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ON OPERATORS WITH SINGLE-VALUED EXTENSION PROPERTY

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Abstract

In this paper, we show that if $T \in L(X)$ and $S \in L(Y)$ are quasi-similar, then $\mathcal{D}(T) = \mathcal{D}(S)$, where $\mathcal{D}(T) := \{\lambda \in \mathbb{C} : T$ fails to have SVEP at $\lambda\}$. In particular, T has the SVEP if and only if S has the SVEP. We also study the SVEP for T, S, ST, and TS in the case that T and S satisfy the operator equations $TST = T^2$ and $STS = S^2$. Moreover, we proved that for every $T, S \in L(X)$, ST does not have the SVEP at 0 if and only if there exists non-zero $Sx_0 \in Ker(T)$ such that $Sx_0 \in X_{ST}(\phi)$.

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1. Introduction and Preliminaries

The single-valued extension property is a unifying theme for a wide variety of bounded linear operators. The single-valued extension property (abbreviated SVEP) was first introduced by Dunford [8, 9] and has received a more systematic treatment in Dunford-Schwartz [10]. It also plays an important role in local spectral theory; see the monograph of Colojoarvă and Foiás [7] and Laursen and Neumann [16].

Throughout this paper, let X be a complex Banach space and let $L(X)$ denote the Banach algebra of all bounded linear operators on X, equipped with the usual operator norm. For an operator $T \in L(X)$, let T^*, $\sigma(T)$, $\rho(T)$, $\sigma_p(T)$ and $Ker(T)$ denote the adjoint operator acting on the dual space X^*, the spectrum, the resolvent set, the point spectrum and the kernel of T, respectively.

The *surjectivity spectrum* $\sigma_{sur}(T)$ of $T \in L(X)$ is defined as the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not surjective. It is well-known that $\sigma_{sur}(T) = \sigma_{ap}(T^*)$, and $\sigma_{sur}(T)$ is a compact subset of $\mathbb{C}$ that contains the boundary of $\sigma(T)$.

Definition 1.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$, if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to X$ which satisfies the equation $(\lambda I - T) f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the constant function $f \equiv 0$.

An operator $T \in L(X)$ is said to have the *single-valued extension property* if T has the single-valued extension property at every point $\lambda \in \mathbb{C}$.

Example 1.2. The unilateral left shift operator L on the Hilbert space $\ell_2(\mathbb{N})$ is an example of an operator without SVEP.

Proof. Let $U := \{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\} \bigcup \{\lambda \in \mathbb{C} : 1 < |\lambda|\}$. We define $f : U \to \ell_2(\mathbb{N})$ by $f(\lambda) := \sum_{n=0}^{\infty} \lambda^{n-1} e_{n+1}$ for all $0 < |\lambda| < 1$ and $f(\lambda) = 0$ for all $|\lambda| > 1$. Then clearly, f is analytic and $(\lambda I - L) f(\lambda) = 0$ for all $\lambda \in U$. But $\| f(\lambda) \| = (|\lambda| \sqrt{1 - |\lambda|^2})^{-1} \neq 0$ on $0 < |\lambda| < 1$, and hence L does not have the SVEP.

The *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ such that there exists an analytic function $f : U \to X$ which satisfies the equation

$$(\lambda I - T) f(\lambda) = x \quad \text{for all } \lambda \in U.$$

Note that the resolvent function $R(\lambda, T) := (\lambda I - T)^{-1} (\lambda \in \rho(T))$ is an analytic function, and hence $(\lambda I - T) R(\lambda, T) x = x$ for all $\lambda \in \rho(T)$. This means that $\rho(T) \subseteq \rho_T(x)$ for all $x \in X$ and $\rho_T(x)$ is open. Another important consequence of the SVEP is the existence of a maximal analytic extension f of $R(\lambda, T) x = (\lambda I - T)^{-1} x$ to the set $\rho_T(x)$ for every $x \in X$. This function identically verifies the equation

$$(\lambda I - T) f(\lambda) = x \quad \text{for every } \lambda \in \rho_T(x)$$

and $f(\lambda) = (\lambda I - T)^{-1} x$ for all $\lambda \in \rho(T)$.

