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CAYLEY FUZZY GRAPHS

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Abstract

In this paper, we introduce a class of Cayley fuzzy graphs and then study its various graph theoretic properties in terms of algebraic properties. Moreover, we introduce the concepts: α-connectedness, weakly α-connectedness, semi α-connectedness, locally α-connectedness, quasi-α-connectedness and strength of connectivity in fuzzy graphs and then study these concepts in terms of algebraic properties.

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1. Introduction

The study of vertex-transitive graphs has a long and rich history in discrete mathematics. Prominent examples of vertex-transitive graphs are Cayley graphs which are important in both theory as well as applications. For example, Cayley graphs are good models for interconnection networks. Articles [1] and [9] give a survey. Cayley graphs are useful for studying structure of groups and the relationships between elements with respect to subsets of these groups. Cayley graphs are also useful in semigroup theory, for establishing which elements are $\mathcal{L}$ and $\mathcal{R}$ related. In this paper, we introduce a class of Cayley fuzzy graphs induced by groups. We prove that all Cayley fuzzy graphs are vertex-transitive graphs. We also discuss some fuzzy graph properties in terms of algebraic properties. Whenever the word graph is used in this paper, it will be referring to a digraph unless otherwise stated. We follow [11] for standard terminology and notation in fuzzy set theory. Here, we need the following:

Let X be any set. Then a fuzzy subset A of X is a function $A : X \to [0, 1]$. For a fuzzy subset A of X and for $\alpha \in [0, 1]$, $\{x : A(x) \geq \alpha\}$ is called α-*cut* of A and $\{x : A(x) > \alpha\}$ is called the *strong* α-*cut* of A. They are, respectively, denoted by A_α and A_α^+. For a fuzzy subset A of X, the support of A is the set $\{x \in X : A(x) > 0\}$ and is denoted by $\operatorname{supp}(A)$. It can be noted that $\operatorname{supp}(A) = A_0^+$.

Let S and T be two sets and let μ and ν be fuzzy subsets of S and T, respectively. Then a fuzzy relation ρ from μ to ν is a fuzzy subset of $S \times T$ such that $\rho(x, y) \leq \mu(x) \wedge \nu(y)$ for all $x \in S$ and $y \in T$ [11].

1. If $S = T$ and $\mu = \nu$, then ρ is said to be a *fuzzy relation* on μ.

2. If $\mu(x) = 1$ for all $x \in S$ and $\nu(y) = 1$ for all $y \in T$, then ρ is said to be a *fuzzy relation* from S into T.

3. If $S = T$ and $\mu(x) = \nu(x) = 1$ for all $x \in X$, then ρ is said to be a *fuzzy relation* on S.

Let ρ be a fuzzy relation from a fuzzy subset μ of S into a fuzzy subset ν of T and ω be a fuzzy relation from ν into a fuzzy subset ξ of U. Then the composition of ρ and ω denoted by $(\rho \circ \omega)$ is a fuzzy relation from μ into ξ defined by $(\rho \circ \omega)(x, z) = \vee\{\rho(x, y) \vee \omega(y, z) : y \in T\}$ for all $x \in S$ and $z \in U$. For all $n \in \mathbb{N}$, $\rho^{n+1} = \rho^n \circ \rho$, and ρ^∞ is defined by $\rho^\infty(x, y) = \vee\{\rho^k(x, y) : k = 1, 2, ...\}$ for all $x, y \in S$. For any fuzzy relation ρ on S, we define ρ^{-1} as the fuzzy relation given by $\rho^{-1}(x, y) = \rho(y, x)$ for all $x, y \in S$. Let ρ be a fuzzy relation on a fuzzy subset μ of S. Then ρ is said to be:

(1) *reflexive* if $\rho(x, x) = \mu(x)$ for all $x \in S$;

(2) *symmetric* if $\rho(x, y) = \rho(y, x)$ for all $x, y \in S$;

(3) *antisymmetric* if $\rho(x, y) = \rho(y, x)$ if and only if $x = y$;

(4) *transitive* if $\rho^2 \leq \rho$;

(5) a *fuzzy preorder* if it is reflexive and transitive;

(6) a *fuzzy partial order* if it is reflexive, antisymmetric and transitive;

(7) a *fuzzy equivalence relation* if it is reflexive, symmetric and transitive;

(8) a *fuzzy linear order* if it is a partial order and $(\rho \vee \rho^{-1})(x, y) > 0$ for all $x, y \in S$.

A fuzzy directed graph (fuzzy digraph) G is a triplet (V, μ, ρ), where V is a non-empty set, μ is a fuzzy subset of V and ρ is a fuzzy relation on μ. In case $\mu = \chi_V$, where χ_V is the characteristic function on V, then the fuzzy digraph (V, μ, ρ) is simply denoted by $G = (V, \rho)$. Furthermore, G is said to be a *fuzzy graph* if the fuzzy relation is symmetric. In this sequel, we consider fuzzy digraphs of the form $G = (V, \rho)$. Let $G = (V, \rho)$ and $G' = (V', \rho')$ be two fuzzy digraphs. Then G is said to be *isomorphic* to G' if there is a bijection $f : V \to V'$ such that for all $u, v \in V$, $\rho(u, v) = \rho'(f(u), f(v))$. Here f is called an *isomorphism* from G into G'. An isomorphism from a fuzzy digraph G onto itself is called an *automorphism*. Observe that, if $(V, *)$ is a group and ν is a fuzzy subset of V, then $R : V \times V \to [0, 1]$ defined by $R(x, y) = \nu(x^{-1}y)$ for all $x, y \in V$ is a fuzzy relation on V.

