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RELATIVELY COMPACT SETS IN THE REDUCED C^*-ALGEBRAS OF COXETER GROUPS

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Abstract

We characterize relatively norm compact sets in the regular C^*-algebra of finitely generated Coxeter groups using a geometrically defined positive semigroup acting on the algebra.

1. Introduction

Let (X, d) be a compact metric space, $x_0 \in X$. In $C(X)$, the continuous complex valued functions on X, consider the convex, balanced and closed set

$$\mathcal{K} = \{f : |f(x) - f(y)| \leq d(x, y),\ f(x_0) = 0\}.$$

The Arzela-Ascoli theorem shows that $\mathcal{K}$ is relatively compact. On the other hand, this theorem can be thought to compare any relatively compact set against this special set.

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In the non-commutative context, this has been made precise by Antonescu and Christensen [1] as follows:

Let A be a unital, separable C^*-algebra and $\mathcal{S}$ be the set of its states endowed with the w^*-topology.

Definition 1. $\mathcal{K} \subset A$ is called a *metric set* if it is convex, balanced *norm compact* and *separates the states* of A.

Lemma 2 [1]**.** *If* $\mathcal{K} \subset A$ *is a metric set*, *then*

$$d(\varphi, \psi) := \sup_{x \in \mathcal{K}} |\varphi(x) - \psi(x)|, \quad \varphi, \psi \in \mathcal{S}$$

defines a metric on $\mathcal{S}$, *which generates the* w^**-topology*.

Their *general non-commutative Arzela-Ascoli Theorem* reads as follows:

Theorem 3 [1]**.** *Let* A *be a unital* C^**-algebra*, $\mathcal{K} \subset A$ *be a metric set*. *Then* $\mathcal{H} \subset A$ *is relatively compact if and only if* $\mathcal{H}$ *is bounded and for all* $\varepsilon > 0$, *there exists* $N > 0$ *such that*

$$\mathcal{H} \subset B_\varepsilon(0) + N\mathcal{K} + \mathbb{C}Id,$$

where $B_\varepsilon(0) \subset A$ *is the ball of radius* ε *around* 0.

Our aim here is to give an example of some such set $\mathcal{K}$ in the reduced C^*-algebra $A = C_\lambda^*(G)$ of a finitely generated Coxeter group G.

Let G, S be a Coxeter group and l be the length function associated to the generating set S. (For the convenience to the readers in the next two sections, we recall some notions and assertions related to the regular C^*-algebra of Coxeter groups.)

Theorem 4.

$$\mathcal{K} = \{\lambda(f) : \|\lambda(f)\| \leq 1 \textit{ and } \|\lambda(l \cdot f)\| \leq 1\}$$

is relatively compact in $C_\lambda^*(G)$.

The proof of this theorem is given in our last section.

Since the set $\mathcal{K}$ in $C_\lambda^*(G)$ separates the states, is convex and balanced, an application of the theorem of Antonescu and Christensen characterizes relatively compact subsets of $C_\lambda^*(G)$ as follows:

Corollary 1. *A set $\mathcal{H} \subset C_\lambda^*(G)$ is relatively compact if and only if it is bounded and for all $\varepsilon > 0$, there is an $m \in \mathbb{N}$ such that*

$$\mathcal{H} \subset m\mathcal{K} + \mathbb{C}\lambda(\delta(e)) + B_e(0),$$

where $B_\varepsilon(0) \subset C_\lambda^*(G)$ is the ball of radius ε *and center* 0.

2. Coxeter Group

Definition 5. A pair (G, S) is a Coxeter group if S is a finite generating subset of the group G with the following presentation:

$$s^2 = e, \qquad s \in S,$$

$$(st)^{m(s,t)} = e, \quad s, t \in S, s \neq t,$$

where $m(s, t) \in \{2, 3, 4, ..., \infty\}$.

A specific tool for working with Coxeter groups is their *geometric representation*.

Let $V = \oplus_{s \in S} \mathbb{R}\alpha_s$ be an abstract real vector space with basis $\{\alpha_s : s \in S\}$. Define a bilinear form on it:

$$B(\alpha_s, \alpha_t) = \begin{cases} 1, & s = t, \\ -\cos \dfrac{\pi}{m(s,t)}, & m(s, t) \neq \infty, \\ -1, & m(s, t) = \infty. \end{cases}$$

For $s \in S$, define a reflection by $\sigma_s\xi = \xi - 2B(\alpha_s, \xi)\alpha_s$. Then

$\circ$: $V = \mathbb{R}\alpha_s \oplus H_s$, where $H_s = \{\xi : B(\alpha_s, \xi) = 0\}$ is stabilized pointwise by σ_s and $\sigma_s\alpha_s = -\alpha_s$.

$\circ$: $s \mapsto \sigma_s$ extends multiplicatively to a representation $\sigma : G \to Gl(V)$ of the Coxeter group.

$\circ$: σ is faithful and $\sigma(G)$ is a discrete subgroup of $Gl(V)$.