The *local spectrum* $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbb{C} \backslash \rho_T(x)$. Obviously, we have $\sigma_T(x) \subseteq \sigma(T)$ and $\sigma_T(x)$ is closed. It is well known that if $\sigma(T) \neq \sigma_{sur}(T)$, then T does not have the SVEP.

It is clear that the SVEP is inherited by restrictions on closed invariant subspaces, i.e., if $T \in L(X)$ has the SVEP at λ_0 and M is a closed T-invariant subspace, then $T|M$ has the SVEP at λ_0. Moreover, $\sigma_T(x) \subseteq \sigma_{T|M}(x)$ for every $x \in M$.

Proposition 1.3. *Let X be a complex Banach space and $T \in L(X)$. If $\sigma_p(T)$ does not cluster at λ_0, then T has the SVEP at λ_0.*

Proof. Let U be a neighborhood of λ_0 such that $\lambda I - T$ is injective for every $\lambda \in U \backslash \{\lambda_0\}$. Let $f : V \to X$ be an analytic function defined on a neighborhood V of λ_0 such that

$$(\lambda I - T) f(\lambda) = 0 \quad \text{for all } \lambda \in V.$$

We may assume that $V \subseteq U$. Then $f(\lambda) \in Ker(\lambda I - T) = \{0\}$ for every $\lambda \in V \backslash \{\lambda_0\}$, and hence $f(\lambda) = 0$ for all $\lambda \in V \backslash \{\lambda_0\}$. From the continuity of f at λ_0, we conclude that $f(\lambda_0) = 0$. Hence, $f \equiv 0$ on V and therefore T has the SVEP at λ_0.

The same argument shows that if $\sigma_{ap}(T)$ does not cluster at λ_0, then T has the SVEP at λ_0. Moreover, every operator $T \in L(X)$ has the SVEP at an isolated point of the spectrum $\sigma(T)$. From these facts it follows that every quasi-nilpotent operator has the SVEP. More generally, if $\sigma_p(T)$ has empty interior, then T has the SVEP. In particular, any operator with a real spectrum has the SVEP.

For every subset F of $\mathbb{C}$, *the analytic spectral subspace of T* associated with F is the set

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

For an arbitrary operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, let $\mathcal{X}_T(F)$ denote the space of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \backslash F \to X$ with $(\lambda I - T) f(\lambda) = x$ for all $\lambda \in \mathbb{C} \backslash F$. It is clear that

$$\sigma_T(x) = \bigcap \{F \subseteq \mathbb{C} : F \text{ is closed and } x \in \mathcal{X}_T(F)\}.$$

Evidently, $X_T(F)$ and $\mathcal{X}_T(F)$ are (not necessarily closed) hyperinvariant linear subspaces of X with $\mathcal{X}_T(F) \subseteq X_T(F)$ for every closed $F \subseteq \mathbb{C}$.

It is clear that the zero operator has an empty local spectrum. The following proposition shows that if $T \in L(X)$ has the SVEP, then 0 is the unique element of *X* having empty local spectrum. It is found in [16].

Proposition 1.4. *Let* $T \in L(X)$ *and F be a closed subset of* $\mathbb{C}$. *Then* $\mathcal{X}_T(\phi) = \{0\}$ *and* $\mathcal{X}_T(F) = \mathcal{X}_T(F \cap \sigma(T))$. *Moreover, the following assertions are equivalent*:

(a) *T has the SVEP*.

(b) $\mathcal{X}_T(F) = X_T(F)$ *for all closed* $F \subseteq \mathbb{C}$.

(c) $X_T(\phi)$ *is closed*.

(d) $X_T(\phi) = \{0\}$.

For an arbitrary subset $F \subset \mathbb{C}$, let $E_T(F)$ denote the largest linear subspace *Y* of *X* for which

$$(\lambda I - T)Y = Y \text{ for all } \lambda \in \mathbb{C} \backslash F.$$

The spaces $E_T(F)$ are known as the *algebraic spectral subspaces* of *T*.