Let $G = (V, \rho)$ be a fuzzy digraph. If $u \in V$, then the in-degree of u, denoted by $\operatorname{ind}(u)$, is defined by

$$\operatorname{ind}(u) = \sum_{v \in V} \rho(v, u).$$

Similarly, the out-degree of u, denoted by $\operatorname{outd}(u)$, is defined by

$$\operatorname{outd}(u) = \sum_{v \in V} \rho(u, v).$$

A fuzzy digraph in which each vertex has the same out-degree r is called an *out-regular digraph* with index of out-regularity r. In-regular digraphs are similarly defined. Let $G = (V, R)$ be a fuzzy digraph. Let k and k' be two positive numbers. Then G is said to be (k, k')*-regular* if $\operatorname{ind}(u) = k$ and $\operatorname{outd}(u) = k'$ for all $u \in V$. A fuzzy digraph is said to be *regular* if it is (k, k)-regular for some positive number k. Let $G = (V, R)$ be a fuzzy digraph. Then a path (directed path) of length n in G from a vertex x to a vertex y is a sequence of distinct vertices $x = x_0, x_1, \ldots, x_n = y$ such that $R(x_{i-1}, x_i) > 0$ for $1 \leq i \leq n$. A fuzzy digraph $G = (V, R)$ is said to be: (i) *connected* (*strongly connected*) if for all $x, y \in V$, there is a directed path from x to y, (ii) *weakly connected* if $G' = (V, R \vee R^{-1})$ is connected, (iii) *semi-connected* if for all $x, y \in V$, there is a directed path from x to y or there is a directed path from y to x in G, (iv) *locally connected*, if for any $x, y \in V$, there is a directed path from x to y whenever there is directed path from y to x in G, (v) *quasi-connected* (*quasi-strongly connected*) if for every pair $x, y \in V$, there is some $z \in V$ such that there is a directed path from z to x and there is a directed path from z to y, (vi) *Hasse diagram*, if G is connected and for any path $x_0, x_1, ..., x_n$, $n \geq 2$ from x_0 to x_n in G, $R(x_0, x_n) = 0$ and (vii) *complete* if $R(x, y) = 1$ for all $x, y \in V$. A vertex x in G is said to be a *source* in G if there is a directed path from X to every other vertex in G.

Let $G = (V, R)$ be a fuzzy graph. The distance between two points u and v in G, $d(u, v)$, is the length of the shortest path from u to v. If there is no path from u to v, then we define $d(u, v) = \infty$. The diameter of a fuzzy graph $G = (V, R)$, denoted by $\operatorname{diam}(G)$, is defined as

$$\operatorname{diam}(G) = \sup\{d(u, v) : u, v \in V\}.$$

Let $(V, *)$ be a group and A be any subset of V. Then the Cayley graph induced by $(V, *, A)$ is the graph $G = (V, R)$, where $R = \{(x, y) : x^{-1}y \in A\}$.

2. Cayley Fuzzy Graphs

In this section, we introduce Cayley fuzzy graph and prove that all Cayley fuzzy graphs are vertex-transitive and hence regular.

We start with the following:

Definition 2.1. Let $(V, *)$ be a group and ν be a fuzzy subset of V. Then the fuzzy relation R defined on V by $R(x, y) = \nu(x^{-1} * y)$ for all $x, y \in V$ induces a fuzzy graph $G = (V, R)$ called the *Cayley fuzzy graph* induced by the triple $(V, *, \nu)$.

Example 1. Let us consider the group $\mathbb{Z}_4$ and take $V = \{0, 1, 2, 3\}$. Define $\nu : V \to [0, 1]$ by $\nu(0) = 1$, $\nu(1) = \frac{1}{2}$, $\nu(2) = \frac{1}{4}$ and $\nu(3) = 0$. Then the Cayley fuzzy graph $G = (V, R)$ induced by $(\mathbb{Z}_4, \nu)$ is given by the following table and Figure 1.

A	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
B	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
$a^{-1}b$	0	1	2	3	3	0	1	2	2	3	0	1	1	2	3	0
$\nu(a^{-1}b)$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0	1

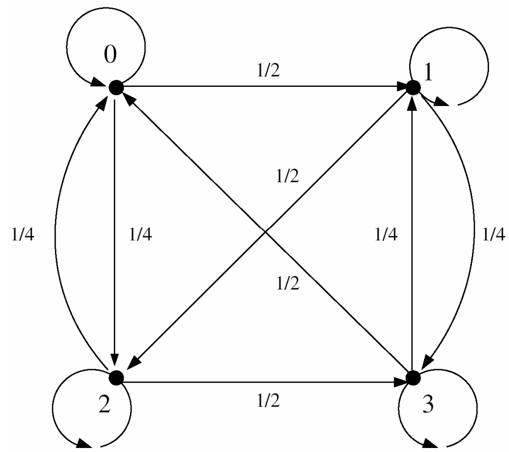

Figure 1. The fuzzy Cayley graph $G = (V, R)$.

Observe that Cayley fuzzy graphs are actually fuzzy digraphs. Furthermore, the relation R in the above definition describes the strength of each directed edge. We define a fuzzy graph G to be *vertex-transitive*, if for $x, y \in V$, there is an automorphism f on G such that $f(x) = y$. Let G denote fuzzy graph $G = (V, R)$ induced by the triple $(V, *, \nu)$. First, we will show that G is vertex-transitive.

Theorem 2.2. *The Cayley fuzzy graph G is vertex-transitive.*

Proof. Let $a, b \in V$. Define $\psi : V \to V$ by $\psi(x) = ba^{-1}x$ for all $x \in V$. Clearly, ψ is a bijection onto itself.

Furthermore, we have, for each $x, y \in V$,

$$R(\psi(x), \psi(y)) = R(ba^{-1}x, ba^{-1}y)$$

$$= \nu((ba^{-1}x)^{-1}(ba^{-1}y))$$

$$= \nu(x^{-1}y)$$

$$= R(x, y).$$

Thus, ψ is an automorphism on G. Also, $\psi(a) = b$. Hence G is vertex-transitive. This completes the proof. □

Theorem 2.3. *Every vertex-transitive fuzzy graph is in-regular and out-regular*.

Proof. Let $G = (V, R)$ be any vertex-transitive fuzzy graph and $u, v \in V$. Then there is an automorphism f on G such that $f(u) = v$. Note that

$$\mathrm{ind}(u) = \sum_{x \in V} R(x, u)$$

$$= \sum_{x \in V} R(f(x), f(u))$$

$$= \sum_{f(x) \in V} R(f(x), v)$$

$$= \sum_{y \in V} R(y, v) = \mathrm{ind}(v).$$

Similarly, we can prove that $\mathrm{outd}(u) = \mathrm{outd}(v)$. Hence G is in-regular and out-regular. □

From Theorems 2.2 and 2.3, we have the following.

Theorem 2.4. *Cayley fuzzy graphs are in-regular and out-regular*.

2.1. Basic results

In this subsection, we express many algebraic properties in terms of fuzzy graph properties. Let G denote fuzzy graph $G = (V, R)$ induced by the triple $(V, *, \nu)$. Then we have the following results.

Theorem 2.5. *The fuzzy relation R is reflexive if and only if* $\nu(1) = 1$.

Proof. Observe that R is reflexive if and only if $R(x, x) = 1$ for all $x \in V$. Now $R(x, x) = \nu(x^{-1}x) = \nu(1)$ for all $x \in V$. Therefore, R is reflexive if and only if $\nu(1) = 1$. □

Theorem 2.6. *The fuzzy relation R is symmetric if and only if* $\nu(x) = \nu(x^{-1})$ *for all* $x \in V$.

