Far East Journal of Mathematical Sciences (FJMS)
Volume 72, Number 1, 2013, Pages 37-56
Published Online: December 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

MESHLESS LOCAL PETROV-GALERKIN METHOD FOR THE TWO-DIMENSIONAL DIFFUSION PROBLEM WITH NON-LOCAL BOUNDARY CONDITION

T. Techapirom[1] **and A. Luadsong**[1,2,*]

[1]Department of Mathematics
Faculty of Science
King Mongkut's University of Technology Thonburi (KMUTT)
126 Pracha-utid Road, Bangmod, Toongkru
Bangkok, 10140, Thailand

[2]Centre of Excellence in Mathematics
CHE, Si Ayutthaya, Bangkok 10400, Thailand
e-mail: anirut.lua@kmutt.ac.th

Abstract

In this paper, a meshless local Petrov-Galerkin (MLPG) method is presented to treat the diffusion equation with non-local boundary condition on a square domain. General papers use a penalty parameter imposed on Dirichlet's boundary conditions for mixed problems. For this method, we use a penalty parameter imposed at Dirichlet's and Neumann's boundary conditions for mixed problems with non-local boundary conditions in each sub-domain. This implementation is verified to be efficient, more accurate and truly meshless.

2010 Mathematics Subject Classification: 65M99, 65N99.

Keywords and phrases: MLS, MLPG, diffusion equation, Dirichlet's and Neumann's boundary conditions, non-local boundary condition.

*Corresponding author

Received August 20, 2012

1. Introduction

MLPG method was first discovered by Atluri and Zhu [5] in 1998. This method has been described in textbooks [3, 12] and allows for the freedom to choose a test function. This method has been applied widely and very successfully in recent years. The method is based on local weak forms and moving least squares (MLS) approximation, and approaches the true solution of the problem. The main advantage of the MLPG method over the widely used finite element method and meshless methods is that it does not require a "finite element mesh" for the purposes of either interpolation or integration, thereby reducing costs. Effective implementations of meshless methods are key to success [2, 4-6, 8-11].

Finding the numerical solution with non-local boundary conditions is of importance in many fields of science and engineering research such as chemical diffusion, diffusion equation, thermoelasticity, heat conduction processes, heat transfer, control theory, medical schemes, and so on [16]. It is most widely used and very important in thermoelasticity. In 1963, Cannon [7] first introduced non-local boundary condition problems, and most investigations developed various problems as one-dimensional or two-dimensional, with the Dirichlet's or the Neumann's boundary condition, and in 2010, Abbasbandy and Shirzadi [1, 2] researched the MPLG method for a two-dimensional diffusion equation with the Neumann's boundary condition and non-classical boundary conditions, and a meshless method for a two-dimensional diffusion equation with an integral condition. The proposed method worked very well for the two-dimensional diffusion equations with a non-classical boundary condition, because of its simplicity and high accuracy.

The purpose of this article is to develop an efficient, more accurate and truly meshless method for solving the following two-dimensional time-dependent diffusion problem with non-local boundary conditions which had been studied by Abbasbandy and Shirzadi [2]. The diffusion equation can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad \forall (x, y, t) \in \Omega \times [0, T], \tag{1}$$

with initial condition

$$u(x, y, 0) = f(x, y), \quad \Omega = \{(x, y) | 0 \le x, y \le 1\} \tag{2}$$

and boundary conditions

$$\frac{\partial u(0, y, t)}{\partial x} = g_0(y, t), \quad 0 \le t \le T, \quad 0 \le y \le 1, \tag{3}$$

$$\frac{\partial u(1, y, t)}{\partial x} = g_1(y, t), \quad 0 \le t \le T, \quad 0 \le y \le 1, \tag{4}$$

$$u(x, 1, t) = h_1(x, t), \quad 0 \le t \le T, \quad 0 \le x \le 1, \tag{5}$$

$$u(x, 0, t) = h_0(x)\mu(t), \quad 0 \le t \le T, \quad 0 \le x \le 1, \tag{6}$$

and the non-local boundary condition

$$\int_\Omega u(x, y, t) d\Omega = m(t), \quad 0 \le x \le 1, \quad 0 \le y \le 1, \tag{7}$$

where f, g_0, g_1, h_0, h_1 and m are given functions, while the functions u and μ are unknowns. The non-local boundary condition is variable-separable, with spatial dependence given by $h_0(x)$ and time dependence given by $\mu(t)$.

Abbasbandy and Shirzadi [2] used a meshless local Petrov-Galerkin (MLPG) method to treat parabolic partial differential equations with the Dirichlet's and Neumann's conditions with a non-classical boundary. A difficulty in implementing the MLPG method to impose essential boundary conditions is that the moving least squares (MLS) trial functions do not pass through the nodal values. To overcome this difficulty, they used the MLPG method only inside the domain, while at boundaries, they used finite difference schemes in all boundary conditions. We present techniques for use on square domains using classical MLPG and penalty parameters imposed at the Dirichlet's and Neumann's boundary conditions for mixed problem with non-local boundary conditions, such that this implementation can be verified to be efficient, more accurate, and truly meshless.

It should be noted that the MLPG method has been successfully applied to diffusion problems with Dirichlet or Neumann boundary conditions in many papers such as [1, 2, 9, 14]. In this paper, we consider problems under a non-local boundary condition as in equation (7). Also, we impose the Dirichlet's and Neumann's boundary conditions, and we impose a non-local boundary condition in LWF with the classical MLPG method, making our method truly meshless.