Recall that a linear subspace *Y* of *X* is said to be *T-divisible* if $(\lambda I - T)Y = Y$ for all $\lambda \in \mathbb{C}$. Evidently, $E_T(\phi)$ is the largest *T*-divisible linear subspace. It is easily seen that $X_T(F) \subseteq E_T(F) = E_T(F \cap \sigma(T))$ for all $F \subseteq \mathbb{C}$, and $E_T(\sigma_{sur}(T)) = X$.

They were first introduced by Johnson and Sinclair [12] in the context of automatic continuity theory, but then proved to be a useful tool in local spectral theory as well, [13-16]. In the theory of automatic continuity of intertwining linear transformations, it is essential to avoid non-trivial divisible subspaces.

Lemma 1.5. *Let* $T \in L(X)$, $S \in L(Y)$ *and* $A \in L(X, Y)$ *such that* $SA = AT$. *Then we have*

(a) $AE_T(F) \subseteq E_S(F)$ *and* $AX_T(F) \subseteq Y_S(F)$ *for all closed* $F \subseteq \mathbb{C}$.

(b) $AX \subseteq E_S(\sigma_{sur}(T) \cap \sigma_{sur}(S))$. *In particular, if* $\sigma_{sur}(T) \cap \sigma_{sur}(S) = \varnothing$ *and* $\{0\}$ *is the only S-divisible linear subspace of Y, then A is zero operator.*

Proof. (a) Clearly, $(\lambda I - T)E_T(F) = E_T(F)$ for all $\lambda \in \mathbb{C}\backslash F$. Hence, we have

$$(\lambda I - S)AE_T(F) = A(\lambda I - T)E_T(F) = AE_T(F) \text{ for all } \lambda \in \mathbb{C}\backslash F.$$

By the maximality of $E_S(F)$, we have

$$AE_T(F) \subseteq E_S(F).$$

We claim that $AX_T(F) \subseteq Y_S(F)$. It suffices to show that $\sigma_S(Ax) \subseteq \sigma_T(x)$ for all $x \in X$. Let $\lambda \notin \sigma_T(x)$. Then there exists an analytic function $f : U \to X$ on a neighborhood U of λ which satisfies the equation

$$(\mu I - T) f(\mu) = x \text{ for all } \mu \in U.$$

Thus, we have $(\mu I - S)Af(\mu) = A(\mu I - T) f(\lambda) = Ax$ for all $\mu \in U$. This means that $\lambda \in \rho_S(Ax)$ and hence $\lambda \notin \sigma_S(Ax)$.

(b) Clearly, we have

$$AX = AE_T(\sigma_{sur}(T)) \subseteq E_S(\sigma_{sur}(T)) = E_S(\sigma_{sur}(T) \cap \sigma_{sur}(S)).$$

If $T, S \in L(X)$ satisfy the equations $TST = T^2$ and $STS = S^2$, then clearly, $TE_{ST}(F) \subseteq X_T(F)$ and $TE_{ST}(F) \subseteq E_TS(F)$ for all closed $F \subseteq \mathbb{C}$.

It is clear that $E_T(\phi) = (\lambda I - T)^n E_T(\phi) \subseteq (\lambda I - T)^n X$ for all $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$. Thus, we have

$$E_T(\phi) \subseteq \bigcap_{n \in \mathbb{N}, \lambda \in \mathbb{C}} (\lambda I - T)^n X.$$

It follows that $E_T(\phi) = \{0\}$ whenever the operator T is normal, hyponormal, generalized scalar, subscalar, or isometry.

We have the following corollary:

Corollary 1.6. *Let* X *be a complex Banach space and* $T \in L(X)$. *If* $E_T(\phi) = \{0\}$, *then* T *has the SVEP.*

Proof. By Proposition 1.4 and Lemma 1.5, it is clear.