Proof. Suppose that R is symmetric. Then, for any $x \in V$,

$$\nu(x) = \nu(x^{-1}x^2) = R(x, x^2) = R(x^2, x) = \nu(x^{-1}x^{-1}x) = \nu(x^{-1}).$$

Conversely, suppose that $\nu(x) = \nu(x^{-1})$ for all $x \in V$. Then, for all $x, y \in V$,

$$R(x, y) = \nu(x^{-1}y) = \nu(y^{-1}x) = R(y, x).$$

This implies that R is symmetric. □

Theorem 2.7. *The fuzzy relation R is antisymmetric if and only if*

$$\{x : \nu(x) = \nu(x^{-1})\} = \{1\}.$$

Proof. Suppose that R is antisymmetric and let $x \in V$. Then $\nu(x) = \nu(x^{-1})$ implies that $R(1, x) = R(x, 1)$ and hence $x = 1$. Thus, $\{x : \nu(x) = \nu(x^{-1})\} = \{1\}$. Conversely, suppose that $\{x : \nu(x) = \nu(x^{-1})\} = \{1\}$. Then, for any $x, y \in V$,

$$R(x, y) = R(y, x) \Leftrightarrow \nu(x^{-1}y) = \nu(y^{-1}x).$$

This implies that $\nu(x^{-1}y) = \nu((x^{-1}y)^{-1})$. That is, $x^{-1}y = 1$. Equivalently, $x = y$. Hence R is antisymmetric. □

Definition 2.8. Let $(S, *)$ be a semigroup. Let A be a fuzzy subset of S. Then A is said to be *fuzzy sub-semigroup* of S if for all $a, b \in S$, $A(ab) \geq A(a) \wedge A(b)$.

Theorem 2.9. *The fuzzy relation R is transitive if and only if ν is a fuzzy sub-semigroup of* $(V, *)$.

Proof. Suppose that R is transitive and let $x, y \in V$. Then $R^2 \subseteq R$. Also, we have $R(1, x) = \nu(x)$. This implies that

$$\vee\{R(1, z) \wedge R(z, xy) : z \in V\} = R^2(1, xy) \leq R(1, xy) = \nu(xy).$$

That is, $\vee\{\nu(z) \wedge \nu(z^{-1}(xy)) : z \in V\} \leq \nu(xy)$. Hence $\nu(x) \wedge \nu(y) \leq \nu(xy)$. Thus, ν is a fuzzy sub-semigroup of $(V, *)$.

Conversely, suppose that ν is a fuzzy sub-semigroup of $(V, *)$. That is, $\nu(xy) \geq \nu(x) \wedge \nu(y)$ for all $x, y \in V$. Then, for any $x, y \in V$,

$$\begin{aligned} R^2(x, y) &= \vee\{R(x, z) \wedge R(z, y) : z \in V\} \\ &= \vee\{\nu(x^{-1}z) \wedge \nu(z^{-1}y) : z \in V\} \\ &\leq \nu(x^{-1}y) = R(x, y). \end{aligned}$$

Thus, $R^2 \subseteq R$ and hence R is transitive. □

Theorem 2.10. *The fuzzy relation R is a partial order if and only if ν is a fuzzy sub-semigroup of* $(V, *)$ *satisfying*:

(i) $\nu(1) = 1$ *and*

(ii) $\{x : \nu(x) = \nu(x^{-1})\} = \{1\}$.

Theorem 2.11. *The fuzzy relation R is a linear order if and only if ν is a fuzzy sub-semigroup of* $(V, *)$ *satisfying*:

(i) $\nu(1) = 1$,

(ii) $\{x : \nu(x) = \nu(x^{-1})\} = \{1\}$ *and*

(iii) $\{x : \nu(x) \vee \nu(x^{-1}) > 0\} = V$.

Proof. Suppose that R is a linear order. Then, by Theorem 2.10, the conditions (i) and (ii) are satisfied. Now, for any $x \in V$, $(R \vee R^{-1})(1, x) > 0$. This implies that $R(1, x) \vee R(x, 1) > 0$. Hence $\nu(x) \vee \nu(x^{-1}) > 0$.

Conversely, suppose that the conditions (i), (ii) and (iii) hold. Then, by Theorem 2.10, R is a partial order. Now, for any $x, y \in V$, we have $(x^{-1}y), (y^{-1}x) \in V$. Then, by condition (iii), $\nu(x^{-1}y) \vee \nu(y^{-1}x) > 0$. That is, $R(x, y) \vee R(y, x) > 0$. Hence $(R \vee R^{-1})(x, y) > 0$. Thus, R is a linear order. □

Theorem 2.12. *The fuzzy relation R is an equivalence relation if and only if ν is a fuzzy sub-semigroup of $(V, *)$, satisfying*:

(i) $\nu(1) = 1$ *and*

(ii) $\nu(x) = \nu(x^{-1})$ *for all* $x \in V$.

Theorem 2.13. *G is a Hasse diagram if and only if G is connected and*

$$\nu(x_1 x_2 \cdots x_n) = 0$$

for any collection $x_0, x_1, x_2, x_3, ..., x_n$ *of vertices in V with* $n \geq 2$ *and* $\nu(x_i) > 0$ *for* $i = 1, 2, ..., n$.

Proof. Suppose that G is a Hasse diagram and let $x_1, x_2, ..., x_n$ be vertices in V with $n \geq 2$ and $\nu(x_i) > 0$ for $i = 1, 2, ..., n$. Then it is obvious that $R(x_1 x_2 \cdots x_{i-1}, x_1 x_2 \cdots x_i) = \nu(x_i) > 0$ for $i = 1, 2, ..., n$, where $x_0 = 1$. Thus, $(1, x_1, x_1 x_2, ..., x_1 x_2 \cdots x_n)$ is a path from 1 to $x_1 x_2 \cdots x_n$. Since G is a Hasse diagram, we have $R(1, x_1 x_2 \cdots x_n) = 0$. This implies that $\nu(x_1 x_2 \cdots x_n) = 0$.