2. The MLS Approximation Scheme

Using the moving least squares (MLS) approximation scheme, we can approximate the trial function for the displacement at each point. Consider a subdomain $\Omega_{\mathbf{x}}$, the neighborhood of a point $\mathbf{x}$ and denote as the domain of definition of the MLS approximation for the trial function at $\mathbf{x}$, which is located in the problem domain $\Omega_{\mathbf{x}}$. To approximate the distribution of function u in $\Omega_{\mathbf{x}}$ over a number of randomly located nodes $\mathbf{x}_i$, $i = 1, 2, ..., n$, the moving least squares approximation $u^h(\mathbf{x})$ of u, $\forall \mathbf{x} \in \Omega_{\mathbf{x}}$, can be defined by

$$u^h(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})\mathbf{a}(\mathbf{x}); \quad \forall \mathbf{x}, \tag{8}$$

where $\mathbf{P}^T(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), ..., p_m(\mathbf{x})]$ is a complete monomial basis of order m; and $\mathbf{a}(\mathbf{x})$ is a vector containing coefficients $a_j(\mathbf{x})$, $j = 1, 2, ..., m$, which are functions of the space coordinates $\mathbf{x}$. For example, for a two-dimensional problem, $\mathbf{P}^T(\mathbf{x}) = [1, x, y]$ and $\mathbf{P}^T(\mathbf{x}) = [1, x, y, x^2, xy, y^2]$, for linear basis $(m = 3)$ and quadratic basis $(m = 6)$, respectively.

The coefficient vector $\mathbf{a}(\mathbf{x})$ is determined by minimizing a weighted discrete L_2 norm, defined as

$$J(x) = \sum_{i=1}^{n} w_i(\mathbf{x})[p^T(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - \hat{u}_i]^2 = [\mathbf{P} \cdot \mathbf{a}(\mathbf{x}) - \hat{\mathbf{u}}]^T W[\mathbf{P} \cdot \mathbf{a}(\mathbf{x}) - \hat{\mathbf{u}}], \tag{9}$$

where $w_i(\mathbf{x})$ is the weight function associated with the node i, with $w_i(\mathbf{x}) > 0$ for all $\mathbf{x}$ in the support of $w_i(\mathbf{x})$, $\mathbf{x}_i$ denotes the value of $\mathbf{x}$ at node i, n is the number of nodes in $\Omega_{\mathbf{x}}$ for which the weight functions $w_i(\mathbf{x}) > 0$, the matrices $\mathbf{P}$ and $\mathbf{W}$ are defined as

$$\mathbf{P} = \begin{pmatrix} P^T(\mathbf{x}_1) \\ P^T(\mathbf{x}_2) \\ \cdots \\ P^T(\mathbf{x}_n) \end{pmatrix}_{n \times m}, \quad \mathbf{W} = \begin{pmatrix} w_1(\mathbf{x}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_n(\mathbf{x}) \end{pmatrix},$$

and $\hat{\mathbf{u}}^T = [\hat{u}_1, \hat{u}_2, ..., \hat{u}_n]$. Here it should be noted that $\hat{u}_i$, $i = 1, 2, ..., n$, in (9) are the fictitious nodal values, and not the nodal values of the unknown trial function $u^h(\mathbf{x})$ in general. The stationary of J in (9) with respect to $\mathbf{a}(\mathbf{x})$ leads to the following linear relation between $\mathbf{a}(\mathbf{x})$ and $\hat{\mathbf{u}}$:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\hat{\mathbf{u}}, \tag{10}$$

where the matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are defined by

$$\mathbf{A}(\mathbf{x}) = \mathbf{P}^T\mathbf{W}\mathbf{P} = \sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i), \tag{11}$$

$$\mathbf{B}(\mathbf{x}) = \mathbf{P}^T\mathbf{W} = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), ..., w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)]. \tag{12}$$

The MLS approximation is well defined only when the matrix $\mathbf{A}$ in (10) is non-singular. It can be seen that this is the case if and only if the rank of $\mathbf{P}$ equals m. A necessary condition for a well-defined MLS approximation is that at least m weight functions are non-zero (i.e., $n > m$) for each sample point $\mathbf{x} \in \Omega$ and that the nodes in $\Omega_{\mathbf{x}}$ are not arranged in a special pattern such as on a straight line. Here, a sample point may be a nodal point under consideration or a quadrature point.

Solving for $\mathbf{a}(\mathbf{x})$ from (10) and substituting it into (8) gives a relation which may be written as the form of an interpolation function similar to that used in FEM as

$$u^h(\mathbf{x}) = \Phi^T(\mathbf{x}) \cdot \hat{\mathbf{u}} = \sum_{i=1}^{n} \Phi_i(\mathbf{x}) \hat{u}_i, \quad u^h(\mathbf{x}) \equiv u_i, \quad \mathbf{x} \in \Omega_{\mathbf{x}}, \tag{13}$$

and essentially $u_i \neq \hat{u}_i$, where

$$\Phi^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \tag{14}$$

or

$$\phi_i(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) [\mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x})]. \tag{15}$$

Usually $\phi_i(\mathbf{x})$ is called the *shape function* of the MLS approximation corresponding to nodal point y_i. From (12) and (14), it may be seen that $\phi_i(\mathbf{x}) = 0$ when $w_i(\mathbf{x}) = 0$. In practical applications, $w_i(\mathbf{x})$ is generally chosen such that it is non-zero over the support of nodal points y_i. The support of the nodal point y_i is usually taken to be a circle of radius r_i, centered at y_i. The fact that $\phi_i(\mathbf{x}) = 0$, for $\mathbf{x}$ not in the support of nodal point y_i preserves the local character of the moving least squares approximation.

