}{g_i(\nu)}\right\}^{k_i}\right]^{-1}$$

$$\leq \tau\left[\frac{\lambda}{\left\{\prod_{i=1}^{m} g_i(\mu)\right\}^{k_i}} + \frac{1-\lambda}{\left\{\prod_{i=1}^{m} g_i(\nu)\right\}^{k_i}}\right]^{-1}$$

$$= \tau\left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]^{-1}$$

$$\Rightarrow g(\lambda\mu + (1-\lambda)\nu) \preccurlyeq \left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]^{-1}$$

$\Rightarrow g$ is H-convex.

We can prove similarly for H-concave mappings.

Theorem 4.5. *Let* $g_i : F \to F$, $i = 1, 2, \ldots, m$ *be H-convex* (*H-concave*) *mappings and let* $k_i > 0$, $i = 1, \ldots, m$ *be positive constants. Then*

$$g = \prod_{i=1}^{m} [g_i]^{k_i}$$

is H-convex (*H-concave*) *on* Γ.

Proof. Similar to the proof of Theorem 4.4.

Theorem 4.6. *The reciprocal* $h : F \to F$ *of an H-concave function* $g : F \to F$ *is convex and conversely.*

Proof. Let $g : F \to F$ be an H-concave function. So

$$g(\lambda\mu + (1-\lambda)\nu) \succcurlyeq \left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]^{-1},$$

$$\tau(g(\lambda\mu + (1-\lambda)\nu)) \geq \tau\left(\left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]^{-1}\right)$$

$$= \frac{1}{\tau\left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]}.$$

[As the inverse of a fuzzy number μ is a fuzzy set μ^{-1} and $\mu^{-1}(x) = \mu\left(\frac{1}{x}\right)$].

$$\Rightarrow \frac{1}{\tau(g(\lambda\mu + (1-\lambda)\nu))} \leq \tau\left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]$$

$$\Rightarrow \frac{1}{g(\lambda\mu + (1-\lambda)\nu)} \preccurlyeq \frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}$$

$$\Rightarrow h(\lambda\mu + (1-\lambda)\nu) \preccurlyeq \lambda h(\mu) + (1-\lambda) g(\nu)$$

$\Rightarrow h$ is convex.

Theorem 4.7. *If* $g : K \to F$ *is H-convex for* $i = 1, \ldots, m,$ $K \subset R^n,$ $\nu_i \in K,$ $k_i \geq 0$ *and* $\sum_{i=1}^{m} k_i = 1,$ *then* $g\left(\sum_{i=1}^{m} k_i \nu_i\right) \preccurlyeq \prod_{i=1}^{m} [g(\nu_i)]^{k_i}.$

Proof. The proof follows from the definition of *H*-convex functions.

Theorem 4.8. *If* $g : F \to F$ *is H-convex for* $i = 1, \ldots, m,$ $\nu_i \in F,$ $k_i \geq 0$ *and* $\sum_{i=1}^{m} k_i = 1,$ *then* $g\left(\sum_{i=1}^{m} k_i \nu_i\right) \preccurlyeq \prod_{i=1}^{m} [g(\nu_i)]^{k_i}.$

Theorem 4.9. *Let K be a nonempty convex subset of* $R^n,$ *and let f be a fuzzy mapping on K. Then the following conditions are equivalent*:

1. $f : K \to F$ *is H-convex*;

2. *For* $x, y \in K$ *and* $\lambda \in (0, 1),$

$$\tau(f(\lambda x + (1 - \lambda) y)) \leq \left[\frac{\lambda}{\tau(f(x))} + \frac{1 - \lambda}{\tau(f(y))}\right]^{-1};$$