Conversely, suppose that for any collection $x_1, x_2, ..., x_n$ of vertices in V with $n \geq 2$ and $\nu(x_i) > 0$ for $i = 1, 2, ..., n$, we have $\nu(x_1 x_2 \cdots x_n) = 0$. Let $(x_0, x_1, x_2, ..., x_n)$ be a path in G from x_0 to x_n with $n \geq 2$. Then

$R(x_{i-1}, x_i) > 0$ for $i = 1, 2, ..., n$. Therefore, $\nu(x_{i-1}^{-1}x_i) > 0$ for $i = 1, 2, ..., n$. Now consider the elements $x_0^{-1}x_1, x_1^{-1}x_2, ..., x_{n-1}^{-1}x_n$ in V. Then, by assumption, $\nu(x_0^{-1}x_1x_1^{-1}x_2, ..., x_{n-1}^{-1}x_n) = 0$. That is, $\nu(x_0^{-1}x_n) = 0$. Hence, $R(x_0, x_n) = 0$. Thus, G is a Hasse diagram. □

Theorem 2.14. *For all* $u, v \in V$, *we have*

$$\mathrm{ind}(u) = \sum_{v \in V} \nu(v) = outd(u).$$

That is, *Cayley fuzzy graphs are regular*.

Proof. By Theorem 2.4, it suffices to consider the in-degree and out-degree of the vertex 1. Observe that

$$\mathrm{ind}(1) = \sum_{v \in V} R(v, 1) = \sum_{v \in V} \nu(v^{-1}) = \sum_{v \in V} \nu(v) = \sum_{v \in V} R(1, v) = \mathrm{outd}(1).$$

This completes the proof. □

Theorem 2.15. *G is complete if and only if* $\nu \geq \chi_{V-\{1\}}$, *where* $\chi_{V-\{1\}}$ *is the characteristic function of* $V - \{1\}$.

Proof. Suppose that G is complete. Then, for every $x, y \in V$, $x \neq y$, $R(x, y) = 1$. This implies that $\nu(x^{-1}y) = 1$ for all $x, y \in V$ with $x \neq y$. Therefore, for any $x \in V$ with $x \neq 1$, $\nu(x) = \nu(1x) = 1$. Thus, $\nu \geq \chi_{V-\{1\}}$.

Conversely, suppose that $\nu \geq \chi_{V-\{1\}}$. Then, for any $x, y \in V$ with $x \neq y$, we have $R(x, y) = \nu(x^{-1}y) = 1$. Hence G is complete. □

Theorem 2.16. *Let G be finite and connected*. *Then* $diam(G)$ *is the least positive integer n such that for any* $x \in V$, *there exist* $x_1, x_2, ..., x_n \in V$ *with* $\nu(x_i) > 0$ *for* $i = 1, 2, ..., n$ *and* $x = x_1x_2 \cdots x_n$.

Proof. Suppose n is the least positive integer such that for any $x \in V$, there exist elements $x_1, x_2, ..., x_n \in V$ with $\nu(x_i) > 0$ for $i = 1, 2, ..., n$ and

$x = x_1 x_2 \cdots x_n$. Note that, for any $x, y \in V$, we have $x^{-1}y \in V$. Then, by assumption, there exist $x_1, x_2, ..., x_n \in V$ with $\nu(x_i) > 0$ for $i = 1, 2, ..., n$ and $x^{-1}y = x_1 x_2 \cdots x_n$. Therefore, $y = xx_1 x_2 \cdots x_n$. Consider the sequence

$$x, xx_1, ..., xx_1 \cdots x_n = y. \tag{1}$$

Observe that $R(xx_1 \cdots x_{i-1}, xx_1 \cdots x_i) = \nu(x_i) > 0$ for $i = 1, 2, ..., n$, where $x_0 = x$. Thus, (1) represents a path of length *n* from *x* to *y*. Since *x* and *y* are arbitrary, $\operatorname{diam}(G) \leq n$. Since *n* is the least positive integer such that for any $x \in V$, there exist $x_1, x_2, ..., x_n \in V$ with $\nu(x_i) > 0$ for $i = 1, 2, ..., n$ and $x = x_1 x_2 \cdots x_n$. It is clear that there exists an $x \in V$ such that for any collection of $n - 1$ elements, say $x_1, x_2, ..., x_{n-1}$ with $\nu(x_i) > 0$ for $i = 1, 2, ..., n - 1$, we have

$$x \neq x_1 x_2 \cdots x_{n-1}. \tag{2}$$

Suppose that $\operatorname{diam}(G) \leq n - 1$. Then there is a path $1, x_1, x_2, ..., x_m$ from 1 to *x* of length *m*, where $m \leq n - 1$. Then $\nu(x_{i-1}^{-1} x_i) = R(x_{i-1}, x) > 0$ for $i = 1, 2, ..., m$, where $x_0 = 1$. Also, note that $(x_1)(x_1^{-1} x_2) \cdots (x_{m-1}^{-1} x_m) = x$. Thus, $x = (x_1)(x_1^{-1} x_2) \cdots (x_{m-1}^{-1} x_m) x_{m+1} \cdots x_{n-1}$, where $x_{m+1} = x_{m+2} = \cdots = x_{n-1} = 1$, a contradiction to (3). Thus, $\operatorname{diam}(G) > n - 1$. Since $\operatorname{diam}(G) \leq n$, it follows that $\operatorname{diam}(G) = n$. □

3. Cayley Graph Induced by Cayley Fuzzy Graphs

Let $(V, *)$ be a group, ν be a fuzzy subset of *V* and $G = (V, R)$ be the Cayley fuzzy graph induced by $(V, *, \nu)$. Also, for any $\alpha \in [0, 1]$, let ν_α be the α-cut of ν and ν_α^+ be the strong α-cut of ν. Define S_{ν_α} and $S_{\nu_\alpha^+}$ as

$S_{\nu_\alpha} = \{(x, y) \in V \times V : x^{-1}y \in \nu_\alpha\}$ and

$$S_{\nu_\alpha^+} = \{(x, y) \in V \times V : x^{-1}y \in \nu_\alpha^+\}.$$

Then it is clear that for any $\alpha \in [0, 1]$, the Cayley fuzzy graph induced by $(V, *, \nu)$ induces the Cayley graphs (V, S_{ν_α}) and $(V, S_{\nu_\alpha^+})$.

Now it can be noted that for any $\alpha \in [0, 1]$, $S_{\nu_\alpha} = R_\alpha$ and $S_{\nu_\alpha^+} = R_\alpha^+$. Thus, for any $\alpha \in [0, 1]$, the Cayley fuzzy graph (V, R) induces the Cayley graphs (V, R_α) and (V, R_α^+). □

Remark 1. Let $G = (V, R)$ be any fuzzy graph. Then G is connected (weakly connected, semi-connected, locally connected or quasi-connected) if and only if the induced graph (V, R_0^+) is connected (weakly connected, semi-connected, locally connected or quasi-connected).