Let $C^q(\Omega)$ be the space of qth continuously differentiable functions on Ω. If $w_i(\mathbf{x}) \in C^q(\Omega)$ and $p_j(\mathbf{x}) \in C^s(\Omega)$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$, then $\phi_i(\mathbf{x}) \in C^r(\Omega)$ with $r = \min(q, s)$.

The partial derivatives of $\phi_i(\mathbf{x})$ are obtained as

$$\phi_{i,k} = \sum_{j=1}^{m} [p_{j,k}(\mathbf{A}^{-1}\mathbf{B})_{ji} + p_j(\mathbf{A}^{-1}\mathbf{B}_{,k} + \mathbf{A}^{-1}_{,k}\mathbf{B})_{ji}], \tag{16}$$

in which $\mathbf{A}_{,k}^{-1} = (\mathbf{A}^{-1})_{,k}$ represents the derivative of the inverse of $\mathbf{A}$ with respect to x_k, which is given by $\mathbf{A}_{,k}^{-1} = -\mathbf{A}^{-1}\mathbf{A}_{,k}\mathbf{A}^{-1}$, where $(\cdot)_{,i}$ denotes $\frac{\partial(\cdot)}{\partial x_i}$.

In this paper, the Gaussian weight function is used as

$$w_i(\mathbf{x}) = \begin{cases} \dfrac{\exp\left[-\left(\dfrac{d_i}{c_i}\right)^2\right] - \exp\left[-\left(\dfrac{r_i}{c_i}\right)^2\right]}{1 - \exp\left[-\left(\dfrac{r_i}{c_i}\right)^2\right]}, & 0 \le d_i \le r_i, \\ 0, & d_i \ge r_i, \end{cases} \tag{17}$$

where $d_i = \| x - x_i \|$, c_i is a constant controlling the shape of the weight function w_i and r_i is the size of the support domain.

The size of support, r_i, of the weight function w_i associated with node i should be chosen such that r_i should be large enough to have sufficient number of nodes covered in the domain of definition of every sample point $(n \ge m)$ to ensure the regularity of $\mathbf{A}$. A very small r_i may result in a relatively large numerical error in using Gauss numerical quadrature to calculate the entries in the system matrix. On the other hand, r_i should also be small enough to maintain the local character of the MLS approximation.

3. Local Weak Form

The MLPG method constructs the weak form over local sub-domains such as Ω_s, which is a small region taken for each node in the global domain Ω and may be of any geometric shape and size. In this paper, they are taken to be of circular shape. Because the weak form is constructed over local sub-domains, the formulation is called the "*local weak formulation*".

The local weak form of (1) for $x_i = (x^i, y^i) \in \Omega_s^i$ can be written as follows:

$$\int_{\Omega_s^i}\left(\frac{\partial u}{\partial t}-\nabla^2 u\right)v_i d\Omega = 0, \tag{18}$$

where v_i is a test function. Using the divergence theorem and the property

$$(\nabla^2 u)v_i = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}v_i\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}v_i\right)-\left(\frac{\partial u}{\partial x}\frac{\partial v_i}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v_i}{\partial y}\right)$$

yields the following expression:

$$\int_{\Omega_s^i}\frac{\partial u}{\partial t}v_i d\Omega-\int_{\partial\Omega_s^i}\frac{\partial u}{\partial n}v_i d\Gamma+\int_{\Omega_s^i}\frac{\partial u}{\partial x}\frac{\partial v_i}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v_i}{\partial y}d\Omega = 0, \tag{19}$$

where Ω_s^i is a circle of radius r_0 centered at $\mathbf{x}_i$, $\partial\Omega_s^i$ is the boundary of Ω_s^i, $n=(n_1, n_2)$ is the outward unit normal to the boundary $\partial\Omega_s^i = \Gamma_{su}^i \cup \Gamma_{sq_0}^i \cup \Gamma_{sq_1}^i$,

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}n_1+\frac{\partial u}{\partial y}n_2,$$

and yields the following expression:

$$\int_{\Omega_s^i}\frac{\partial u}{\partial t}v_i d\Omega-\int_{\Gamma_{su}^i}\frac{\partial u}{\partial n}v_i d\Gamma-\int_{\Gamma_{sq_0}^i}\frac{\partial u}{\partial n}v_i d\Gamma-\int_{\Gamma_{sq_1}^i}\frac{\partial u}{\partial n}v_i d\Gamma$$

$$+\int_{\Omega_s^i}\frac{\partial u}{\partial x}\frac{\partial v_i}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v_i}{\partial y}d\Omega = 0. \tag{20}$$