We dualise the representation σ to obtain the adjoint representation

$$\sigma^*(g)f(\xi) = f(\sigma(g^{-1})\xi),\ f \in V^*,\ \xi \in V.$$

For $s \in S$, let Z_s be the hyperplane $Z_s = \{f \in V^* : f(\alpha_s) = 0\}$, and A_s the halfspace $A_s = \{f \in V^* : f(\alpha_s) > 0\}$; define a family of hyperplanes in $V^*\mathcal{H} = \bigcup_{g \in G} gZ_s$. Denote $C = \bigcap_{s \in S} A_s$ the intersection of the halfspaces, its closure $D = \overline{C} \setminus \{0\}$ is called the *fundamental chamber* usually considered as a subset of the union of its translates $U = \bigcup_{g \in G} gD$, the *Tit's cone*.

The following facts hold true:

(i) C is a simplicial cone, its faces are the sets $Z_s \cap D$,

(ii) U is a convex cone, D a fundamental domain for the action of G on it,

(iii) A closed line segment $[u, v] \subset U$ meets only finitely many members of $\mathcal{H}$,

(iv) moreover, for any $c \in C$:

$$card(\{Z \in \mathcal{H} : [gc, c] \cap Z \neq \varnothing\}) = l(g),$$

where $l(g) = \inf\{k : g = s_1 \dots s_k,\ s_i \in S\}$ denotes the usual length with respect to the generating set S. This construction, due to Tits, was used by Bożejko et al. [3], we recall sketching the proof of their theorem.

Theorem 6 [3]. *For* $t > 0$,

$$\varphi_t : g \mapsto e^{-tl(g)}$$

is a positive definite function on G.

Proof.

$$l(g^{-1}h) = card(\{Z \in \mathcal{H} : [hc, gc] \cap Z \neq \varnothing\})$$

$$= \sum_{Z \in \mathcal{H}} |\chi_h(Z) - \chi_g(Z)|^2,$$

where $c \in C$ is arbitrary and χ_h is the characteristic function of $N^h = \{Z \in \mathcal{H} : [hc, c] \cap Z \neq \varnothing\}$. Hence $l(\cdot)$ is negative definite and, by a theorem of Schoenberg [9] (we only need the part already due to Schur [10]), $e^{-tl(\cdot)}$ is positive definite, see, e.g., [2, Theorem 7.8]. □

3. Regular Representation

For functions $f, h : G \to \mathbb{C}$, their convolution is defined by:

$$f * h(y) = \sum_{x \in G} f(x) h(x^{-1}y).$$

For summable $f : G \to \mathbb{C}$, we denote $\lambda(f) : l^2(G) \to l^2(G)$ the associated convolution operator $\lambda(f)h = f * h$. The regular (or reduced) C^*-algebra $C^*_\lambda(G)$ is the just the operator norm closure of $\{\lambda(f) : f \in l^1(G)\}$. Denote for $g \in G$ δ_g, the point mass one in $g \in G$, then $\lambda(\delta_g)$ is just left translation by g^{-1} on $l^2(G)$ and we are just dealing with the integrated version of the left regular representation. Since for $A \in C^*_\lambda(G)$, there is a unique $f = A\delta_e \in l^2$, we abuse notation to denote $A = \lambda(f)$.

The Tits cone with its division by the hyperplanes can be seen as a subset of a cubical building. This allows to estimate certain convolution operator norms. The first example of such an estimation was given for the free group on two generators by Haagerup [6] and accordingly such inequalities are called *Haagerup inequality*. Versions more appropriate for our purpose appear in [4, 5, 8, 11]:

Theorem 7. *A Coxeter group is a group of rapid decay*: *there is* $C > 0$ *and* $k \in \mathbb{N}$ *such that*

$$\| \lambda(f) \| \leq C\left(\sum_{g} | f(g) |^2 (1 + l(g))^{2k} \right)^{\frac{1}{2}}.$$

A consequence of this theorem is the following lemma due to Haagerup [6, 7]. For convenience, we recall their proofs.

Lemma 8. *If* $\varphi : G \to G$ *is such that* $\sup_g | \varphi(g) | (1 + l(g))^k < \infty$, *then for all* $\lambda(f) \in C^*_\lambda(G)$,

$$\| \lambda(\varphi \cdot f) \| \leq C \sup_g | \varphi(g) | (1 + l(g))^k \| \lambda(f) \|.$$

Here C and k are the constants in the Haagerup inequality.

Proof. From $\lambda(f)\delta_e = f$, *we have* $\left(\sum_g | f(g) |^2 \right)^{\frac{1}{2}} = \| f \|_2 \leq \| \lambda(f) \|$ and by the Haagerup inequality:

$$\begin{aligned} \| \lambda(\varphi \cdot f) \|^2 &\leq C \sum_g | \varphi(g) f(g) |^2 (1 + l(g))^{2k} \\ &\leq \sum_g | f(g) |^2 C \sup_g | \varphi(g) |^2 (1 + l(g))^{2k}. \end{aligned}$$ □

Lemma 9. *There is a sequence of finitely supported functions* (ψ_m) *such that for* $\lambda(f) \in C^*_\lambda(G)$:

- $\lambda(\psi_m \cdot f) \to \lambda(f)$ *as* $n \to \infty$
- $\| \lambda(\psi_m \cdot f) \| \leq 3 \| \lambda(f) \|$.