However, the Volterra operator T on the Banach space $X := C([0, 1])$, defined by

$$(Tf)(t) := \int_0^t f(s)ds \quad \text{for all } f \in C([0, 1]) \text{ and } t \in [0, 1].$$

Then T is injective, compact and quasi-nilpotent. It follows that $X_T(\phi) = \{0\}$. By Proposition 1.4, T has the SVEP. On the other hand, it is easy to check that

$$E_T(\phi) = \bigcap_{n=1}^{\infty} T^n X = \{f \in C^{\infty}([0, 1]) : f^{(n)}(0) = 0 \text{ for all } n \in \mathbb{N}\}.$$

Hence $E_T(\phi)$ is non-trivial.

2. Main Results

For an arbitrary operator $T \in L(X)$, let

$$\mathcal{D}(T) := \{\lambda \in \mathbb{C} : T \text{ fails to have SVEP at } \lambda\}.$$

Obviously, $\mathcal{D}(T)$ is empty precisely when T has SVEP. Moreover, it follows readily from the identity theorem for analytic functions that $\mathcal{D}(T)$ is open, and therefore contained in the interior of the spectrum $\sigma(T)$. Obviously, $\sigma(T) = \sigma_{sur}(T) \cup \mathcal{D}(T)$ and in particular, $\sigma_{sur}(T)$ contains the boundary of $\mathcal{D}(T)$.

An operator $A \in L(X, Y)$ is said to be a *quasi-affinity* if A is injective and has dense range. Recall that $T \in L(X)$ and $S \in L(Y)$ are *quasi-similar* if there exist quasi-affinities $A \in L(X, Y)$ and $B \in L(Y, X)$ for which $SA = AT$ and $TB = BS$.

Theorem 2.1. *Suppose that* $T \in L(X)$ *and* $S \in L(Y)$ *are quasi-similar. Then* $\mathcal{D}(T) = \mathcal{D}(S)$. *In particular, T has the SVEP if and only if S has the SVEP.*

Proof. Let $A \in L(X, L)$ and $B \in L(Y, X)$ be quasi-affinities for which $SA = AT$ and $TB = BS$. Let $\lambda_0 \notin \mathcal{D}(T)$ and let $f : U \to Y$ be an analytic function defined on an open neighborhood U of λ_0 such that

$$(\lambda I - S) f(\lambda) = 0 \quad \text{for all } \lambda \in U.$$

Then $0 = B(\lambda I - S) f(\lambda) = (\lambda I - T) Bf(\lambda)$ for all $\lambda \in U$. Since T has the SVEP at λ_0, and $Bf(\lambda)$ is analytic, we have $Bf(\lambda) = 0$ for all $\lambda \in U$. It follows from injectivity of B that $f \equiv 0$ on U. Thus, S has the SVEP at λ_0 and hence $\lambda_0 \notin \mathcal{D}(S)$. Conversely, let $\lambda_0 \notin \mathcal{D}(S)$ and let $g : V \to X$ be an analytic function defined on an open neighborhood V of λ_0 such that

$$(\lambda I - T) g(\lambda) = 0 \quad \text{for all } \lambda \in V.$$

Then $0 = A(\lambda I - T) g(\lambda) = (\lambda I - S) Ag(\lambda)$ for all $\lambda \in V$. Since S has the SVEP at λ_0, and $Ag(\lambda)$ is analytic, $Ag(\lambda) = 0$ for all $\lambda = V$. It follows from injectivity of A that $f \equiv 0$ on V. Thus, T has the SVEP at λ_0 and hence $\lambda_0 \notin \mathcal{D}(T)$.

The case that $S, T \in L(X)$ satisfy the operator equations

$$TST = T^2 \quad \text{and} \quad STS = S^2$$

has been first studied in [20], and more recently it has been investigated by some other authors, [2, 19]. In this case, T, S, ST and TS share many spectral properties and local spectral properties as decomposability, property (β), Dunford's property (C) and SVEP [4-6]. If $S, T \in L(X)$ satisfy the operator equations $TST = T^2$, $STS = S^2$, and $0 \notin \sigma(T) \cap \sigma(S)$, then clearly $T = I = S$. For this reason, we shall assume that *T* and *S* are not invertible.