Substituting trial function $u^h(\mathbf{x}, t)=\sum_{j=1}^{n}\phi_j(\mathbf{x})\hat{u}_j(t)$ and $q\equiv\frac{\partial u}{\partial n}$ into (20) yields

$$\int_{\Omega_s^i}v_i\sum_{j=1}^{n}\phi_j\frac{\partial\hat{u}_j}{\partial t}d\Omega-\int_{\Gamma_{su}^i}v_i\sum_{j=1}^{n}\frac{\partial\phi_j}{\partial n}\hat{u}_j d\Gamma-\int_{\Gamma_{sq_0}^i}g_0 v_i d\Gamma$$

$$-\int_{\Gamma_{sq_1}^i}g_1 v_i d\Gamma+\int_{\Omega_s^i}\sum_{j=1}^{n}\frac{\partial\phi_j}{\partial x}\frac{\partial v_i}{\partial x}\hat{u}_j+\sum_{j=1}^{n}\frac{\partial\phi_j}{\partial y}\frac{\partial v_i}{\partial y}\hat{u}_j d\Omega = 0, \tag{21}$$

$$\sum_{j=1}^{n}\int_{\Omega_s^i} v_i\phi_j d\Omega \frac{\partial \hat{u}_j}{\partial t} - \sum_{j=1}^{n}\int_{\Gamma_{su}^i} v_i \frac{\partial \phi_j}{\partial n} d\Gamma \hat{u}_j$$

$$+ \sum_{j=1}^{n}\int_{\Omega_s^i} \frac{\partial \phi_j}{\partial x}\frac{\partial v_i}{\partial x} + \frac{\partial \phi_j}{\partial y}\frac{\partial v_i}{\partial y} d\Omega \hat{u}_j$$

$$= \int_{\Gamma_{sq0}^i} g_0 v_i d\Gamma + \int_{\Gamma_{sq1}^i} g_1 v_i d\Gamma, \tag{22}$$

where $i = 1, 2, 3, ..., n$. By substituting (13) into (20), governing equations are transformed into the discretized system, written in the matrix form as

$$\mathbf{K}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{F}\mathbf{U} = \mathbf{C}(t), \tag{23}$$

where **K**, **F** and **C** are matrices, which are described in the following:

$$\mathbf{K} = [K_{ij}], \quad K_{ij} = \int_{\Omega_s^i} \phi_j v_i d\Omega, \tag{24}$$

$$\mathbf{F} = [F_{ij}], \quad F_{ij} = \int_{\Omega_s^i} \frac{\partial \phi_j}{\partial x}\frac{\partial v_i}{\partial x} + \frac{\partial \phi_j}{\partial y}\frac{\partial v_i}{\partial y} d\Omega$$

$$+ \int_{\Gamma_{su0}^i} v_i \frac{\partial \phi_j}{\partial y} d\Gamma - \int_{\Gamma_{su1}^i} v_i \frac{\partial \phi_j}{\partial y} d\Gamma, \tag{25}$$

$$\mathbf{C}(t) = [C_i(t)], \quad C_i(t) = \int_{\Gamma_{sq0}^i} g_0 v_i d\Gamma + \int_{\Gamma_{sq1}^i} g_1 v_i d\Gamma. \tag{26}$$

There is a problem for MLS at boundary conditions because the trial function does not pass through the nodal values, so we impose a penalty parameter at the Dirichlet's and the Neumann's boundary conditions with $\alpha >> 1$:

$$\alpha\int_{\Gamma_{su0}^i} (u - h_0)v_i d\Gamma + \alpha\int_{\Gamma_{su1}^i} (u - h_1\mu)v_i d\Gamma + \alpha\int_{\Gamma_{sq0}^i} \left(\frac{\partial u}{\partial x} - g_0\right)v_i d\Gamma$$

$$+ \alpha\int_{\Gamma_{sq1}^i} \left(\frac{\partial u}{\partial x} - g_1\right)v_i d\Gamma = 0. \tag{27}$$

Substituting trial function $u^h(\mathbf{x}) = \sum_{j=1}^{n} \phi_j(\mathbf{x})\hat{u}_j$ in (27), yields

$$\sum_{j=1}^{n}\Bigg[\alpha\int_{\Gamma^i_{su0}} \phi_j(\mathbf{x})v_i d\Gamma + \alpha\int_{\Gamma^i_{sq0}} \frac{\partial\phi_j}{\partial x} v_i d\Gamma + \alpha\int_{\Gamma^i_{su1}} \phi_j(\mathbf{x})v_i d\Gamma$$

$$+ \alpha\int_{\Gamma^i_{sq1}} \frac{\partial\phi_j}{\partial x} v_i d\Gamma\Bigg]\hat{u}_j - \left(\alpha\int_{\Gamma^i_{su0}} h_0 v_i d\Gamma\right)\mu$$

$$= \alpha\int_{\Gamma^i_{su1}} h_1 v_i d\Gamma + \alpha\int_{\Gamma^i_{su0}} g_0 v_i d\Gamma + \alpha\int_{\Gamma^i_{sq1}} g_1 v_i d\Gamma. \tag{28}$$

By substituting (13) into (27), governing equations are transformed into the discretized system, written in the matrix form as

$$\mathbf{D}\hat{\mathbf{U}} - \mathbf{E}\hat{\mu} = \mathbf{H}(t), \tag{29}$$

where **D**, **E** and **H** are matrices, which are described in the following:

$$\mathbf{D} = [D_{ij}], \quad D_{ij} = \alpha\int_{\Gamma^i_{su0}} \phi_j(\mathbf{x})v_i d\Gamma + \alpha\int_{\Gamma^i_{sq0}} \frac{\partial\phi_j}{\partial x} v_i d\Gamma$$

$$+ \alpha\int_{\Gamma^i_{su1}} \phi_j(\mathbf{x})v_i d\Gamma + \alpha\int_{\Gamma^i_{sq1}} \frac{\partial\phi_j}{\partial x} v_i d\Gamma, \tag{30}$$

$$\mathbf{E} = [E_{ij}], \quad E_{ij} = \alpha\int_{\Gamma^i_{su0}} h_0 v_i d\Gamma, \tag{31}$$

$$\mathbf{H}(t) = [H_i(t)], \quad H_i(t) = \alpha\int_{\Gamma^i_{su1}} h_1 v_i d\Gamma$$

$$+ \alpha\int_{\Gamma^i_{sq0}} g_0 v_i d\Gamma + \alpha\int_{\Gamma^i_{sq1}} g_1 v_i d\Gamma. \tag{32}$$