We now observe the following definition and lemma to study different types of connectedness of G.

Definition 3.1. Let $(S, *)$ be a semigroup and let A be a fuzzy subset of S. Then the fuzzy sub-semigroup generated by A is the smallest fuzzy sub-semigroups of S which contains A and is denoted by $\langle A \rangle$.

Example 2. Consider $S = Z_4$ and ν as in Example 1. Then $\langle \nu \rangle$ is given by $\langle \nu \rangle (0) = 1$, and $\langle \nu \rangle (y) = \frac{1}{2}$ for $y = 1, 2, 3$.

Remark 2. Let $(S, *)$ be a semigroup and let A be a fuzzy subset of S. Then the fuzzy sub-semigroup generated by A is the meet of all fuzzy sub-semigroups of S which contains A.

Lemma 3.2. *Let $(S, *)$ be a semigroup and ν be a fuzzy subset of S. Then the fuzzy subset $\langle \nu \rangle$ is precisely given by*

$$\langle \nu \rangle (x) = \vee\{\nu(x_1) \wedge \nu(x_2) \wedge \cdots \wedge \nu(x_n) : x = x_1 x_2 \cdots x_n \text{ with a finite positive integer } n,\ x_i \in S \text{ and } \nu(x_i) > 0 \text{ for } i = 1, 2, \dots, n\}$$

for any $x \in S$.

Proof. Let ν' be the fuzzy subset of S defined by $(\nu')(x) = \vee\{\nu(x_1) \wedge \nu(x_2) \wedge \cdots \wedge \nu(x_n) : x = x_1 x_2 \cdots x_n$ with a finite positive integer n, $x_i \in S$ and $\nu(x_i) > 0$ for $i = 1, 2, ..., n\}$ for any $x \in S$. If $y \in S$ and $\nu(y) > 0$, then by definition of ν', it is clear that $\nu'(y) \geq \nu(y)$. Thus, we have

$$\nu \leq \nu'. \tag{3}$$

Let $x, y \in S$. If $\nu(x) = 0$ or $\nu(y) = 0$, then $\nu(x) \wedge \nu(y) = 0$. Therefore, $\nu'(xy) \geq \nu(x) \wedge \nu(y)$. Again, if $\nu(x) \neq 0$ and $\nu(y) \neq 0$, then by definition of ν', we have $\nu'(xy) \geq \nu(x) \wedge \nu(y)$. Hence ν' is a fuzzy sub-semigroup of S containing ν. Now let A be any fuzzy sub-semigroup of S containing ν. Then, for any $x \in S$ with $x = x_1 x_2 \cdots x_n$ with a finite positive integer n, $x_i \in S$ and $\nu(x_i) > 0$ for $i = 1, 2, ..., n$, we have

$$A(x) \geq A(x_1) \wedge A(x_2) \wedge \cdots \wedge A(x_n) \geq \nu(x_1) \wedge \nu(x_2) \wedge \cdots \wedge \nu(x_n).$$

Thus, $A(x) \geq \vee\{\nu(x_1) \wedge \nu(x_2) \wedge \cdots \wedge \nu(x_n) : x = x_1 x_2 \cdots x_n$ with a finite positive integer n, $x_i \in S$ and $\nu(x_i) > 0$ for $i = 1, 2, ..., n\}$ for any $x \in S$. Hence $A(x) \geq \nu'(x)$ for all $x \in S$. Thus, $\nu' \leq A$. Thus, ν' is the meet of all fuzzy sub-semigroups containing ν. □

Theorem 3.3. *Let $(S, *)$ be a semigroup and ν be a fuzzy subset of S. Then, for any $\alpha \in [0, 1]$, $\langle \nu_\alpha \rangle = \langle \nu \rangle_\alpha$ and $\langle \nu_\alpha^+ \rangle = \langle \nu \rangle_\alpha^+$, where $\langle \nu_\alpha \rangle$ denotes the sub-semigroup generated by ν_α and $\langle \nu \rangle$ denotes the fuzzy sub-semigroup generated by ν.*

Proof. Observe that

$$\begin{aligned} x \in \langle \nu_\alpha \rangle &\Leftrightarrow \text{there exist } x_1, x_2, ..., x_n \text{ in } \nu_\alpha \text{ such that } x = x_1 x_2 \cdots x_n \\ &\Leftrightarrow \text{there exist } x_1, x_2, ..., x_n \text{ in } S \text{ such that } \nu(x_i) \geq \alpha \text{ for all} \\ &\qquad i = 1, 2, ..., n \text{ and } x = x_1 x_2 \cdots x_n \end{aligned}$$

$$\Leftrightarrow \langle \nu \rangle (x) \geq \alpha$$

$$\Leftrightarrow x \in \langle \nu \rangle_{\alpha}.$$

Therefore, $\langle \nu_{\alpha} \rangle = \langle \nu \rangle_{\alpha}$. Similarly, we have $\langle \nu_{\alpha}^{+} \rangle = \langle \nu \rangle_{\alpha}^{+}$. □

Remark 3. Let $(S, *)$ be a semigroup and ν be a fuzzy subset of S. Then, by Theorem 3.3, we have $\langle \mathrm{supp}(\nu) \rangle = \mathrm{supp}\langle \nu \rangle$.

4. Connectedness in Cayley Fuzzy Graphs

Let G denote the Cayley fuzzy graph $G = (V, R)$ induced by the triple $(V, *, \nu)$. Then we have the following results.

Theorem 4.1. *Let A be any subset of a set V' and $G' = (V', R')$ be the Cayley graph induced by the triplet $(V', *, A)$. Then G' is connected if and only if $\langle A \rangle \supseteq V - \{1\}$.*

Theorem 4.2. *G is connected if and only if $\mathrm{supp}\langle \nu \rangle \supseteq V - \{1\}$.*

Proof. By Remarks 1, 3 and by Theorem 4.1,

$$\begin{aligned} G \text{ is connected } &\Leftrightarrow (V, R_0^{+}) \text{ is connected} \\ &\Leftrightarrow \langle \nu_0^{+} \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \langle \mathrm{supp}(\nu) \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \mathrm{supp}\langle \nu \rangle \supseteq V - \{1\}. \end{aligned}$$

This completes the proof. □

Theorem 4.3. *Let A be any subset of a set V' and $G' = (V', R')$ be the Cayley graph induced by the triplet $(V', *, A)$. Then G' is weakly connected if and only if $\langle A \cup A^{-1} \rangle \supseteq V - \{1\}$, where $A^{-1} = \{x^{-1} : x \in A\}$.*

Definition 4.4. Let $(S, *)$ be a group and A be a fuzzy subset of S. Then we define A^{-1} as the fuzzy subset of S given by $A^{-1}(x) = A(x^{-1})$ for all $x \in S$.