Theorem 2.2. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. Then any one of T, S, TS and ST has the SVEP at λ_0 implies they all have the SVEP at λ_0.*

Proof. It suffices to show that if *T* has the SVEP at λ_0, then *S*, *TS* and *ST* are have the SVEP at λ_0. Suppose that *T* has the SVEP at λ_0. At first, we claim that *ST* has the SVEP at λ_0. Let $f : U \to X$ be an analytic function defined on an open neighborhood *U* of λ_0 such that

$$(\lambda I - ST) f(\lambda) = 0 \text{ for all } \lambda \in U. \tag{1}$$

Then $0 = T(\lambda I - ST) f(\lambda) = (\lambda I - T)Tf(\lambda)$ for all $\lambda \in U$. The SVEP of *T* at λ_0 entails that $Tf(\lambda) = 0$ for all $\lambda \in U$, and hence $STf(\lambda) = 0$ for all $\lambda \in U$. In equation (1), we deduce $f(\lambda) = 0$ for all $0 \neq \lambda \in U$. By the continuity of *f*, $f(\lambda) = 0$ for all $\lambda \in U$. Hence, *ST* has the SVEP at λ_0.

We have to show that *TS* has the SVEP at λ_0. Let $g : V \to X$ be an analytic function defined on an open neighborhood *V* of λ_0 such that

$$(\lambda I - TS) g(\lambda) = 0 \text{ for all } \lambda \in V. \tag{2}$$

Then $0 = S(\lambda I - TS) f(\lambda) = (\lambda I - ST)Sg(\lambda)$ for all $\lambda \in V$. It follows from the SVEP of *ST* at λ_0 that $Sg(\lambda) = 0$ for all $\lambda \in V$, and hence $TSg(\lambda) = 0$ for all $\lambda \in U$. In equation (2), we deduce $g(\lambda) = 0$ for all $0 \neq \lambda \in V$. By the continuity of g, $g(\lambda) = 0$ for all $\lambda \in V$, and so $g(\lambda) = 0$ for all $\lambda \in V$. Hence, *TS* has the SVEP at λ_0.

Finally, we have to show that *S* has the SVEP at λ_0. Let $h : O \to X$ be an analytic function defined on an open neighborhood *O* of λ_0 such that

$$(\lambda I - S) h(\lambda) = 0 \text{ for all } \lambda \in O. \tag{3}$$

Then $0 = S(\lambda I - S)h(\lambda) = (\lambda I - ST)Sh(\lambda)$ for all $\lambda \in O$. It follows from the SVEP of ST at λ_0 that $Sh(\lambda) = 0$ for all $\lambda \in O$. In equation (1), we deduce $h(\lambda) = 0$ for all $0 \neq \lambda \in O$. By the continuity of h, $h(\lambda) = 0$ for all $\lambda \in O$. Hence S has the SVEP at λ_0.

Corollary 2.3. *Let $T, S \in L(X)$ be such that $TST = T^2$ and $STS = S^2$. Then the following assertions are equivalent*:

(a) *T has the SVEP*.

(b) *TS has the SVEP*.

(c) *ST has the SVEP*.

(d) *S has the SVEP*.

Theorem 2.4. *Let $T, S \in L(X)$. Then ST does not have the SVEP at 0 if and only if there exists non-zero $Sx_0 \in Ker(T)$ such that $Sx_0 \in X_{ST}(\phi)$.*

Proof. Suppose that there exists non-zero $Sx_0 \in Ker(T)$ such that $Sx_0 \in X_{ST}(\phi)$. We may assume that $\| Sx_0 \| = 1$. It is clear that $Sx_0 \in K(ST)$. By definition of $K(ST)$, there exists a sequence (a_n) in X and $c > 0$ so that

$$a_0 = Sx_0, \quad STa_n = a_{n-1} \text{ and } \| a_n \| \leq c^n \text{ for all } n \geq 1.$$