To discrete time variable, the Crank-Nicolson technique is applied to (23) which yields

$$\mathbf{K}\frac{\hat{\mathbf{U}}^{k+1}-\hat{\mathbf{U}}^{k}}{\Delta t}+\frac{\mathbf{F}}{2}(\mathbf{U}^{k+1}-\mathbf{U}^{k})=\mathbf{C}^{k+\frac{1}{2}}. \tag{33}$$

3.1. Time discretization I

Equation (29) is discretized at time level $k+\frac{1}{2}$, obtaining:

$$\mathbf{D}\hat{\mathbf{U}}^{k+\frac{1}{2}}-\mathbf{E}\hat{\mu}^{k+\frac{1}{2}}=\mathbf{H}^{k+\frac{1}{2}}, \tag{34}$$

$$\frac{\mathbf{D}}{2}(\hat{\mathbf{U}}^{k+1}+\hat{\mathbf{U}}^{k})-\frac{\mathbf{E}}{2}(\hat{\mu}^{k+1}+\hat{\mu}^{k})=\mathbf{H}^{k+\frac{1}{2}}. \tag{35}$$

Subtracting (35) from (33), we obtain:

$$\mathbf{K}\frac{\hat{\mathbf{U}}^{k+1}-\hat{\mathbf{U}}^{k}}{\Delta t}+\frac{\mathbf{F}}{2}(\hat{\mathbf{U}}^{k+1}+\hat{\mathbf{U}}^{k})-\frac{\mathbf{D}}{2}(\hat{\mathbf{U}}^{k+1}+\hat{\mathbf{U}}^{k})+\frac{\mathbf{E}}{2}(\hat{\mu}^{k+1}+\hat{\mu}^{k})$$

$$=\mathbf{C}^{k+\frac{1}{2}}-\mathbf{H}^{k+\frac{1}{2}}, \tag{36}$$

$$\left(\mathbf{K}+\frac{\Delta t}{2}\mathbf{F}-\frac{\Delta t}{2}\mathbf{D}\right)\hat{\mathbf{U}}^{k+1}+\frac{\Delta t}{2}\mathbf{E}\hat{\mu}^{k+1}$$

$$=\Delta t\left(\mathbf{C}^{k+\frac{1}{2}}-\mathbf{H}^{k+\frac{1}{2}}\right)+\left(\mathbf{K}-\frac{\Delta t}{2}\mathbf{F}+\frac{\Delta t}{2}\mathbf{D}\right)\hat{\mathbf{U}}^{k}-\frac{\Delta t}{2}\mathbf{E}\hat{\mu}^{k}. \tag{37}$$

From (37), assuming that $\hat{\mu}_i^k$, for $i = 1, 2, ..., N$ and $\hat{\mu}^k$ are known, our aim is to compute $\hat{\mu}_i^{k+1}$, for $i = 1, 2, ..., N$ and $\hat{\mu}^{k+1}$. Now we have $N+1$ unknowns so that we need one equation to compute these unknowns, which can be obtained from the non-local boundary condition from (7), and substituting trial function $u^h(\mathbf{x}) = \sum_{j=1}^{n}\phi_j(\mathbf{x})\hat{\mu}_j$, yields:

$$\int_{\Omega}(u^h)^{k+1}(\mathbf{x})d\Omega = \int_{\Omega}\sum_{j=1}^{n}\phi_j(\mathbf{x})\hat{u}_j^{k+1}d\Omega$$

$$= \sum_{j=1}^{n}\left[\int_{\Omega}\phi_j(\mathbf{x})d\Omega\right]\hat{u}_j^{k+1} = m^{k+1}, \tag{38}$$

which written in the matrix form as

$$\mathbf{S}\hat{\mathbf{U}}^{k+1} = m^{k+1}, \tag{39}$$

where **S** is a matrix, which is described in the following:

$$\mathbf{S} = [S_j], \quad S_j = \int_{\Omega}\phi_j(\mathbf{x})d\Omega. \tag{40}$$

Equations (37) and (39) can be written in the matrix form as follows:

$$\widetilde{\mathbf{A}}\hat{\mathbf{U}}^{k+1} = \widetilde{\mathbf{B}}\hat{\mathbf{U}}^{k} + \widetilde{\mathbf{C}}, \tag{41}$$

where

$$\widetilde{\mathbf{A}} = \begin{pmatrix} \mathbf{K} + \frac{\Delta t}{2}\mathbf{F} - \frac{\Delta t}{2}\mathbf{D} & \frac{\Delta t}{2}\mathbf{E} \\ \mathbf{S} & 0 \end{pmatrix}, \tag{42}$$

$$\widetilde{\mathbf{B}} = \begin{pmatrix} \mathbf{K} - \frac{\Delta t}{2}\mathbf{F} + \frac{\Delta t}{2}\mathbf{D} & -\frac{\Delta t}{2}\mathbf{E} \\ \mathbf{O}^T & 0 \end{pmatrix}, \tag{43}$$

$$\widetilde{\mathbf{C}} = \begin{pmatrix} \Delta t\left(\mathbf{C}^{k+\frac{1}{2}} - \mathbf{H}^{k+\frac{1}{2}}\right) \\ m^{k+1} \end{pmatrix}. \tag{44}$$