3. *The epigraph*

$$epi(f) = \{(x, \mu) \mid x \in K, \mu \in F, f(x) \preccurlyeq \mu\}$$

of $f : K \to F$ *is a convex subset of* $R^n \times \nu.$

Proof. Given $f : K \to F$ is *H*-convex

$$\Rightarrow f : K \to F \text{ is convex}$$

$$\Rightarrow f(\lambda x + (1 - \lambda) y) \preccurlyeq \lambda f(x) + (1 - \lambda) f(y)$$

$$\Rightarrow \tau(f(\lambda x + (1 - \lambda) y)) \leq \lambda \tau(f(x)) + (1 - \lambda) \tau(f(y))$$

$$\leq \lambda \tau(\mu) + (1 - \lambda) \tau(\nu) \text{ (as } (x, \mu), (y, \nu) \in epi(f))$$

$$= \tau(\lambda \mu + (1 - \lambda) \nu) \quad \text{for} \quad \lambda \in (0, 1),$$

$$f(\lambda x + (1 - \lambda) y) \preccurlyeq \lambda \mu + (1 - \lambda) \nu,$$

$$(\lambda x + (1 - \lambda) y, \lambda\mu + (1 - \lambda)\nu)$$

$$= \lambda(x, \mu) + (1 - \lambda)(y, \nu) \in epi(f)$$

$$\Rightarrow epi(f) \text{ is a convex subset of } R^n \times \nu.$$

Theorem 4.10. *Let C be a nonempty convex subset of F, and let g be a fuzzy mapping on C. Then if* $g : C \to F$ *is H-convex, then for* $\mu, \nu \in C$ *and* $\lambda \in (0, 1)$,

$$\tau(g(\lambda\mu + (1 - \lambda)\nu)) \leq \lambda\tau(g(\mu)) + (1 - \lambda)\tau(g(\nu))$$

and the epigraph

$$epi(g) = \{(\mu, \omega) | \mu \in C, \omega \in F, g(\mu) \preccurlyeq \omega\}$$

of $g : C \to F$ *is a convex subset of* $\nu \times \nu$.

Proof. Similar to the proof of Theorem 4.9.

Theorem 4.11. *Let K be a nonempty convex subset of* R^n *and let* $g : K \to F$ *be a fuzzy mapping on K. Then if the hypograph*

$$hyp(g) = \{(x, \nu) : x \in K, \nu \in F, g(x) \succcurlyeq \nu\}$$

is a convex subset of $R^n \times \nu$, *then* $g : K \to F$ *is H-concave.*

Proof. Given $hyp(g)$ is a convex subset of $R^n \times \nu$

$\Rightarrow g(\lambda x + (1 - \lambda) y) \succcurlyeq \lambda g(x) + (1 - \lambda) g(y)$ for $x, y \in K, \lambda \in (0, 1)$

$\Rightarrow g : K \to F$ is concave

$\Rightarrow g : K \to F$ is H-concave.

Theorem 4.12. *Let C be a nonempty convex subset of F and let* $g : C \to F$ *be a fuzzy mapping on C. Then if the hypograph*

$$hyp(g) = \{(x, \nu) : x \in C, \nu \in F, g(x) \succcurlyeq \nu\}$$

is a convex subset of $\nu \times \nu$, *then g is H-concave.*

Proof. Similar to the proof of Theorem 4.11.

Definition 4.1. Let K be a nonempty convex subset of R^n. A fuzzy mapping $g : K \to F$ is said to be *quasiconvex* if for $\mu, \nu \in K$ and $\lambda \in (0, 1)$,

$$\tau(g(\lambda\mu + (1 - \lambda)\nu)) \leq \max\{\tau(g(\mu)), \tau(g(\nu))\},$$

i.e.,

$$\tau(g(\mu)) \leq \tau(g(\nu)) \Rightarrow \tau(g(\lambda\mu + (1 - \lambda)\nu)) \leq \tau(g(\nu))$$

and *quasiconcave* if for $\mu, \nu \in K$ and $\lambda \in (0, 1)$,

$$\tau(g(\lambda\mu + (1 - \lambda)\nu)) \geq \min\{\tau(g(\mu)), \tau(g(\nu))\},$$

i.e.,

$$\tau(g(\mu)) \geq \tau(g(\nu)) \Rightarrow \tau[g(\lambda\mu + (1 - \lambda)\nu)] \geq \tau(g(\nu)).$$

Theorem 4.13. *Let* $g : K \to F$ *be an H-concave function. Then g is strictly quasiconcave provided* $\frac{\lambda}{g(\mu)} + \frac{1 - \lambda}{g(\nu)} \prec \frac{1}{g(\mu)}$ *and* $g(\mu) \succ g(\nu)$, *where* $0 < \lambda < 1$ *and* $\mu, \nu \in F$.

