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NEW OSCILLATION CRITERIA FOR SYSTEM OF DIFFERENCE EQUATIONS

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Abstract

In this paper, we provide sufficient conditions for the oscillation of every solution of the systems of difference equations

$$\Delta y_n + \sum_{i=1}^{m} P_i y_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots,$$

where $P_i \in \mathbb{R}^{r \times r}$ and $k_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, m$. The conditions are given in terms of the eigenvalues of the P_i matrix for $i = 1, 2, \ldots, m$.

1. Introduction

The concept of the oscillatory behaviour of solutions of difference equations has been extensively investigated, see [1, 5] and the references

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cited therein. In [5], Ladas established the theorems for the oscillatory behaviour of all solutions for the following difference equations:

$$\Delta y_n + py_{n-k} = 0, \quad n = 0, 1, 2, \ldots$$

and

$$\Delta y_n + \sum_{i=1}^{m} p_i y_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots,$$

where $p_i \in \mathbb{R}$ and $k_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, m$.

In [1], Chuanxi et al. obtained the oscillatory behaviour of all solutions of linear autonomous system of difference equations

$$\Delta y_n + Py_{n-k} = 0, \quad n = 0, 1, 2, \ldots,$$

where $P \in \mathbb{R}^{r\times r}$ and $k \in \mathbb{Z}$. Furthermore, they obtained sufficient conditions for the oscillation of all solutions of the system of difference equations

$$\Delta y_n + \sum_{i=1}^{m} P_i y_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

by using logarithmic norm of P which is denoted by $\mu(P)$ and defined by

$$\mu(P) = \max_{\|\xi\|=1} (P\xi, \xi),$$

where $(,)$ is an inner product in $\mathbb{R}^r$ and $\|\xi\| = (\xi, \xi)^{\frac{1}{2}}$, $P_i \in \mathbb{R}^{r\times r}$ and $k_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, m$.

In the present paper, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1) without using logarithmic norm. Our result improves the known results in [1] and [5]. We believe that the results of this paper are more useful than the results of [1]. Because, in [1], to obtain the oscillatory results of equation (1.1), are required, the calculations of eigenvalues, eigenvectors and logarithmic norms of the matrices P_i for $i = 1, 2, \ldots, m$.

By a solution of equation (1.1), we mean a sequence $\{y_n\}$ of vectors in $\mathbb{R}^r$ for $n = 0, 1, 2, \ldots$ which satisfies equation (1.1). A sequence of real numbers $\{y_n\}$ is said to *oscillate* if the terms y_n are not eventually positive or eventually negative. Let $\{y_n\}$ be a solution of equation (1.1) with $y_n = [y_n^1, y_n^2, \ldots, y_n^r]^T$ for $n = 0, 1, 2, \ldots$. We say that the solution $\{y_n\}$ oscillates componentwise or simply oscillates if each component $\{y_n^i\}$ oscillates. Otherwise the solution is called *nonoscillatory*.

Let

$$k = \max\{0, k_1, k_2, \ldots, k_m\} \quad \text{and} \quad l = \max\{1, -k_1, -k_2, \ldots, -k_m\}.$$

Then equation (1.1) can be written in the form

$$\Delta y_n + \sum_{j=-k}^{l} Q_j y_{n+j} = 0, \quad n = 0, 1, 2, \ldots. \tag{1.2}$$

Then equation (1.2) is a difference equation of order $(k + l)$. If $k \geq 0$ and $l = 1$, then we say that equation (1.2) is a *delay difference equation*. When $k = 0$ and $l \geq 2$, equation (1.2) is called an *advanced difference equation*. When $k \geq 1$ and $l \geq 2$, then equation (1.2) is of the mixed type.

For the purpose of existence and uniqueness of solutions, we should assume that

$$\left.\begin{array}{l} \text{if } l = 1, \text{ then } \det(Q_1 + I) \neq 0 \\ \text{if } k = 0 \text{ and } l \geq 2, \text{ then } \det Q_l \neq 0 \end{array}\right\}. \tag{1.3}$$

Let $a_{-k}, \ldots, a_{l-1}$ be $(k + l)$ given vectors in $\mathbb{R}^r$. Then under assumption (1.3), equation (1.2) has a unique solution $\{y_n\}$ which satisfies the initial conditions

$$y_i = a_i, \quad i = -k, \ldots, l - 1.$$

We need the following lemma, which is proved in [3].

Lemma 1.1. *Assume that* $Q_1, Q_2, ..., Q_k \in \mathbb{R}^{r\times r}$ *and suppose that condition* (1.3) *is satisfied. Then the following statements are equivalent*:

(a) *Every solution of equation* (1.2) *oscillates componentwise.*

(b) *The characteristic equation of* (1.2),

$$\det\left[\gamma I - I + \sum_{j=-k}^{l} \gamma^j Q_j\right] = 0 \tag{1.4}$$

has no positive roots.