3.2. Time discretization II

The second scheme, we discretize (29) at time level $k+1$, we obtain:

$$\mathbf{D}\hat{\mathbf{U}}^{k+1} - \mathbf{E}\hat{\mu}^{k+1} = \mathbf{H}^{k+1}. \tag{45}$$

Subtracting (45) from (33), we obtain:

$$\mathbf{K}\frac{\hat{\mathbf{U}}^{k+1}-\hat{\mathbf{U}}^{k}}{\Delta t}+\frac{\mathbf{F}}{2}(\hat{\mathbf{U}}^{k+1}+\hat{\mathbf{U}}^{k})-\mathbf{D}\hat{\mathbf{U}}^{k+1}+\mathbf{E}\hat{\mu}^{k+1}$$

$$=\mathbf{C}^{k+\frac{1}{2}}-\mathbf{H}^{k+1}, \tag{46}$$

$$\left(\mathbf{K}+\frac{\Delta t}{2}\mathbf{F}-\Delta t\mathbf{D}\right)\hat{\mathbf{U}}^{k+1}+\Delta t\mathbf{E}\hat{\mu}^{k+1}$$

$$=\Delta t\left(\mathbf{C}^{k+\frac{1}{2}}-\mathbf{H}^{k+1}\right)+\left(\mathbf{K}-\frac{\Delta t}{2}\mathbf{F}\right)\hat{\mathbf{U}}^{k}. \tag{47}$$

So we need one equation same the first scheme, that can be obtained from the non-local boundary condition from (7):

$$\int_{\Omega}(u^{h})^{k+1}(\mathbf{x})d\Omega=\int_{\Omega}\sum_{j=1}^{n}\phi_{j}(\mathbf{x})\hat{u}_{j}^{k+1}d\Omega$$

$$=\sum_{j=1}^{n}\left[\int_{\Omega}\phi_{j}(\mathbf{x})d\Omega\right]\hat{u}_{j}^{k+1}=m^{k+1}, \tag{48}$$

which written in the matrix form as

$$\mathbf{S}\hat{\mathbf{U}}^{k+1}=m^{k+1}, \tag{49}$$

where $\mathbf{S}$ is a matrix, which is described in the following:

$$\mathbf{S}=[S_{j}],\quad S_{j}=\int_{\Omega}\phi_{j}(\mathbf{x})d\Omega. \tag{50}$$

Equations (47) and (49) can be written in the matrix form as follows:

$$\widetilde{\mathbf{A}}\hat{\mathbf{U}}^{k+1}=\widetilde{\mathbf{B}}\hat{\mathbf{U}}^{k}+\widetilde{\mathbf{C}}, \tag{51}$$

where

$$\widetilde{\mathbf{A}}=\begin{pmatrix}\mathbf{K}+\frac{\Delta t}{2}\mathbf{F}-\Delta t\mathbf{D} & \Delta t\mathbf{E}\\ \mathbf{S} & 0\end{pmatrix}, \tag{52}$$

$$\widetilde{\mathbf{B}} = \begin{pmatrix} \mathbf{K} - \frac{\Delta t}{2}\mathbf{F} & \mathbf{O} \\ \mathbf{O}^T & 0 \end{pmatrix}, \tag{53}$$

$$\widetilde{\mathbf{C}} = \begin{pmatrix} \Delta t\left(\mathbf{C}^{k+\frac{1}{2}} - \mathbf{H}^{k+1}\right) \\ m^{k+1} \end{pmatrix}. \tag{54}$$

4. Numerical Experiments

Two schemes are presented in this section. The numerical solutions in the first scheme uses a time step of $k + \frac{1}{2}$ with regular and irregular nodes and the second scheme uses a time step of $k + 1$ with regular nodes. The results are presented in Tables 1, 2 and 3, respectively. For both schemes, we use the quadratic basis in the MLS approximations. The relative error is reported in the tables, defined as follows:

$$\| u \|_R = \sqrt{\frac{\sum_{i=1}^{n} (u_i - \widehat{u}_i)^2}{\sum_{i=1}^{n} (u_i)^2}},$$

$$\| \mu \|_\infty = \max| \widehat{\mu}_i - \mu_i |.$$

For both schemes with non-local boundary conditions, we consider (1)-(7) with the following conditions:

$$f(x, y) = \exp(x + y),$$

$$g_0(y, t) = \exp(y + 2t),$$

$$g_1(y, t) = \exp(1 + y + 2t),$$

$$h_1(x, t) = \exp(1 + x + 2t),$$

$$h_0(x) = \exp(x),$$

$$m(t) = \exp(2t)(\exp(2) - 2\exp(1) + 1),$$

for which the exact solution is

$$u(x, y, t) = \exp(x + y + 2t), \quad \mu(t) = \exp(2t).$$

Table 1 compares the results obtained for u and μ, between imposing at Dirichlet's boundary condition and imposing at Dirichlet's and Neumann's boundary conditions, at several time levels with $\Delta t = 0.1$ on regular nodes in Figure 1 $(N = 441$ nodes$)$ from time step $k + \frac{1}{2}$.