Theorem 4.5. *G is weakly connected if and only if*

$$\operatorname{supp}(\langle \nu \vee \nu^{-1} \rangle) \supseteq V - \{1\}.$$

Proof. By Remarks 1, 3 and by Theorem 4.3,

$$\begin{aligned} G \text{ is weakly connected} &\Leftrightarrow (V, R_0^+) \text{ is weakly connected} \\ &\Leftrightarrow \langle \nu_0^+ \cup (\nu_0^+)^{-1} \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \langle \operatorname{supp}(\nu) \cup \operatorname{supp}((\nu)^{-1}) \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \langle \operatorname{supp}(\nu \vee (\nu)^{-1}) \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \operatorname{supp}(\langle \nu \vee \nu^{-1} \rangle) \supseteq V - \{1\}. \end{aligned}$$

This completes the proof. □

Theorem 4.6. *Let A be any subset of a set V' and $G' = (V', R')$ be the Cayley graph induced by the triplet $(V', *, A)$. Then G' is semi-connected if and only if $\langle A \rangle \cup \langle A^{-1} \rangle \supseteq V - \{1\}$, where $A^{-1} = \{x^{-1} : x \in A\}$.*

Theorem 4.7. *Let G be semi-connected if and only if*

$$\operatorname{supp}(\langle \nu \rangle \vee \langle \nu^{-1} \rangle) \supseteq V - \{1\}.$$

Proof. By Remarks 1, 3 and by Theorem 4.6,

$$\begin{aligned} G \text{ is semi-connected} &\Leftrightarrow (V, R_0^+) \text{ is semi-connected} \\ &\Leftrightarrow \langle \nu_0^+ \rangle \cup \langle (\nu_0^+)^{-1} \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \langle \operatorname{supp}(\nu) \rangle \cup \langle \operatorname{supp}((\nu)^{-1}) \rangle \supseteq V - \{1\} \end{aligned}$$

$$\Leftrightarrow \operatorname{supp}\langle \nu \rangle \cup \operatorname{supp}\langle \nu^{-1} \rangle \supseteq V - \{1\}$$

$$\Leftrightarrow \operatorname{supp}(\langle \nu \rangle \vee \langle \nu^{-1} \rangle) \supseteq V - \{1\}.$$

This completes the proof. □

Theorem 4.8. *Let $G' = (V', R')$ be the Cayley graph induced by the triplet $(V', *, A)$. Then G' is locally connected if and only if $\langle A \rangle = \langle A^{-1} \rangle$, where $A^{-1} = \{x^{-1} : x \in A\}$.*

Theorem 4.9. *Let G be locally connected if and only if*

$$\operatorname{supp}(\langle \nu \rangle) = \operatorname{supp}(\langle \nu^{-1} \rangle).$$

Proof. By Remarks 1, 3 and by Theorem 4.8,

$$\begin{aligned} G \text{ is locally connected} &\Leftrightarrow (V, R_0^+) \text{ is locally connected} \\ &\Leftrightarrow \langle \nu_0^+ \rangle = \langle (\nu_0^+)^{-1} \rangle \\ &\Leftrightarrow \langle \operatorname{supp}(\nu) \rangle = \langle \operatorname{supp}((\nu)^{-1}) \rangle \\ &\Leftrightarrow \operatorname{supp}\langle \nu \rangle = \operatorname{supp}\langle \nu^{-1} \rangle. \end{aligned}$$

This completes the proof. □

Theorem 4.10 [12]**.** *A finite digraph G' has a source if and only if it is quasi-connected.*

Theorem 4.11. *Let $G' = (V', R')$ be the Cayley graph induced by the triplet $(V', *, A)$, where V' is finite. Then G' is quasi-connected if and only if it is connected.*

Proof. We know that every connected graph is quasi-connected. Thus, we have to prove only the other part. First, note that G' is finite. Thus, by Theorem 4.10, G' has a source, say z. Then, for any $x \in V'$ with $x \neq z$, there is a directed path from z to x. Thus, it is clear that $z^{-1}x \in \langle A \rangle$ for every

$x \in V$ with $x \neq z$. Hence $\langle A \rangle \supseteq V - \{1\}$. Hence, by Theorem 4.1, G' is connected. □

Theorem 4.12. *A finite Cayley fuzzy graph G is quasi-connected if and only if it is connected.*

Proof. By Remarks 1, 3 and by Theorem 4.11,

$$\begin{aligned} G \text{ is quasi-connected} &\Leftrightarrow (V, R_0^+) \text{ is quasi-connected} \\ &\Leftrightarrow (V, R_0^+) \text{ is connected} \\ &\Leftrightarrow G \text{ is connected.} \end{aligned}$$

This completes the proof. □

5. Strength of Connectedness in Cayley Fuzzy Graphs

In this section, we introduce the concepts: α-connectedness, weakly α-connectedness, semi α-connectedness, locally α-connectedness, quasi-α-connectedness and strength of connectivity of fuzzy graphs.

Definition 5.1. Let $P = (x_0, x_1, ..., x_n)$ be a path in a fuzzy graph $G = (V, \rho)$. Then the strength of a path P in G, denoted $strength(P)$, is defined as [10]

$$strength(P) = \wedge_{i=1}^{n} \rho(x_{i-1}, x_i).$$

Let $\alpha \in (0, 1]$. Then we define the following:

Definition 5.2. Let $G = (V, \rho)$ be a fuzzy graph. Then G is said to be: (i) α-*connected* if for every pair of vertices $x, y \in G$, there is a path P from x to y such that $strength(P) \geq \alpha$, (ii) *weakly* α-*connected* if the fuzzy graph $(V, R \vee R^{-1})$ is α-connected, (iii) *semi* α-*connected* if for every $x, y \in V$, there is a path of strength greater than or equal to α from x to y or from y to x in G and (iv) *locally* α-*connected* if for every pair of vertices x and y, there is a path P of strength greater than or equal to α from x to y whenever there is a path P' of strength greater than or equal to α from y to x. (v) *quasi* α-*connected* if for every pair $x, y \in V$, there is some $z \in V$ such that there is a directed path from z to x of strength greater than or equal to α and there is a directed path from z to y of strength greater than or equal to α.