Obviously, $STa_0 = STSx_0 = 0$. Let $U := \{\lambda \in \mathbb{C} : |\lambda| < c^{-1}\}$. Then for every $\lambda \in U$,

$$\| \lambda^{n+1} a_n \| \leq c^n |\lambda|^{n+1} \to 0 \text{ as } n \to \infty,$$

it follows that

$$(\lambda I - ST)\left(\sum_{k=0}^{n} \lambda^k a_k\right) = \lambda^{n+1} a_n \to 0 \text{ as } n \to \infty.$$

So $f(\lambda) := \sum_{n=0}^{\infty} \lambda^n a_n$ defines an analytic function on U that satisfies $(\lambda I - ST) f(\lambda) = 0$ for all $\lambda \in U$. Since $f(0) = Sx_0 \neq 0$, it follows that ST does not have the SVEP at 0.

In order to prove opposite inclusion, suppose that for every non-zero $Sx \in Ker(T)$, we have $\sigma_{ST}(Sx) \neq \varnothing$. By Proposition 3.1 of [5], for every non-zero $Sx \in Ker(T)$, we have $\sigma_{TS}(x) \neq \varnothing$. At first, we have to show that TS has the SVEP at 0. Let $f : V \to X$ be an analytic function in a neighborhood V of 0 such that $(\lambda I - TS) f(\lambda) = 0$ for every $\lambda \in V$. By Lemma 1.2.14 of [16],

$$\sigma_{TS}(f(\lambda)) = \sigma_{TS}(0) = \varnothing \quad \text{for all } \lambda \in V.$$

Since f is analytic, there exists a sequence $(a_n) \subseteq X$ such that $f(\lambda) = \sum_{n=0}^{\infty} \lambda^n a_n$. We claim that $a_n = 0$ for all $n \geq 0$. Clearly, $TSa_0 = TSf(0) = 0$, so $Sa_0 \in KerT$. Moreover, from the equalities $\sigma_{TS}(f(\lambda)) = \sigma_{TS}(0) = \varnothing$ for every $\lambda \in V$, we obtain $\sigma_{TS}(f(0)) = \sigma_{TS}(a_0) = \varnothing$, and therefore by assumption we conclude that $a_0 = 0$. For every non-zero $\lambda \in V$, we obtain

$$0 = (\lambda I - TS) f(\lambda) = (\lambda I - TS)\left(\sum_{n=1}^{\infty} \lambda^n a_n\right) = \lambda(\lambda I - TS)\left(\sum_{n=0}^{\infty} \lambda^n a_{n+1}\right)$$

and therefore

$$(\lambda I - TS)\left(\sum_{n=0}^{\infty} \lambda^n a_{n+1}\right) = 0 \quad \text{for every non-zero } \lambda \in V.$$

Since $\lambda I - TS$ is continuous, thus we have

$$(\lambda I - TS)\left(\sum_{n=0}^{\infty} \lambda^n a_{n+1}\right) = 0 \quad \text{for every non-zero } \lambda \in V.$$

At this point, by using the same argument as in the first part of the proof, it is possible to show that $a_1 = 0$, and by iterating this procedure we conclude that $a_2 = a_3 = \cdots = 0$. This shows that $f \equiv 0$ on V and therefore TS has the SVEP at 0. Finally, we have to show that ST has the SVEP at 0.

Let $g : W \to X$ be an analytic function on a neighborhood W of 0 such that

$$(\lambda I - ST)g(\lambda) = 0 \quad \text{for every } \lambda \in W. \tag{1}$$

Then $(\lambda I - TS)Tg(\lambda) = T(\lambda I - ST)g(\lambda) = 0, \lambda \in W$. Now since TS has the SVEP at 0, we have $Tg(\lambda) = 0$ for all $\lambda \in W$, and hence $STg(\lambda) = 0$ for every $\lambda \in W$. In equation (1), we deduce that $g(\lambda) = 0$ for all $0 \neq \lambda \in W$. By the continuity of g, $g(\lambda) = 0$ for all $\lambda \in W$. Hence, ST has the SVEP at 0.

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