Table 2 compares the results obtained for u and μ, between imposing at Dirichlet's boundary condition and imposing at Dirichlet's and Neumann's boundary conditions at several time levels, with $\Delta t = 0.1$ on irregular nodes in Figure 2 $(N = 121$ nodes$)$ from time step $k + \frac{1}{2}$.

Table 3 compares the results obtained for u and μ, between imposing at Dirichlet's boundary condition and imposing at Dirichlet's and Neumann's boundary condition at several time levels with $\Delta t = 0.1$ on regular nodes from time step $k + 1$.

In each case above $T = 1$, the relative errors have increased slightly because of the accumulated error in each time step. For $T > 1$, the results will have the same manner as $T = 1$.

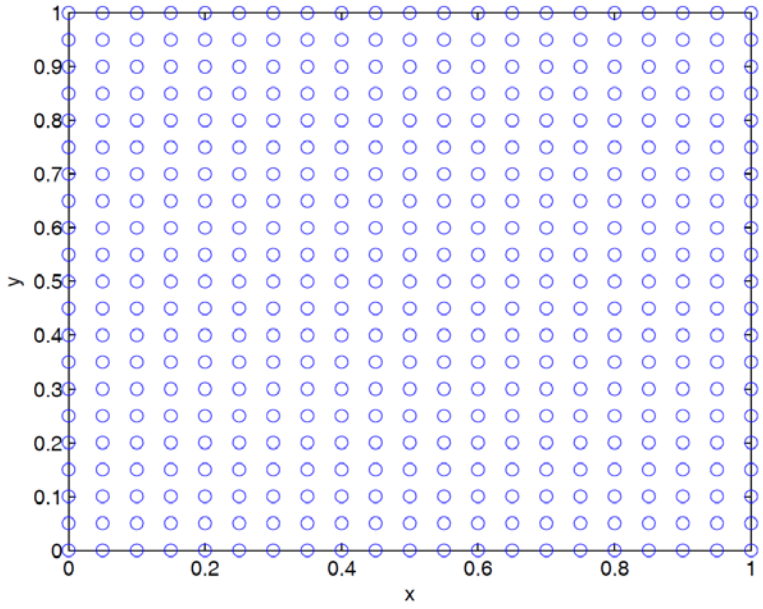

Figure 1. Regular nodes in the interval $[0, 1] \times [0, 1]$.

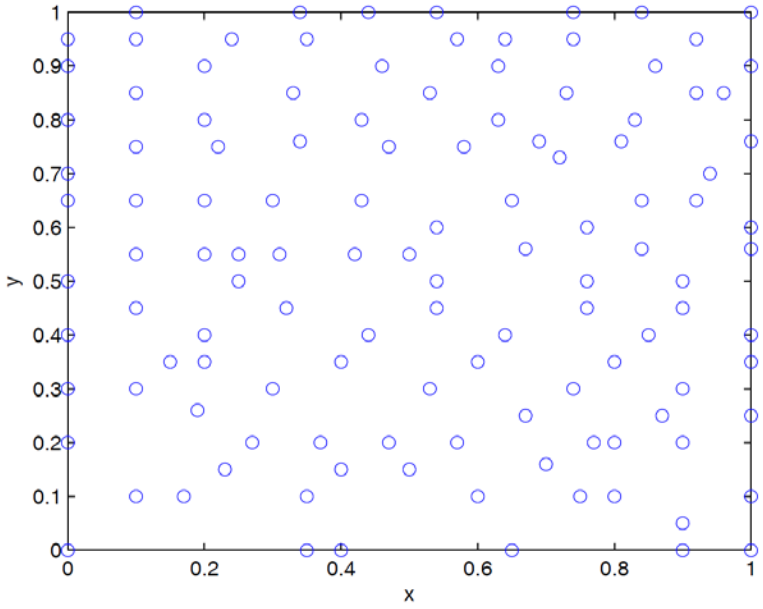

Figure 2. Irregular nodes in the interval $[0, 1] \times [0, 1]$.

Table 1. Relative error for diffusion problem with non-local BC imposed at Dirichlet's boundary conditions, compared with imposing at Dirichlet's and Neumann's boundary conditions time step $k+\frac{1}{2}$ with regular nodes $[21\times 21]$

t	Imposing on Dirichlet's BC		Imposing on both types of BC	
	$r_0\times 10^{-1}$	$\|\mu\|_\infty\times 10^{-1}$	$r_0\times 10^{-3}$	$\|\mu\|_\infty\times 10^{-2}$
0.1	1.3320	5.2	11.2128	6.9
0.2	1.0053	3.4	4.0078	1.4
0.3	1.1706	6.3	9.1762	1.0
0.4	1.0509	6.0	5.1589	8.7
0.5	1.1300	8.8	8.2067	3.6
0.6	1.0695	9.4	5.7409	11.5
0.7	1.1121	12.7	7.6579	7.0
0.8	1.0797	14.4	6.0962	15.6
0.9	1.1031	18.6	7.3275	12.0
1	1.0854	21.9	6.3234	21.8