2. Sufficient Conditions for Oscillation of (1.1)

In this section, we obtain sufficient conditions for the oscillation of all solutions of the linear equation with the matrix coefficients of $P_1, P_2, ..., P_m$,

$$\Delta y_n + \sum_{i=1}^{m} P_i y_{n-ki} = 0, \quad n = 0, 1, 2, \ldots.$$

The conditions will be given in terms of the k_i and eigenvalues of the matrices P_i for each $i = 1, 2, ..., m.$

Throughout this paper, we will use the convention that $0^0 = 1.$

Theorem 2.1. *Let* $P_i \in R^{r\times r}$, $k_i \in \{0, 1, 2, ...\}$ *or* $k_i \in \{..., -3, -2, -1\}$ *for* $i = 1, 2, ..., m$. *Suppose that condition* (1.3) *is satisfied. Then every solution of equation* (1.1) *oscillates* (*componentwise*) *provided that*

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{(k_i + 1)^{k_i+1}}{k_i^{k_i}}\right) > 1, \tag{2.1}$$

where $\lambda(P)$ *denotes for any real eigenvalues of P.*

Proof. If $k_i = k$ for all $i = 1, 2, ..., m,$ then every solution of equation (1.1) oscillates if and only if

$$\lambda\left(\sum_{i=1}^{m} P_i\right) > \frac{k^k}{(k+1)^{k+1}},$$

which is given in [1]. Thus, we assume to the case when $k_i \neq k$ for all $i = 1, 2, ..., m$. First, assume that $k_i \in \{0, 1, 2, ...\}$ for $i = 1, 2, ..., m$. Assume, for the sake of contradiction, that characteristic equation of equation (1.1),

$$\det\left[(\gamma - 1)I + \sum_{i=1}^{m} P_i \gamma^{-k_i}\right] = 0 \tag{2.2}$$

has a γ_0 positive root. If $\gamma_0 \in (1, \infty)$, then equation (2.2) becomes

$$\det\left[I + \sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(\gamma_0 - 1)}\right] = 0$$

and

$$\det\left[I - \sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1 - \gamma_0)}\right] = 0.$$

Hence we say that all eigenvalues of the matrix $\sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1-\gamma_0)}$ are one. That is

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1 - \gamma_0)}\right) = 1.$$

But, by condition (2.1), this is impossible, indeed we observe that for $i = 1, 2, ..., m$,

$$\lim_{\gamma_0 \to \infty} \frac{\gamma_0^{-k_i}}{(1 - \gamma_0)} = -\infty \quad \text{and} \quad \lim_{\gamma_0 \to 1^+} \frac{\gamma_0^{-k_i}}{(1 - \gamma_0)} = 0,$$

then from (2.1), we have

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1-\gamma_0)}\right) < 0.$$

If $\gamma_0 \in (0, 1)$, then equation (2.2) becomes

$$\det\left[I - \sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1-\gamma_0)}\right] = 0,$$

this implies that

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1-\gamma_0)}\right) = 1.$$

Otherwise, we have for $i = 1, 2, \ldots, m$,

$$\inf_{0<\gamma_0<1} \frac{\gamma_0^{-k_i}}{(1-\gamma_0)} = \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}}.$$

Then we get

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}}\right) \le 1$$

which is a contradiction to (2.1).

If $\gamma_0 = 1$, then equation (2.2) becomes

$$\det\left[\sum_{i=1}^{m} P_i\right] = 0,$$

and also

$$\det\left[\sum_{i=1}^{m} P_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}}\right] = 0,$$

which means that at least eigenvalue of the matrix $\sum_{i=1}^{m} P_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}}$ is zero. So, we obtain a contradiction to (2.1).

Next, assume that $k_i \in \{..., -3, -2, -1\}$ for $i = 1, 2, ..., m$. If $\gamma_0 \in (0, 1)$, then

$$\det\left[I - \sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1-\gamma_0)}\right] = 0.$$

Hence we say that

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1-\gamma_0)}\right) = 1,$$

but, by condition (2.1), this is impossible, indeed we observe that for $i = 1, 2, ..., m$,

$$\lim_{\gamma_0 \to 0^+} \frac{\gamma_0^{-k_i}}{(1-\gamma_0)} \quad \text{and} \quad \lim_{\gamma_0 \to 1^-} \frac{\gamma_0^{-k_i}}{(1-\gamma_0)} = \infty,$$

then, under condition (2.1), we have

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1-\gamma_0)}\right) < 0.$$

If $\gamma_0 \in (1, \infty)$, then equation (2.2) becomes

$$\det\left[I + \sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(\gamma_0-1)}\right] = 0$$

and

$$\det\left[I - \sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1-\gamma_0)}\right] = 0,$$

this implies that

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{\gamma_0^{-k_i}}{(1-\gamma_0)}\right) = 1,$$

but, by condition (2.1), this is impossible, indeed, we have for $i = 1, 2, ..., m$,

$$\max_{\gamma_0 > 1} \frac{\gamma_0^{-k_i}}{(1-\gamma_0)} = \frac{(k_i + 1)^{k_i+1}}{k_i^{k_i}},$$

then we get

$$\lambda\left(\sum_{i=1}^{m} P_i \frac{(k_i + 1)^{k_i+1}}{k_i^{k_i}}\right) \leq 1$$

which is a contradiction to (2.1).

If $\gamma_0 = 1,$ then we also obtain a contradiction to (2.1) as above. Thus the proof is complete. □

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