Proof. Since, by Theorem 6, the functions φ_t are positive definite, they define contractive (i.e., norm non-increasing) multipliers on the regular C^*-algebra. Let

$$\varphi_{n,t} = \begin{cases} e^{-tl(g)}, & \text{if } l(g) \leq n, \\ 0, & \text{else.} \end{cases}$$

Then

$$\begin{aligned} \| \lambda(\varphi_{n,t} \cdot f) - \lambda(f) \| &\leq \| \lambda(\varphi_{n,t} \cdot f) - \lambda(\varphi_t \cdot f) \| \\ &\quad + \| \lambda(\varphi_t \cdot f) - \lambda(f) \| \\ &\leq C \sup_{l>n} e^{-tl}(1+l)^k \| \lambda(f) \| \\ &\quad + \| \lambda(\varphi_t \cdot f) - \lambda(f) \|. \end{aligned}$$

Since $\sup_{l>n} e^{-tl}(1+l)^k \to 0$ as $n \to \infty$, we can extract the ψ_m from the $\varphi_{n,t}$. □

4. Relatively Compact Sets

First, we notice that the positive definite functions $\varphi_t : g \mapsto e^{-tl(g)}$ define a C_0-semigroup of multipliers on $C^*_\lambda(G)$ given by $M_t : C^*_\lambda(G) \to C^*_\lambda(G)$, $\lambda(f) \mapsto \lambda(\varphi_t \cdot f)$.

Lemma 10. *$M : t \mapsto M_t$ is a C_0-semigroup of contractions on $C^*_\lambda(G)$.*

Proof. Since φ_t is positive definite,

$$\| M_t \| = \varphi_t(e) = 1.$$

For finitely supported *f*, everything is elementary and now an approximation proves the assertion. □

Lemma 11. *The generator D of the semigroup* M_t *is given by*

$$D(\lambda(f)) = -\lambda(l \cdot f),$$

$$Dom(D) = \{\lambda(f) : \lambda(l \cdot f) \in C^*_\lambda(G)\}.$$

Proof. We have

$$\lambda(\varphi_t \cdot \delta_g) = e^{-tl(g)}\lambda(\delta_g),$$

hence the assertion is clear for finitely supported $f = \sum_g f(g)\delta_g$.

Now as a generator of a C_0-contraction semigroup, the operator *D* has a closed graph. But if

$$\lambda(f) \text{ and } \lambda(l \cdot f) \in C^*_\lambda(G),$$

then for the finitely supported ψ_m as above:

$$\lambda(\psi_m \cdot f) \to \lambda(f)$$

and

$$\lambda(\psi_m \cdot l \cdot f) \to \lambda(l \cdot f). \qquad \square$$

Proof of Theorem 4. We shall show that for $\varepsilon > 0$, there exists a finite dimensional bounded set

$$\widetilde{\mathcal{K}}_\varepsilon \subset C^*_\lambda(G)$$

such that for all $f \in \mathcal{K}$,

$$\text{dist}(f, \widetilde{\mathcal{K}}_\varepsilon) \leq \varepsilon.$$

(This shows that $\mathcal{K}$ is totally bounded.)

We have for $f \in \mathcal{K}$:

$$\lambda(\varphi_t \cdot f) - \lambda(f) = M_t(\lambda(f)) - \lambda(f) = \int_0^t D(M_s(\lambda(f)))ds.$$

Hence

$$\begin{aligned} \| \lambda(\varphi_t \cdot f) - \lambda(f) \| &\le t \sup_{s<t} \| \lambda(le^{-sl} \cdot f) \| \\ &\le t \| \lambda(l \cdot f) \| \le t \end{aligned}$$

and

$$\begin{aligned} \| \lambda(\varphi_t \cdot f) - \lambda(\varphi_{n,t} \cdot f) \| &\le C \sup_{l>n} e^{-tl}(1+l)^k \| \lambda(f) \| \\ &\le C \sup_{l>n} e^{-tl}(1+l)^k. \end{aligned}$$

Taking first t small and then n large, we have an approximation to $\lambda(f)$ by certain $\lambda(\varphi_{n,t} \cdot f)$ up to ε uniformly in $\lambda(f) \in \mathcal{K}$. Further, for this n,

$$\begin{aligned} \| \lambda(\varphi_{n,t} \cdot f) \| &\le \| \lambda(\varphi_t \cdot f) - \lambda(\varphi_{n,t} \cdot f) \| + \| \lambda(\varphi_t \cdot f) \| \\ &\le C(\sup_{l>n} e^{-tl}(1+l)^k + 1) \| \lambda(f) \| \\ &\le C(\sup_{l>n} e^{-tl}(1+l)^k + 1). \end{aligned}$$

So these $\lambda(\varphi_{n,t} \cdot f)$ are from a bounded set and all have their support in words of length at most n. The functions with support in this finite set give rise to a finite dimensional subspace of $C_\lambda^*(G)$. □

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