Observe that if $\alpha, \beta \in (0, 1]$, $\alpha < \beta$ and G is β-connected, then G is also α-connected. Thus, a finite fuzzy graph G is connected if it is α-connected for some $\alpha \in (0, 1]$. But for infinite fuzzy graphs, this is not true. For example, consider the graph $G = (\mathbb{N}, R)$, where $R(m, n) = \frac{1}{n}$ if $n - m = 1$, $R(m, n) = 1$ if $m = n$ and $R(m, n) = 0$ otherwise. Then G is not α-connected for any $\alpha \in (0, 1]$ but it is connected. A fuzzy graph $G = (V, R)$ is said to be α-*complete* if $R(x, y) \geq \alpha$ for all $x, y \in V$. Observe that any complete fuzzy graph is α-complete for all $\alpha \in [0, 1]$.

Definition 5.3. Let $G = (V, \rho)$ be a fuzzy graph and let $x, y \in V$. Let $\mathscr{P}_G(x, y)$ denote the set of all paths in G from x to y. Then the strength of connectedness between x and y, denoted $CONN_G(x, y)$, is defined as

$$CONN_G(x, y) = \bigvee_{p \in \mathscr{P}_G(x, y)} strength(P).$$

We define the *strength connectivity* of G as

$$SC(G) = \bigwedge_{x, y \in V} CONN_G(x, y).$$

5.1. Different types of α-connectedness in Cayley fuzzy graphs

In this subsection, we prove the following theorems based on different types of α-connectedness. Let $(V, *)$ be a group, ν be a fuzzy subset of V and $G = (V, R)$ be the Cayley fuzzy graph induced by $(V, *, \nu)$. Also, for any $\alpha \in [0, 1]$, let ν_α be the α-cut of ν. Then

$$R_\alpha = \{(x, y) \in V \times V : x^{-1}y \in \nu_\alpha\}.$$

Remark 4. Let $G = (V, R)$ be any fuzzy graph. Then G is α-connected (weakly α-connected, semi α-connected, locally α-connected or quasi α-connected) if and only if the induced graph (V, R_α) is connected (weakly connected, semi-connected, locally connected or quasi-connected).

Theorem 5.4. *The Cayley fuzzy graph G is* α*-connected if and only if* $\langle \nu \rangle_\alpha \supseteq V - \{1\}$.

Proof. By Remark 4 and by Theorems 3.3, 4.1, we have

$$\begin{aligned} G \text{ is } \alpha\text{-connected} &\Leftrightarrow (V, R_\alpha) \text{ is connected} \\ &\Leftrightarrow \langle \nu_\alpha \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \langle \nu \rangle_\alpha \supseteq V - \{1\}. \end{aligned}$$

This completes the proof. □

Theorem 5.5. *Let x and y be any two vertices of the Cayley fuzzy graph* $G = (V, R)$ *induced by* $(V, *, \nu)$. *Then* $CONN_G(x, y) = \langle \nu \rangle (x^{-1}y)$.

Proof. Let $\alpha \in (0, 1]$. Suppose that $CONN_G(x, y) = \alpha$. Then, for any $\varepsilon > 0$, there exists a path, say $P = (x, x_1, x_2, ..., x_n, y)$ from x to y in G such that strength$(P) > \alpha - \varepsilon$. This implies that

$$R(x_{i-1}, x_i) > \alpha - \varepsilon \text{ for all } i = 1, 2, ..., n+1,$$

where $x_0 = x$, $x_{n+1} = y$. This implies that $\nu(x_{i-1}^{-1}x_i) > \alpha - \varepsilon$ for all $i =$ $1, 2, ..., n+1$. Observe that $x^{-1}y$ can be written as

$$x^{-1}y = (x^{-1}x_1)(x_1^{-1}x_2)\cdots(x_n^{-1}y).$$

Then $\langle \nu \rangle (x^{-1}y) \geq \nu(x^{-1}x) \wedge \cdots \wedge \nu(x_n^{-1}y) > \alpha - \varepsilon$. Since ε is arbitrary,

$$\langle \nu \rangle (x^{-1}y) \geq \alpha = CONN_G(x, y). \tag{4}$$

Since $x^{-1}y \in V$, by Lemma 3.2, there exist $x_1, x_2, ..., x_n \in V$ such that

$$x^{-1}y = x_1 x_2 \cdots x_n$$

and $\langle \nu \rangle (x^{-1}y) = \nu(x_1) \wedge \cdots \wedge \nu(x_n)$. Consider the sequence

$$y_0 = x,\ y_1 = xx_1,\ y_2 = xx_1x_2,\ ...,\ y_n = xx_1x_2 \cdots x_n.$$

Then

$$\wedge_{i=1}^{n} R(y_{i-1},\ y_i) = \nu(x_1) \wedge \cdots \wedge \nu(x_n)$$

$$= \langle \nu \rangle (x^{-1}y) \neq 0.$$

This implies that $P' = (y_0,\ y_1,\ ...,\ y_n)$ is a path in G from x to y. Note that

$$CONN_G(x,\ y) = \bigvee_{P \in \mathscr{P}_G(x,\, y)} strength(P) \geq strength(P') = \langle \nu \rangle (x^{-1}y). \quad (5)$$

From equations (4) and (5), we have $CONN_G(x,\ y) = \langle \nu \rangle (x^{-1}y)$.

If $CONN_G(x,\ y) = 0$, then there is no path from x to y in G. If $\langle \nu \rangle (x^{-1}y) \neq 0$, then there is a path from x to y in G of non-zero strength. This is impossible. Hence $\langle \nu \rangle (x^{-1}y) = 0$. Hence

$$CONN_G(x,\ y) = \langle \nu \rangle (x^{-1}y). \qquad \square$$

Remark 5. For the Cayley fuzzy graph $G = (V,\ R)$ induced by $(V,\ *,\ \nu)$, $CONN_G(x,\ y)$ gives the membership grade of $x^{-1}y$ with respect to the fuzzy semigroup generated by ν.

Theorem 5.6. *If* $SC(G) = \alpha$, *then* $\langle \nu \rangle_\alpha \supseteq V - \{1\}$, *that is*,

$$\langle \nu \rangle_{SC(G)} \supseteq V - \{1\}.$$

Proof. The theorem is trivial when $\alpha = 0$. So, assume that $\alpha \neq 0$.