Table 2. Relative error for diffusion problem with non-local BC imposed at Dirichlet's boundary conditions, compared with imposing at Dirichlet's and Neumann's boundary conditions time step $k+\frac{1}{2}$ with irregular nodes $[11\times 11]$

t	Imposing on Dirichlet's BC		Imposing on both types of BC	
	$r_0\times 10^{-1}$	$\|\mu\|_\infty\times 10^{-1}$	$r_0\times 10^{-2}$	$\|\mu\|_\infty\times 10^{-1}$
0.1	1.3259	7.9	5.3081	0.8
0.2	0.7438	5.1	1.7463	0.2
0.3	1.0063	8.5	4.3908	1.1
0.4	0.8422	8.8	2.3207	0.4
0.5	0.9411	12.1	3.9321	1.4
0.6	0.8665	13.5	2.6106	0.8
0.7	0.9200	17.7	3.6561	1.9
0.8	0.8776	20.4	2.7891	1.4
0.9	0.9099	26.1	3.4811	2.7
1	0.8841	30.7	2.9063	2.3

Table 3. Relative error for diffusion problem with non-local BC imposed at Dirichlet's boundary conditions, compared with imposing at Dirichlet's and Neumann's boundary conditions time step $k+1$ with regular nodes $[21\times 21]$

t	Imposing on Dirichlet's BC		Imposing on both types of BC	
	$r_0 \times 10^{-1}$	$\|\mu\|_\infty \times 10^{-1}$	$r_0 \times 10^{-4}$	$\|\mu\|_\infty \times 10^{-3}$
0.1	1.3014	5.5	2.9624	2.2
0.2	1.0434	3.5	1.9259	1.4
0.3	1.1575	6.6	2.3226	2.5
0.4	1.0826	6.2	2.1223	2.5
0.5	1.1320	9.2	2.2270	3.5
0.6	1.0943	9.8	2.1583	3.9
0.7	1.1212	13.3	2.2009	5.1
0.8	1.1000	15.0	2.1712	6.0
0.9	1.1157	19.5	2.1906	7.6
1	1.1032	22.8	2.1769	9.1

5. Conclusions

In this paper, an MLPG method was proposed for the study of two-dimensional diffusion equations for mixed problems with non-local boundary conditions. Two schemes were proposed: the first scheme uses a time step of $k+\frac{1}{2}$ and the second scheme uses a time step of $k+1$. Both implementations were proposed to impose the Dirichlet's and Neumann's boundary conditions on square domains. The numerical results show that the second scheme produces a higher accuracy than imposing at only Dirichlet's boundary condition for mixed problem with regular nodes. The non-local integral boundary condition was discretized using Simpson's composite numerical integration rule and the resulting discretized equation was approximated using MLS approximations. Our proposed implementation has higher accuracy and is truly meshless. Implementing our proposed methods is a further research opportunity for interested researchers.

Acknowledgements

This research is partially supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand. The authors would like to thank their adviser for providing advice and taking care of this research, and BSRU for providing a scholarship. Finally, the authors would like to thank their parents and family, who have also been encouraging, as well as their friends for taking care of each other.

References

[1] S. Abbasbandy and A. Shirzadi, A meshless method for two-dimensional diffusion equation with an integral condition, Engineering Anal. 34 (2010), 1031-1037.

[2] S. Abbasbandy and A. Shirzadi, MLPG method for two-dimensional diffusion equation with Neumann's and non-classical boundary conditions, Appl. Numer. Math. 61(2) (2011), 170-180.

[3] S. N. Atluri, The Meshless Method (MLPG) for Domain and BIE Discretizations, Tech. Science Press, 2004.

[4] S. N. Atluri and Shengping Shen, The meshless local Petrov-Galerkin (MLPG) method: a simple and less-costly alternative to the finite element and boundary element methods, CMES 3(1) (2002), 11-51.

[5] S. N. Atluri and T. L. Zhu, A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics, Comput. Mech. 22 (1998), 117-127.

[6] S. N. Atluri and T. L. Zhu, The meshless local Petrov-Galerkin (MLPG) approach for solving problems in elasto-statics, Comput. Mech. 25 (2000), 169-179.

[7] J. R. Cannon, The solution of the heat equation subject to the specification of energy, Quart. Appl. Math. 21 (1963), 155-160.

[8] M. Dehghan, The one-dimensional heat equation subject to a boundary integral specification, Appl. Math. 32 (2005), 661-675.

[9] W. X. Hong and T. Quan, Meshless method based on the local weak-forms for steady-state heat conduction problems, MFPE 51 (2008), 3103-3112.

[10] A. J. Katz, Meshless methods for computational fluid dynamics, Dissertation Dept. Aeronautics and Astronautics of Stanford University, 2009.

[11] G. R. Liu, Meshfree Methods: Moving Beyond the Finite Element Method, 2nd ed., USA, 2009, pp. 237-272.

[12] G. R. Liu and Y. T. Gu, An Introduction to Meshfree Methods and their Programming, USA, 2005, pp. 54-131.

[13] J. Martin-Vaquero and J. Vigo-Aguiar, A note on efficient techniques for the second-order parabolic equation subject to non-local conditions, Appl. Math. 59 (2009), 1258-1264.

[14] S. T. Mohyud-Din and A. Yildirim, On two-dimensional diffusion with integral condition, J. King Saud University - Science 23(2) (2011), 121-125.

[15] A. Nealen, An As-Short-As-Possible Introduction to the Least Squares, Weighted Least Squares and Moving Least Squares Methods for Scattered Data Approximation and Interpolation, Discrete Geometric Modeling Group TU Darmstadt, 2004.

[16] A. A. Pisano, A. Sofi and P. Fuschi, Nonlocal integral elasticity: 2D finite element based solutions, Internat. J. Solids Structures 46(21) (2009), 3836-3849.

[17] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, Numerical Recipes in FORTRAN, Second Edition, Cambridge University Press, New York, 1992.