Proof. As g is H-concave

$$g(\lambda\mu + (1 - \lambda)\nu) \succcurlyeq \left[\frac{\lambda}{g(\mu)} + \frac{1 - \lambda}{g(\nu)}\right]^{-1}.$$

Furthermore,

$$g(\mu) \succ g(\nu) \Rightarrow \tau[g(\mu)] > \tau[g(\nu)],$$

$$\tau\left[\frac{\lambda}{g(\mu)} + \frac{1 - \lambda}{g(\nu)}\right] < \tau\left[\frac{1}{g(\mu)}\right]$$

$$\Rightarrow \left(\tau\left[\frac{\lambda}{g(\mu)} + \frac{1 - \lambda}{g(\nu)}\right]\right)^{-1} > \left(\tau\left[\frac{1}{g(\mu)}\right]\right)^{-1}$$

$$\Rightarrow \tau\left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]^{-1} > \tau[g(\mu)]$$

$$\Rightarrow \left[\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)}\right]^{-1} \succ g(\mu) \succ g(\nu)$$

$$\Rightarrow g(\lambda\mu + (1-\lambda)\nu) \succ g(\nu)$$

$$\Rightarrow g \text{ is strictly quasiconcave.}$$

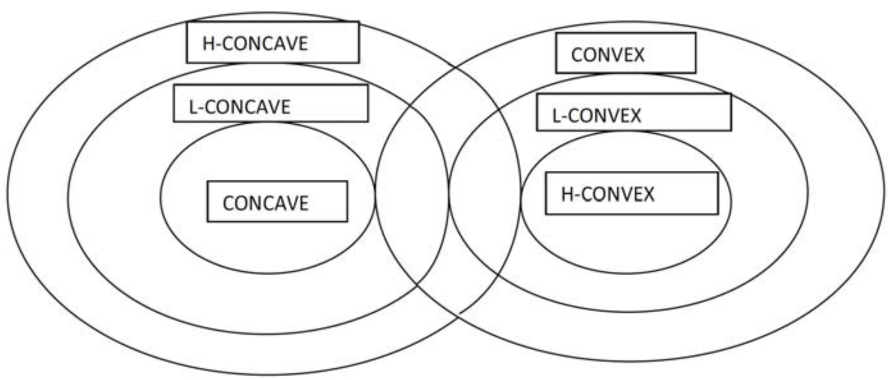
Theorem 4.14. *Let* $g : C \to F$ *be an H-Concave function, where* $C \subseteq F$. *Then g is strictly quasiconcave provided*

$$\frac{\lambda}{g(\mu)} + \frac{1-\lambda}{g(\nu)} \prec \frac{1}{g(\mu)} \text{ and } g(\mu) \succ g(\nu),$$

where $0 < \lambda < 1$ *and* $\mu, \nu \in F$.

Proof. Similar to the proof of Theorem 4.13.

The properties derived in this section are summarized by the following figure.



5. Conclusion

In this paper, we have introduced a new fuzzy convexity known as harmonic fuzzy convexity. We have derived many important results in context to fuzzy harmonic convexity. Fuzzy harmonic convexity is the most generalized version of fuzzy convex mapping. We have only discussed the continuous version of the fuzzy convex mapping. In a subsequent paper we will discuss differentiability and other characteristics of fuzzy harmonic convexity. We assure many results will come out from it. Moreover we will discuss the duality theorem of nonlinear mathematical programming and multiobjective programming problems in context to fuzzy harmonic convexity.

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