From the definition of $SC(G)$, it is obvious that $CONN_G(x,\ y) \geq \alpha$ for all $x,\ y \in V$. In particular, $CONN_G(1,\ x) \geq \alpha$ for all $x \in V$. This implies that there exits a path, say P from 1 to x such that strength$(P) \geq \alpha$. Then one can easily verify that for $x \neq 1$, $\langle \nu \rangle (x) \geq \alpha$. This implies that $x \in \langle \nu \rangle_\alpha$ for all but $x = 1$. Consequently, $\langle \nu \rangle_\alpha \supseteq V - \{1\}$. □

Theorem 5.7. *For any Cayley fuzzy graph G,*

$$SC(G) = \wedge_{\alpha \in [0, 1]} \{\alpha : \langle \nu \rangle_\alpha \subsetneq V\}.$$

Proof. Let $\alpha \in [0, 1]$. If $\langle \nu \rangle_\alpha \subsetneq V$, then there exist $x, y \in V$ such that every path from x to y has strength less than α. This implies that $CONN_G(x, y) < \alpha$. Consequently, $SC(G) < \alpha$. Hence

$$SC(G) \leq \wedge_{\alpha \in [0, 1]} \{\alpha : \langle \nu \rangle_\alpha \neq V\}. \tag{6}$$

Suppose that there is a β such that $SC(G) < \beta < \wedge_{\alpha \in [0, 1]} \{\alpha : \langle \nu \rangle_\alpha \neq V\}$. This implies that $\langle \nu \rangle_\beta = V$,

$$\langle \nu \rangle (x) \geq \beta \text{ for all } x \in V. \tag{7}$$

Let x and y be two elements in V. Then, by equation (7), we have $\langle \nu \rangle (x^{-1}y) \geq \beta$. This implies that there exists a path from x to y of strength greater than or equal to β. That is, $CONN_G(x, y) \geq \beta$ for all $x, y \in V$. In other words, $SC(G) \geq \beta$. This contradiction completes the proof. □

Theorem 5.8. *The Cayley fuzzy graph G is weakly α-connected if and only if*

$$\langle \nu \vee \nu^{-1} \rangle_\alpha \supseteq V - \{1\}.$$

Proof. By Remark 4 and by Theorems 3.3, 4.3, we have

$$\begin{aligned} G \text{ is weakly } \alpha\text{-connected} &\Leftrightarrow (V, R_\alpha) \text{ is weakly connected} \\ &\Leftrightarrow \langle \nu_\alpha \cup \nu_\alpha^{-1} \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \langle (\nu \vee \nu^{-1})_\alpha \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \langle \nu \vee \nu^{-1} \rangle_\alpha \supseteq V - \{1\}. \end{aligned}$$

This completes the proof. □

Theorem 5.9. *The Cayley fuzzy graph G is semi* α*-connected if and only if*

$$(\langle \nu \rangle_\alpha \cup \langle \nu^{-1} \rangle_\alpha) \supseteq V - \{1\}.$$

Proof. By Remark 4 and by Theorems 3.3, 4.6, we have

$$\begin{aligned} G \text{ is semi } \alpha\text{-connected} &\Leftrightarrow (V,\ R_\alpha) \text{ is semi-connected} \\ &\Leftrightarrow \langle \nu_\alpha \rangle \cup \langle \nu_\alpha^{-1} \rangle \supseteq V - \{1\} \\ &\Leftrightarrow \langle \nu \rangle_\alpha \cup \langle \nu^{-1} \rangle_\alpha \supseteq V - \{1\}. \end{aligned}$$

This completes the proof. □

Theorem 5.10. *The fuzzy graph G is locally* α*-connected if and only if*

$$\langle \nu \rangle_\alpha = \langle \nu^{-1} \rangle_\alpha.$$

Proof. By Remark 4 and by Theorems 3.3, 4.8, we have

$$\begin{aligned} G \text{ is locally } \alpha\text{-connected} &\Leftrightarrow (V,\ R_\alpha) \text{ is locally connected} \\ &\Leftrightarrow \langle \nu_\alpha \rangle = \langle \nu_\alpha^{-1} \rangle \\ &\Leftrightarrow \langle \nu \rangle_\alpha = \langle \nu^{-1} \rangle_\alpha. \end{aligned}$$

This completes the proof. □

Theorem 5.11. *A finite Cayley fuzzy graph G is quasi* α*-connected if and only if it is* α*-connected*.

Proof. By Remark 4 and by Theorems 3.3 and 4.11,

$$\begin{aligned} G \text{ is quasi } \alpha\text{-connected} &\Leftrightarrow (V,\ R_\alpha^+) \text{ is quasi-connected} \\ &\Leftrightarrow (V,\ R_\alpha^+) \text{ is connected} \\ &\Leftrightarrow G \text{ is } \alpha\text{-connected}. \end{aligned}$$

This completes the proof. □

Theorem 5.12. *The fuzzy graph G is α-complete if and only if* $\nu_{\alpha} \supseteq V - \{1\}$.

Proof. Suppose that G is α-complete. Then, for every $x, y \in V$ with $x \neq y$, $R(x, y) \geq \alpha$. In particular, $\nu(x) = R(1, x) \geq \alpha$ for all $x \neq 1$. This implies that $x \in \nu_{\alpha}$ for all $x \neq 1$. Since x is arbitrary, $\nu_{\alpha} \supseteq V - \{1\}$.

Conversely, suppose that $\nu_{\alpha} \supseteq V - \{1\}$. Then, for every $x, y \in V$, $x^{-1}y \in V$. This implies that $x^{-1}y \in \nu_{\alpha}$ for $x \neq y$. That is, $R(x, y) = \nu(x^{-1}y) \geq \alpha$ for all $x \neq y$. Hence G is α-complete. □

Theorem 5.13. *If ν is a semigroup and $\alpha, \beta \in [0, 1]$ such that G is α-complete and not β-complete, then either $\alpha < SC(G) \leq \beta$ or $\alpha \leq SC(G) < \beta$.*

Proof. Since G is α-complete and not β-complete, it is clear that $\alpha < \beta$. Now ν is a semigroup implies that $\nu = \langle \nu \rangle$. Then, by Theorem 5.7,

$$\begin{aligned} SC(G) &= \wedge_{\gamma \in [0,1]}\{\gamma : \langle \nu \rangle_{\gamma} \subsetneq V - \{1\}\} \\ &= \wedge_{\gamma \in [0,1]}\{\gamma : \nu_{\gamma} \subsetneq V - \{1\}\} \\ &= \wedge_{\gamma \in [0,1]}\{\gamma : G \text{ is not } \gamma\text{-complete}\}. \end{aligned}$$

Thus, since G is not β-complete, we have $SC(G) \leq \beta$. Also, note that since G is α-complete, $CONN(x, y) \geq \alpha$ for all $x, y \in V$. Hence $SC(G) \geq \alpha$.

From these arguments, it is clear that either $\alpha < SC(G) \leq \beta$ or $\alpha \leq SC(G) < \beta$. □

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